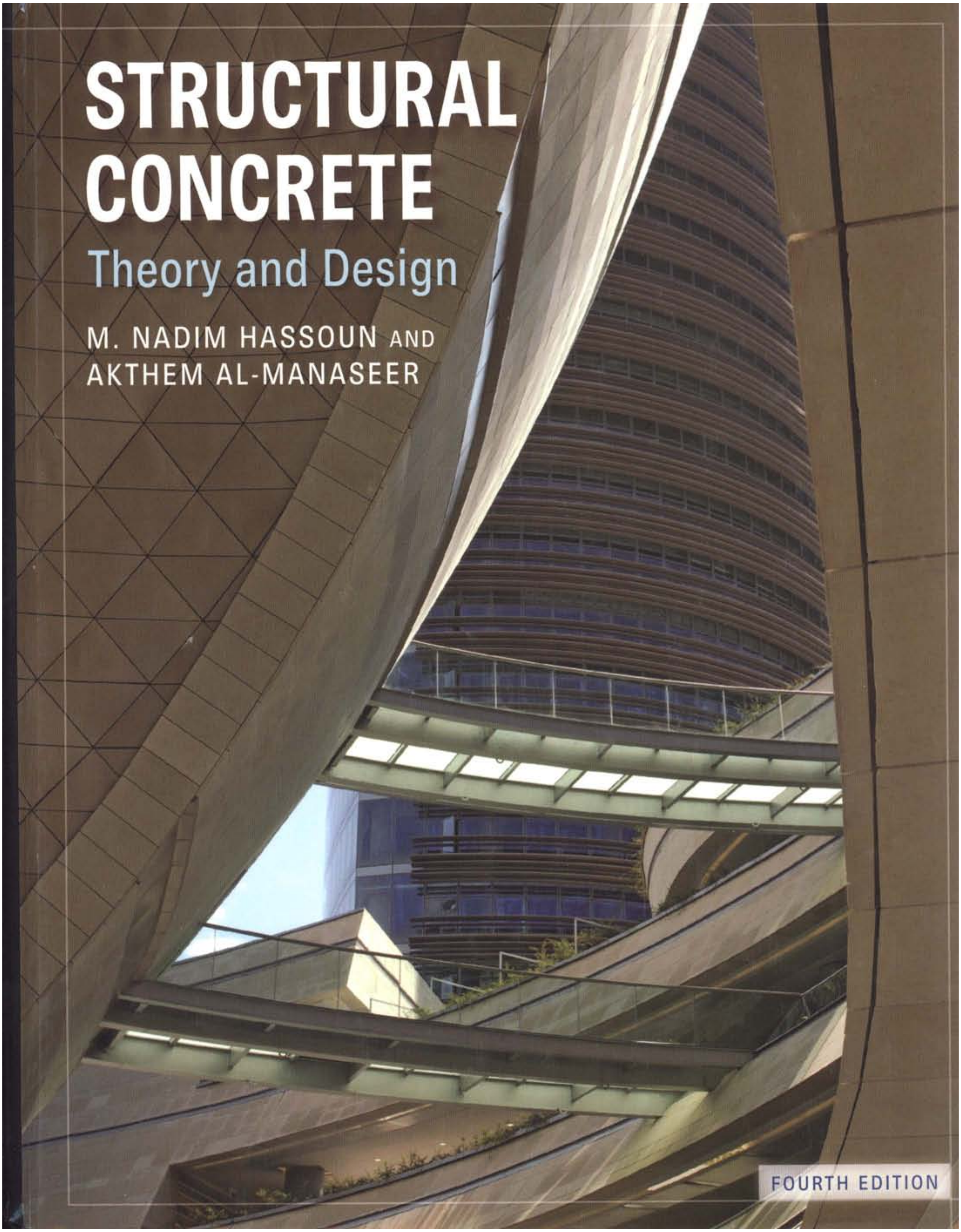
STRUCTURAL CONCRETE

Theory and Design

M. NADIM HASSOUN AND
AKTHEM AL-MANASEER



FOURTH EDITION

Structural Concrete

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Theory and Design

Fourth Edition

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A companion Web site for the book is available at www.wiley.com/college/hassoun. This Web site contains MSExcel spreadsheets that enable students to evaluate different design aspects of concrete members in an interactive environment, and a solutions manual for instructors.

PREFACE

The main objective of a course on structural concrete design is to develop, in the engineering student, the ability to analyze and design a reinforced concrete member subjected to different types of forces in a simple and logical manner using the basic principles of statistics and some empirical formulas based on experimental results. Once the analysis and design procedure is fully understood, its application to different types of structures becomes simple and direct, provided that the student has a good background in structural analysis.

The material presented in this book is based on the requirements of the American Concrete Institute (ACI) Building Code (318–08). Also, information has been presented on material properties, including volume changes of concrete, stress–strain behavior, creep, and elastic and nonlinear behavior or reinforced concrete.

Concrete structures are widely used in the United States and almost all over the world. The progress in the design concept has increased in the last few decades, emphasizing safety, serviceability, and economy. To achieve economical design of a reinforced concrete member, specific restrictions, rules, and formulas are presented in the codes to ensure both safety and reliability of the structure. Engineering firms expect civil engineering graduates to understand the code rules and, consequently, to be able to design a concrete structure effectively and economically with minimum training period or overhead costs. Taking this into consideration, this book is written to achieve the following objectives:

1. To present the material for the design of reinforced concrete members in a simple and logical approach.
2. To arrange the sequence of chapters in a way compatible with the design procedure of actual structures.
3. To provide a large number of examples in each chapter in clear steps to explain the analysis and design of each type of structural member.
4. To provide an adequate number of practical problems at the end of each chapter to achieve a high level of comprehension.
5. To explain the failure mechanism of a reinforced concrete beam due to flexure and to develop the necessary relationships and formulas for design.

6. To explain *why* the code used specific equations and specific restrictions on the design approach based either on a mathematical model or experimental results. This approach will improve the design ability of the student.
7. To provide adequate number of design aids to help the student in reducing the repetitive computations of specific commonly used values.
8. To enhance the student's ability to use a total quality and economical approach in the design of concrete structures and to help the student to design reinforced concrete members with confidence.
9. To explain the nonlinear behavior and the development of plastic hinges and plastic rotations in continuous reinforced concrete structures.
10. To provide a summary at the end of each chapter to help the student to review the materials of each chapter separately.
11. To provide new information on the design of special members, such as beams with variable depth (Chapter 8), stairs (Chapter 18), seismic design utilizing IBC 2006 (Chapter 20), and beams curved in plan (Chapter 21), that are not covered in other books on concrete.
12. To present information on the design of reinforced concrete frames, principles of limit design, and moment redistribution in continuous reinforced concrete structures.
13. To provide examples in SI units in all chapters of the book. Equivalent conversion factors from customary units to SI units are also presented. Design tables in SI units are given in Appendix B.
14. References are presented at the end of most chapters.

The book is an outgrowth of the author's lecture notes, which represent their teaching and industrial experience over the past 28 years. The industrial experience of the authors includes the design and construction supervision and management of many reinforced, prestressed, and precast concrete structures. This is in addition to the consulting work they performed for international design and construction firms, professional registration in the United Kingdom, Canada, and other countries, and a comprehensive knowledge of other European codes on the design of concrete structures.

The book is written to cover two courses in reinforced concrete design. Depending on the proficiency required, the first course may cover Chapters 1 through 11 and part of Chapter 13, whereas the second course may cover the remaining chapters. Parts of the late chapters may also be taught in the first course as needed. A number of optional sections have been included in various chapters. These sections are indicated by an asterisk (*) in the Table of Contents and may easily be distinguished from those that form the basic requirements of the first course. The optional sections may be covered in the second course or relegated to a reading assignment. Brief descriptions of the chapters are given below.

The first chapter of the book presents information on the historical development of concrete, codes of practice, loads and safety provisions, and design philosophy and concepts. The second chapter deals with the properties of concrete as well as steel reinforcement used in the design of reinforced concrete structures, including stress–strain relationships, modulus of elasticity and shear modulus of concrete, shrinkage, creep, fire resistance, high-performance concrete, and fibrous concrete. Because the current ACI Code emphasizes the strength approach based on strain limits, this approach has been adopted throughout the text. Chapters 3 and 4 cover the analysis and design of reinforced concrete sections based on strain limits. The behavior of reinforced concrete beams loaded to failure, the types of flexural failure, and failure mechanism are explained very clearly. It is essential for the student to understand the failure concept and the inherent reserve strength and ductility before using the necessary design formulas.

Chapter 5 covers alternative design methods based on methods described in Appendix A, B, and C of the ACI code. It explains the alternative load factors with the relative strength reduction factors and describes the strut and tie provisions.

Chapter 6 deals with the serviceability of reinforced concrete beams, including deflection and control of cracking. Chapters 7 and 8 cover the bond, development length, shear, and diagonal tension. In Chapter 8, expressions are presented for the design of members of variable depth in addition to prismatic sections and deep beams. It is quite common sometimes to design members with variable depth in actual structures. An example is introduced to explain the design of deep beams using the strut and tie approach.

Chapter 9 covers the design of one-way slabs, including joist-floor systems. Distributions of loads from slabs to beams and columns are also presented in this chapter to enhance the student's understanding of the design loads on each structural component. Chapter 10, 11, and 12 cover the design of axially loaded, eccentrically loaded, and long columns, respectively. Chapter 10 allows the student to understand the behavior of columns, failure conditions, ties and spirals, and other code limitations. Absorbing basic information, the student is introduced in Chapter 11 to the design of columns subjected to compression and bending. New mathematical models are introduced to analyze column sections controlled by compression or tension stresses. Biaxial bending for rectangular and circular columns are introduced using Bresler, PCA, and Hsu methods. Design of long columns is presented in Chapter 12 using the ACI moment-magnifier method.

Chapter 13 and 14 cover the design of footings and retaining walls, whereas Chapter 15 covers the design of reinforced concrete sections for shear and torsion. Torsional theories as well as ACI Code design procedure are explained. Chapter 16 deals with continuous beams and frames. A unique feature of this chapter is the introduction of the design of frames, frame hinges, limit state design collapse mechanism, rotation and plastic hinges, and moment redistribution. Adequate examples are presented to explain these concepts.

Design of two-way slabs introduced in Chapter 17. All types of two-way slabs, including waffle slabs, are presented with adequate examples. Summary of the design procedure is introduced with tables and diagrams. Chapter 18 covers the design of reinforced concrete stairs. Slabtype as well as stepped-type stairs are explained. The second type, although quite common, has not been covered in any text. Chapter 19 covers an introduction to prestressed concrete. Methods of prestressing, fully and partially prestressed concrete design, losses, and shear design are presented with examples. Chapter 20 presents the seismic design and analysis of members utilizing the IBC 2006 and the ACI code. Chapter 21 deals with the design of curved beams. In actual structures curved beams are used frequently. These beams are subjected to flexure, shear, and torsion.

In Appendix A and B of this book, design tables using customary units and SI units are presented.

The photos shown in this book were taken by the authors. We wish to express appreciation to John Gardner and Murat Saatcioglu from the University of Ottawa, Canada, for the photos provided in the seismic chapter.

Our sincere thanks go out to Nadim Wehbe, South Dakota State University, Ahmet Pamuk, Florida A&M University, M. Issa, University of Illinois Chicago, and Faisal Wafa, King Abdul-Aziz University, for their constructive comments to this edition. Our thanks to Basile Rabbat of the Portland Cement Association, Skokie, Illinois, for many discussions on the code interpretation.

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Finally, the book is written to provide basic reference materials on the analysis and design of structural concrete members in a simple, practical, and logical approach. Because this is a required course for seniors in civil engineering, we believe this book will be accepted by reinforced concrete instructors at different universities as well as designers who can make use of the information in their practical design of reinforced concrete structures.

M. Nadim Hassoun *Akthem Al-Manaseer*

NOTATION

c	Distance from extreme compression fiber to neutral axis
c_2	Side of rectangular column measured transverse to the span
C	Cross-sectional constant $\sum (1 - 0.63x/y)x^3 y/3$; compression force
C_c	Compression force in a concrete section with a depth equal to a
C_m	Correction factor applied to the maximum end moment in columns
C_r	Creep coefficient = creep strain per unit stress per unit length
C_s	Force in compression steel
C_t	Factor relating shear and torsional stress properties $= b_w d/\sum x^2 y$
C_w	Compression force in web
C_1	Force in the compression steel
d	Distance from extreme compression fiber to centroid of tension steel
d'	Distance from extreme compression fiber to centroid of compression steel
d_b	Nominal diameter of reinforcing bar
d_c	Distance from tension extreme fiber to center of bar closest to that fiber, used for crack control
d_t	Distance from extreme compression fibers to extreme tension steel
D	Dead load, diameter of a circular section
e	Eccentricity of load
e'	Eccentricity of load with respect to centroid of tension steel
E	Modulus of elasticity, force created by earthquake
E_c	Modulus of elasticity of concrete $= 33w^{1.5}\sqrt{f'_c}$
E_{cb}	Modulus of elasticity of beam concrete
E_{cc}	Modulus of elasticity of column concrete
E_{cs}	Modulus of elasticity of slab concrete
EI	Flexural stiffness of compression member
E_s	Modulus of elasticity of steel $= 29 \times 10^6$ psi $= 2 \times 10^5$ MPa
f	Flexural stress

f_c	Maximum flexural compressive stress in concrete due to service loads
f_{ca}	Allowable compressive stress in concrete (alternate design method)
f'_c	28-day compressive strength of concrete (standard cylinder strength)
f_d	Compressive strength of concrete at transfer (initial prestress)
f_{pc}	Compressive stress in concrete due to prestress after all losses
f_{pe}	Compressive stress in concrete at extreme fiber due to the effective prestressing force after all losses
f_{ps}	Stress in prestress steel at nominal strength
f_{pu}	Tensile strength of prestressing tendons
f_{py}	Yield strength of prestressing tendons
f_r	Modulus of rupture of concrete $= 7.5\lambda\sqrt{f'_c}$ psi
f_s	Stress in tension steel due to service load
f'_s	Stress in the compression steel due to service load
f_{se}	Effective stress in prestressing steel after all losses
f_t	Tensile stress in concrete
f_y	Yield strength of steel reinforcement
F	Lateral pressure of liquids
F_n	Nominal strength of a strut, tie, or nodal zone
F_{ns}	Nominal strength of a strut
F_{nt}	Nominal strength of a tie
G	Shear modulus of concrete (in torsion) $= 0.45E_c$
h	Total depth of beam or slab or column
h_f	Depth of flange in flanged sections
h_p	Total depth of shearhead cross section
H	Lateral earth pressure
I	Moment of inertia
I_b	Moment of inertia of gross section of beam about its centroidal axis
I_c	Moment of inertia of gross section of column
I_{cr}	Moment of inertia of cracked transformed section
I_e	Effective moment of inertia, used in deflection
I_g	Moment of inertia of gross section neglecting steel
I_s	Moment of inertia of gross section of slab
I_{se}	Moment of inertia of steel reinforcement about centroidal axis of section
J	Polar moment of inertia
K	Kip = 1000 lb, a factor used to calculate effective column length
K_b	Flexural stiffness of beam
K_c	Flexural stiffness of column
K_{ec}	Flexural stiffness of equivalent column
K_s	Flexural stiffness of slab
K_t	Torsional stiffness of torsional member
KN	Kilonewton
Ksi	Kip per square inch
ℓ_n	Clear span
ℓ_u	Unsupported length of column
L	Live load, span length
l_d	Development length

l_{dh}	l_{hb} times the applicable modification factor
l_{hb}	Basic development length of a standard hook
l_n	Clear span
l_u	Unsupported length of compression member
l_v	Length of shearhead arm
l_1	Span length in the direction of moment
l_2	Span length in direction transverse to span l_1
M	Bending moment
M_1	Smaller end moment at end of column
M_2	Larger end moment at end of column
M_a	Maximum service load moment
M_b	Balanced moment in columns, used with P_b
M_{cr}	Cracking moment
M_m	Modified moment
M_n	Nominal moment strength $= M_u/\phi$
M_n'	Nominal moment strength using an eccentricity e'
M_o	Total factored moment
M_p	Plastic moment
M_u	Moment strength due to factored loads
M_{u1}	Part of M_u when calculated as singly reinforced
M_{u2}	Part of M_u due to compression reinforcement or overhanging flanges in T- or L-sections
M_u'	Moment strength using an eccentricity e'
M_v	Shearhead moment resistance
n	Modular ratio $= E_s/E_c$
N	Normal force
N_u	Factored normal load
N_1	Normal force in bearing at base of column
NA	Neutral axis
psi	Pounds per square inch
P_{cp}	Outside perimeter of gross area $= 2(x_0 + y_0)$
P_o	Perimeter of shear flow in area A_o
P	Unfactored concentrated load
P_b	Balanced load in column (at failure)
P_c	Euler buckling load
P_n	Nominal axial strength of column for a given e
P_o	Axial strength of a concentrically loaded column
P_s	Prestressing force in the tendon at the jacking end
P_u	Factored load $= \phi P_n$
P_x	Prestressing force in the tendon at any point x
q	Soil-bearing capacity
q_a	Allowable bearing capacity of soil
q_u	Ultimate bearing capacity of soil using factored loads
r	Radius of gyration, radius of a circle
R	Resultant of force system, reduction factor for long columns, or $R = R_u/\phi$
R_u	A factor $= M_u/bd^2$
s	Spacing between bars, stirrups, or ties

SI	International system of units
t	Thickness of a slab
T	Torque, tension force
T_c	Nominal torsional strength provided by concrete
T_{cr}	Cracking torsional moment
T_n	Nominal torsional strength provided by concrete and steel
T_s	Nominal torsional strength provided by reinforcement
T_u	Torque provided by factored load = ϕT_n
u	Bond stress
U	Design strength required to resist factored loads
V	Shear stress produced by working loads
v_c	Shear stress of concrete
v_{cr}	Shear stress at which diagonal cracks develop
v_h	Horizontal shear stress
v_t	Shear stress produced by a torque
v_u	Shear stress produced by factored loads
V	Unfactored shear force
V_c	Shear strength of concrete
V_{ci}	Nominal shear strength of concrete when diagonal cracking results from combined shear and moment
V_{cw}	Nominal shear strength of concrete when diagonal cracking results from excessive principal tensile stress in web
V_d	Shear force at section due to unfactored dead load (d = distance from the face of support)
V_n	Nominal shear strength = $V_c + V_s$
V_p	Vertical component of effective prestress force at section
V_s	Shear strength carried by reinforcement
V_u	Shear force due to factored loads
w	Width of crack at the extreme tension fiber, unit weight of concrete
w_u	Factored load per unit length of beam or per unit area of slab
W	Wind load or total load
x_o	Length of the short side of a rectangular section
x_1	Length of the short side of a rectangular closed stirrup
y_b	Same as y_t, except to extreme bottom fibers
y_o	Length of the long side of a rectangular section
y_t	Distance from centroidal axis of gross section, neglecting reinforcement, to extreme top fiber
y_1	Length of the long side of a rectangular closed stirrup
α	Angle of inclined stirrups with respect to longitudinal axis of beam, ratio of stiffness of beam to that of slab at a joint
α_c	Ratio of flexural stiffness of columns to combined flexural stiffness of the slabs and beams at a joint; $(\Sigma\ K_c)/\Sigma(K_s + K_b)$
α_{ec}	Ratio of flexural stiffness of equivalent column to combined flexural stiffness of the slabs and beams at a joint: $(K_{ec})/\Sigma(K_s + K_b)$
α_m	Average value of a for all beams on edges of a panel
α_v	Ratio of stiffness of shearhead arm to surrounding composite slab section
β	Ratio of long to short side of rectangular footing, measure of curvature in biaxial bending

β_1	Ratio of a/c, where a = depth of stress block and c = distance between neutral axis and extreme compression fibers (This factor is 0.85 for $f_c' \leq 4000$ psi and decreases by 0.05 for each 1000 psi in excess of 4000 psi but is at least 0.65.)
β_a	Ratio of unfactored dead load to unfactored live load per unit area
β_c	Ratio of long to short sides of column or loaded area
β_{dns}	Ratio of maximum factored dead load moment to maximum factored total moment
β_t	Ratio of torsional stiffness of edge beam section to flexural stiffness of slab: $E_{cb}C/2E_{cs}I_s$
γ	Distance between rows of reinforcement on opposite sides of columns to total depth of column h
γ_f	Fraction of unbalanced moment transferred by flexure at slab-column connections
γ_p	Factor for type of prestressing tendon (0.4 or 0.28)
γ_v	Fraction of unbalanced moment transferred by eccentricity of shear at slab-column connections
δ	Magnification factor
δ_{ns}	Moment magnification factor for frames braced against sidesway
δ_s	Moment magnification factor for frames not braced against sidesway
Δ	Deflection
ε	Strain
ε_c	Strain in concrete
ε_s	Strain in steel
ε_s'	Strain in compression steel
ε_y	Yield strain = f_y/E_s
θ	Slope angle
λ	Multiplier factor for reduced mechanical properties of lightweight concrete
λ_Δ	Multiplier for additional long-time deflection
μ	Poisson's ratio; coefficient of friction
ζ	Parameter for evaluating capacity of standard hook
π	A constant equal to approximately 3.1416
ρ	Ratio of the tension steel area to the effective concrete area = A_s/bd
ρ'	Ratio of compression steel area to effective concrete area = A_s'/bd
ρ_1	$(\rho - \rho')$
ρ_b	Balanced steel ratio
ρ_g	Ratio of total steel area to total concrete area
ρ_p	Ratio of prestressed reinforcement A_{ps}/bd
ρ_s	Ratio of volume of spiral steel to volume of core
ρ_w	$A_s/b_w d$
ϕ	Strength-reduction factor
ψ_e	Factor used to modify development length based on reinforcement coating
ψ_s	Factor used to modify development length based on reinforcing size
ψ_t	Factor used to modify development length based on reinforcement location
ω	Tension reinforcing index = $\rho f_y/f'c$
ω'	Compression reinforcing index = $\rho' f_y/f_c'$
ω_p	Prestressed steel index = $\rho_p f_{ps}/f_c'$
ω_{pw}	Prestressed steel index for flanged sections
ω_w	Tension reinforcing index for flanged sections
ω_w'	Compression reinforcing index for flanged sections computed as for ω, ω_p and ω'

CONVERSION FACTORS

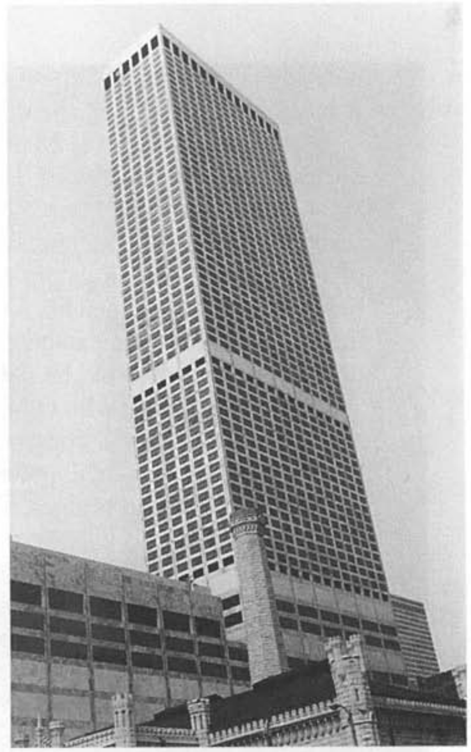
To Convert	to	Multiply By
1. Length		
Inch	Millimeter	25.4
Foot	Millimeter	304.8
Yard	Meter	0.9144
Meter	Foot	3.281
Meter	Inch	39.37
2. Area		
Square inch	Square millimeter	645
Square foot	Square meter	0.0929
Square yard	Square meter	0.836
Square meter	Square foot	10.76
3. Volume		
Cubic inch	Cubic millimeter	16390
Cubic foot	Cubic meter	0.02832
Cubic yard	Cubic meter	0.765
Cubic foot	Liter	28.3
Cubic meter	Cubic foot	35.31
Cubic meter	Cubic yard	1.308
4. Mass		
Ounce	Gram	28.35
Pound (lb)	Kilogram	0.454
Pound	Gallon	0.12

To Convert	to	Multiply By
Short ton (2000 lb)	Kilogram	907
Long ton (2240 lb)	Kilogram	1016
Kilogram	Pound (lb)	2.205
Slug	Kilogram	14.59
5. Density		
Pound/cubic foot	Kilogram/cubic meter	16.02
Kilogram/cubic meter	Pound/cubic foot	0.06243
6. Force		
Pound (lb)	Newton (N)	4.448
Kip (1000 lb)	Kilonewton (kN)	4.448
Newton (N)	Pound	0.2248
Kilonewton (kN)	Kip (K)	0.225
7. Force/length		
Kip/foot	Kilonewton/meter	14.59
Kilonewton/meter	Pound/foot	68.52
Kilonewton/meter	Kip/foot	0.06852
8. Force/area (stress)		
Pound/square inch (psi)	Newton/square centimeter	0.6895
Pound/square inch (psi)	Newton/square millimeter (MPa)	0.0069
Kip/square inch (Ksi)	Meganewton/square meter	6.895
Kip/square inch (Ksi)	Newton/square millimeter	6.895
Pound/square foot	Kilonewton/square meter	0.04788
Pound/square foot	Newton/square meter	47.88
Kip/square foot	Kilonewton/square meter	47.88
Newton/square millimeter	Kip/square inch (Ksi)	0.145
Kilonewton/square meter	Kip/square foot	0.0208
Kilonewton/square meter	Pound/square foot	20.8
9. Moments		
Foot·Kip	Kilonewton·meter	1.356
Inch·Kip	Kilonewton·meter	0.113
Inch·Kip	Kilogram force·meter	11.52
Kilonewton·meter	Foot·Kip	0.7375

Structural Concrete

CHAPTER 1

INTRODUCTION



Water Tower Place, Chicago, 74 stories, tallest concrete building in the United States.

1.1 STRUCTURAL CONCRETE

The design of different structures is achieved by performing, in general, two main steps: (1) determining the different forces acting on the structure using proper methods of structural analysis, and (2) proportioning all structural members economically, considering the safety, stability, serviceability, and functionality of the structure. Structural concrete is one of the materials commonly used to design all types of buildings. Its two component materials, concrete and steel, work together to form structural members that can resist many types of loadings. The key to its performance lies in strengths that are complementary: Concrete resists compression and steel reinforcement resists tension forces.

The term *structural concrete* indicates all types of concrete used in structural applications. Structural concrete may be plain, reinforced, prestressed, or partially prestressed concrete; in addition, concrete is used in composite design. Composite design is used for any structural member, such as beams or columns, when the member contains a combination of concrete and steel shapes.

1.2 HISTORICAL BACKGROUND

The first modern record of concrete is as early as 1760, when John Smeaton used it in Britain in the first lock on the river Calder [1]. The walls of the lock were made of stones filled in with concrete. In 1796, J. Parker discovered Roman natural cement, and 15 years later Vicat burned a mixture of clay and lime to produce cement. In 1824, Joseph Aspdin manufactured portland cement in Wakefield, Britain. It was called portland cement because when it hardened, it resembled stone from the quarries of the Isle of Portland.

In France, François Marte Le Brun built a concrete house in 1832 in Moissac, in which he used concrete arches of 18-ft span. He used concrete to build a school in St. Aignan in 1834 and a church in Corbarièce in 1835. Joseph Louis Lambot [2] exhibited a small rowboat made of reinforced concrete at the Paris Exposition in 1854. In the same year, W. B. Wilkinson of England obtained a patent for a concrete floor reinforced by twisted cables. The Frenchman François Cignet obtained his first patent in 1855 for his system of iron bars, which were embedded in concrete floors and extended to the supports. One year later, he added nuts at the screw ends of the bars, and in 1869, he published a book describing the applications of reinforced concrete.

Joseph Monier, who obtained his patent in Paris on July 16, 1867, was given credit for the invention of reinforced concrete [3]. He made garden tubs and pots of concrete reinforced with iron mesh, which he exhibited in Paris in 1867. In 1873, he registered a patent to use reinforced concrete in tanks and bridges, and four years later, he registered another patent to use it in beams and columns [1].

In the United States, Thaddeus Hyatt conducted flexural tests on 50 beams that contained iron bars as tension reinforcement and published the results in 1877. He found that both concrete and steel can be assumed to behave in a homogeneous manner for all practical purposes. This assumption was important for the design of reinforced concrete members using elastic theory. He used prefabricated slabs in his experiments and considered prefabricated units to be best cast in T-sections and placed side by side to form a floor slab. Hyatt is generally credited with developing the principles upon which the analysis and design of reinforced concrete are now based.

A reinforced concrete house was built by W. E. Ward near Port Chester, New York, in 1875. It used reinforced concrete for walls, beams, slabs, and staircases. P. B. In 1877, Write described in the *American Architect and Building News* the applications of reinforced concrete in Ward's house as a new method in building construction.

E. L. Ransome, head of the Concrete Steel Company in San Francisco, used reinforced concrete in 1879 and deformed bars for the first time in 1884. During 1889–1891, he built the two-story Leland Stanford Museum in San Francisco using reinforced concrete. He also built a reinforced concrete bridge in San Francisco. In 1900, after Ransome introduced the reinforced concrete skeleton, the thick wall system started to disappear in construction. He registered the skeleton type of structure in 1902, using spiral reinforcement in the columns as was suggested by Armand Considére of France. A. N. Talbot, of the University of Illinois, and F. E. Turneaure and M. O. Withney, of the University of Wisconsin, conducted extensive tests on concrete to determine its behavior, compressive strength, and modulus of elasticity.

In Germany, G. A. Wayass bought the French Monier patent in 1879 and published his book on Monier methods of construction in 1887. Rudolph Schuster bought the patent rights in Austria, and the name of Monier spread throughout Europe, which is the main reason for crediting Monier as the inventor of reinforced concrete.

In 1900, the Ministry of Public Works in France called for a committee headed by Armand Considére, chief engineer of roads and bridges, to establish specifications for reinforced concrete, which were published in 1906.

Reinforced concrete was further refined by introducing some precompression in the tension zone to decrease the excessive cracks. This refinement was the preliminary introduction of partial and full prestressing. In 1928, Eugene Freyssinet established the practical technique of using prestressed concrete [4].

The Barwick House, a three-story concrete building built in 1905, Montreal, Canada.

From 1915 to 1935, research was conducted on axially loaded columns and creep effects on concrete; in 1940, eccentrically loaded columns were investigated. Ultimate-strength design started to receive special attention, in addition to diagonal tension and prestressed concrete. The American Concrete Institute Code (ACI Code) specified the use of ultimate-strength design in 1963 and included this method in all later codes. Building codes and specifications for the design of reinforced concrete structures are established in most countries, and research continues on developing new applications and more economical designs.

1.3 ADVANTAGES AND DISADVANTAGES OF REINFORCED CONCRETE

Reinforced concrete, as a structural material, is widely used in many types of structures. It is competitive with steel if economically designed and executed.

The advantages of reinforced concrete can be summarized as follows:

1. It has a relatively high compressive strength.
2. It has better resistance to fire than steel.
3. It has a long service life with low maintenance cost.
4. In some types of structures, such as dams, piers, and footings, it is the most economical structural material.
5. It can be cast to take the shape required, making it widely used in precast structural components. It yields rigid members with minimum apparent deflection.

The disadvantages of reinforced concrete can be summarized as follows:

1. It has a low tensile strength of about one-tenth of its compressive strength.
2. It needs mixing, casting, and curing, all of which affect the final strength of concrete.
3. The cost of the forms used to cast concrete is relatively high. The cost of form material and artisanry may equal the cost of concrete placed in the forms.
4. It has a low compressive strength as compared to steel (the ratio is about 1:10, depending on materials), which leads to large sections in columns of multistory buildings.
5. Cracks develop in concrete due to shrinkage and the application of live loads.

1.4 CODES OF PRACTICE

The design engineer is usually guided by specifications called the codes of practice. Engineering specifications are set up by various organizations to represent the minimum requirements necessary for the safety of the public, although they are not necessarily for the purpose of restricting engineers.

Most codes specify design loads, allowable stresses, material quality, construction types, and other requirements for building construction. The most significant code for structural concrete design in the United States is the Building Code Requirements for Structural Concrete, ACI 318, or the ACI Code. Most of the design examples of this book are based on this code. Other codes of practice and material specifications in the United States include the International Code, the Uniform Building Code, Standard Building Code, National Building Code, Basic Building Code, South Florida Building Code, American Association of State Highway and Transportation Officials (AASHTO) specifications, and specifications issued by the American Society for Testing and Materials (ASTM), American Railway Engineering Association (AREA), and Bureau of Reclamation, Department of the Interior.

Different codes other than those of the United States include the British Standard (BS) Code of Practice for Reinforced Concrete, CP 110 and BS 8110; the National Building Code of Canada; the German Code of Practice for Reinforced Concrete, DIN 1045; Specifications for Steel Reinforcement (U.S.S.R.); and Technical Specifications for the Theory and Design of Reinforced Concrete Structures, CC-BA (France), and the CEB Code (Comitè European Du Beton).

1.5 DESIGN PHILOSOPHY AND CONCEPTS

The design of a structure may be regarded as the process of selecting the proper materials and proportioning the different elements of the structure according to state-of-the-art engineering science and technology. In order to fulfill its purpose, the structure must meet the conditions of safety, serviceability, economy, and functionality. This can be achieved using design approach-based strain limits in concrete and steel reinforcement.

The unified design method (UDM) is based on the strength of structural members assuming a failure condition, whether due to the crushing of the concrete or to the yield of the reinforcing steel bars. Although there is some additional strength in the bars after yielding (due to strain hardening), this additional strength is not considered in the analysis of reinforced concrete members. In this approach, the actual loads, or working loads, are multiplied by load factors to obtain the factored design loads. The load factors represent a high percentage of the factor for safety required in the design. Details of this method are presented in Chapters 3, 4, and 11. The

ACI Code emphasizes this method of design, and its provisions are presented in the body of the Code. The reason for introducing this approach by the ACI Code relates to the fact that different design methods were developed for reinforced and prestressed concrete beams and columns. Also, design procedures for prestressed concrete were different from reinforced concrete. The purpose of the Code approach is to simplify and unify the design requirements for reinforced and prestressed flexural members and compression members.

A second approach for the design of reinforced and prestressed concrete flexural and compression members is called the strength design method, or the alternative provisions (ADM), as introduced in the ACI Code, Appendix B. When this method is used in the design, the designer must adhere to all sections of Appendixes B and C and substitute accordingly for the corresponding sections of the Code. Reinforcement limits, strength reduction factors, load factors, and moment redistribution are affected. The provisions of this method satisfy the Code and are equally acceptable.

A third approach for the design of concrete members is called the strut and tie method (STM). The provisions of this method are introduced in the ACI Code, Appendix A. It applies effectively in regions of discontinuity such as support and load applications on beams. Consequently, the structural element is divided into segments and then analyzed using the truss analogy approach, where the concrete resists compression forces as a strut, while the steel reinforcement resists tensile forces as a tie.

A basic method that is not commonly used is called the working stress design or the elastic design method. The design concept is based on the elastic theory assuming a straight line stress distribution along the depth of the concrete section under service loads. The members are proportioned on the basis of certain allowable stresses in concrete and steel. The allowable stresses are fractions of the crushing strength of concrete and yield strength of steel. This method has been deleted from the ACI Code. The application of this approach is still used in the design of prestressed concrete members under service load conditions, as shown in Chapter 19.

Limit state design is a further step in the strength design method. It indicates the state of the member in which it ceases to meet the service requirements such as losing its ability to withstand external loads, or suffering excessive deformation, cracking, or local damage. According to the limit state design, reinforced concrete members have to be analyzed with regard to three limiting states:

1. Load carrying capacity (safety, stability, and durability)
2. Deformation (deflections, vibrations, and impact)
3. The formation of cracks.

The aim of this analysis is to ensure that no limiting state will appear in the structural member during its service life.

1.6 UNITS OF MEASUREMENT

Two units of measurement are commonly used in the design of structural concrete. The first is the U.S. customary system (lying mostly in its human scale and its ingenious use of simple numerical proportions), and the second is the SI (Le Système International d'Unités), or metric, system.

The metric system is planned to be in universal use within the coming few years. The United States is committed to change to SI units. Great Britain, Canada, Australia, and other countries have been using SI units for several years.

The base units in the SI system are the units of length, mass, and time, which are the meter (m), the kilogram (kg), and the second (s), respectively. The unit of force, a derived unit called the newton (N), is defined as the force that gives the acceleration of 1 meter per second per second (1 m/s^2) to a mass of 1 kg, or $1\,N = 1\,kg \times m/s^2$.

The weight of a body, W, which is equal to the mass, m, multiplied by the local gravitational acceleration, g (9.81 m/s^2), is expressed in newtons (N). The weight of a body of 1 kg mass is $W = mg = 1\,kg \times 9.81\,m/s^2 = 9.81\,N$.

Multiples and submultiples of the base SI units can be expressed through the use of prefixes. The prefixes most frequently used in structural calculations are the kilo (k), mega (M), milli (m), and micro (μ). For example,

$$1\ km = 1000\ m \quad 1\ mm = 0.001\ m \quad 1\ \mu m = 10^{-6}\ m$$

$$1\ kN = 1000\ N \quad 1\ Mg = 1000\ kg = 10^{6}\ g$$

1.7 LOADS

Structural members must be designed to support specific loads.

Loads are those forces for which a given structure should be proportioned. In general, loads may be classified as dead or live.

Dead loads include the weight of the structure (its self-weight) and any permanent material placed on the structure, such as tiles, roofing materials, and walls. Dead loads can be determined with a high degree of accuracy from the dimensions of the elements and the unit weight of materials.

Live loads are all other loads that are not dead loads. They may be steady or unsteady or movable or moving; they may be applied slowly, suddenly, vertically, or laterally, and their magnitudes may fluctuate with time. In general, live loads include the following:

- Occupancy loads caused by the weight of the people, furniture, and goods
- Forces resulting from wind action and temperature changes
- The weight of snow if accumulation is probable
- The pressure of liquids or earth on retaining structures
- The weight of traffic on a bridge
- Dynamic forces resulting from moving loads (impact), earthquakes, or blast loading

The ACI Code does not specify loads on structures; however, occupancy loads on different types of buildings are prescribed by the American National Standards Institute (ANSI) [5]. Some typical values are shown in Table 1.1. Table 1.2 on page 7 shows weights and specific gravity of various materials.

AASHTO and AREA specifications prescribe vehicle loadings on highway and railway bridges, respectively. These loads are given in Refs. 6 and 7.

Snow loads on structures may vary between 10 and 40 lb/ft^2 (0.5 and 2 kN/m^2), depending on the local climate.

Wind loads may vary between 15 and 30 lb/ft^2, depending on the velocity of wind. The wind pressure of a structure, F, can be estimated from the following equation:

$$F = 0.00256 C_s V^2 \tag{1.1}$$

where

V = velocity of air (mi/h)
C_s = shape factor of the structure
F = the dynamic wind pressure (lb/ft^2)

Table 1.1 Typical Uniformly Distributed Design Loads

Occupancy	Contents	Design Live Load lb/ft²	kN/m²
Assembly hall	Fixed seats	60	2.9
	Movable seats	100	4.8
Hospital	Operating rooms	60	2.9
	Private rooms	40	1.9
Hotel	Guest rooms	40	1.9
	Public rooms	100	4.8
	Balconies	100	4.8
Housing	Private houses and apartments	40	1.9
	Public rooms	100	4.8
Institution	Classrooms	40	1.9
	Corridors	100	4.8
Library	Reading rooms	60	2.9
	Stack rooms	150	7.2
Office building	Offices	50	2.4
	Lobbies	100	4.8
Stairs (or balconies)		100	4.8
Storage warehouses	Light	100	4.8
	Heavy	250	12.0
Yards and terraces		100	4.8

Table 1.2 Density and Specific Gravity of Various Materials

Material	Density lb/ft³	kg/m³	Specific Gravity
Building materials			
Bricks	120	1,924	1.8–2.0
Cement, portland, loose	90	1,443	—
Cement, portland, set	183	2,933	2.7–3.2
Earth, dry, packed	95	1,523	—
Sand or gravel, dry, packed	100–120	1,600–1,924	—
Sand or gravel, wet	118–120	1,892–1,924	—
Liquids			
Oils	58	930	0.9–0.94
Water (at 4 °C)	62.4	1,000	1.0
Ice	56	898	0.88–0.92
Metals and minerals			
Aluminum	165	2,645	2.55–2.75
Copper	556	8,913	9.0
Iron	450	7,214	7.2
Lead	710	11,380	11.38
Steel, rolled	490	7,855	7.85
Limestone or marble	165	2,645	2.5–2.8
Sandstone	147	2,356	2.2–2.5
Shale or slate	175	2,805	2.7–2.9
Normal-weight concrete			
Plain	145	2,324	2.2–2.4
Reinforced or prestressed	150	2,405	2.3–2.5

As an example, for a wind of 100 mi/h with $C_s = 1$, the wind pressure is equal to 25.6 lb/ft^2. It is sometimes necessary to consider the effect of gusts in computing the wind pressure by multiplying the wind velocity in Eq. 1.1 by a gust factor, which generally varies between 1.1 and 1.3.

The shape factor, C_s, varies with the horizontal angle of incidence of the wind. On vertical surfaces of rectangular buildings, C_s may vary between 1.2 and 1.3. Detailed information on wind loads can be found in Ref. 5.

1.8 SAFETY PROVISIONS

Structural members must always be proportioned to resist loads greater than the service or actual load in order to provide proper safety against failure. In the strength design method, the member is designed to resist factored loads, which are obtained by multiplying the service loads by load factors. Different factors are used for different loadings. Because dead loads can be estimated quite accurately, their load factors are smaller than those of live loads, which have a high degree of uncertainty. Several load combinations must be considered in the design to compute the maximum and minimum design forces. Reduction factors are used for some combinations of loads to reflect the low probability of their simultaneous occurrences. The ACI Code presents specific values of load factors to be used in the design of concrete structures (see Chapter 3, Section 3.5).

In addition to load factors, the ACI Code specifies another factor to allow an additional reserve in the capacity of the structural member. The nominal strength is generally calculated using accepted analytical procedure based on statistics and equilibrium; however, in order to account for the degree of accuracy within which the nominal strength can be calculated, and for adverse variations in materials and dimensions, a strength reduction factor, ϕ, should be used in the strength design method. Values of the strength reduction factors are given in Chapter 3, Section 3.6.

To summarize the above discussion, the ACI Code has separated the safety provision into an overload or load factor and to an undercapacity (or strength reduction) factor, ϕ. A safe design is achieved when the structure's strength, obtained by multiplying the nominal strength by the reduction factor, ϕ, exceeds or equals the strength needed to withstand the factored loadings (service loads times their load factors). For example,

$$M_u \le \phi M_n \quad \text{and} \quad V_u \le \phi V_n \tag{1.2}$$

where

M_u and V_u = external factored moment and shear forces

M_n and V_n = nominal flexural strength and shear strength of the member, respectively

Given a load factor of 1.2 for dead load and a load factor of 1.6 for live load, the overall safety factor for a structure loaded by a dead load, D, and a live load, L, is

$$\text{Factor of safety} = \frac{1.2D + 1.6L}{D + L}\left(\frac{1}{\phi}\right) = \frac{1.2 + 1.6(L/D)}{1 + (L/D)}\left(\frac{1}{\phi}\right) \tag{1.3}$$

The factor of safety for the various values of ϕ and L/D ratios is shown below.

ϕ	0.9				0.8				0.7			
L/D	0	1	2	3	0	1	2	3	0	1	2	3
Factor of Safety	1.33	1.56	1.63	1.67	1.50	1.74	1.83	1.88	1.71	2.00	2.10	2.15

For members subjected to flexure (beams), with tension-controlled sections, $\phi = 0.9$, and the factor of safety ranges between 1.33 for $L/D = 0$ and 1.67 for $L/D = 3$. These values are less than those specified by the ACI Code 318–99 of 1.56 for $L/D = 0$ and 1.81 for $L/D = 3.0$ based on load factors of 1.4 for the dead load and 1.7 for the live load. This reduction ranges between 17 and 8% respectively.

For members subjected to axial forces (spiral columns), $\phi = 0.7$, and the factor of safety ranges between 1.71 for $L/D = 0$ and 2.15 for $L/D = 3$. The increase in the factor of safety in columns reflects the greater overall safety requirements of these critical building elements.

A general format of Eq. 1.2 may be written as follows [8]:

$$\phi R \geq v_0 \Sigma(v_i Q_i) \tag{1.4}$$

where

R_n = nominal strength of the structural number
ϕ = undercapacity factor (<1.0)
ΣQ_i = sum of load effects
v_i = overload factor
v_0 = analysis factor (>1.0)

The subscript i indicates the load type, such as dead load, live load, and wind load. The analysis factor, v_0, is greater than 1.0 and is introduced to account for uncertainties in structural analysis. The overload factor, v_i, is introduced to account for several factors such as an increase in live load due to a change in the use of the structure and variations in erection procedures. The design concept is referred to as load and resistance factor design (LRFD) [8,9].

1.9 STRUCTURAL CONCRETE ELEMENTS

Structural concrete can be used for almost all buildings, whether single story or multistory. The concrete building may contain some or all of the following main structural elements, which are explained in detail in other chapters of the book:

- *Slabs* are horizontal plate elements in building floors and roofs. They may carry gravity loads as well as lateral loads. The depth of the slab is usually very small relative to its length or width (Chapters 9 and 17).
- *Beams* are long, horizontal or inclined members with limited width and depth. Their main function is to support loads from slabs (Chapters 3 and 4).
- *Columns* are critical members that support loads from beams or slabs. They may be subjected to axial loads or axial loads and moments (Chapters 10 and 11).
- *Frames* are structural members that consist of a combination of beams and columns or slabs, beams, and columns. They may be statically determinate or statically indeterminate frames (Chapter 16).
- *Footings* are pads or strips that support columns and spread their loads directly to the soil (Chapter 13).
- *Walls* are vertical plate elements resisting gravity as well as lateral loads as in the case of basement walls (Chapter 14).

1.10 STRUCTURAL CONCRETE DESIGN

The first step in the design of a building is the general planning carried out by the architect to determine the layout of each floor of the building to meet the owner's requirements. Once the architectural plans are approved, the structural engineer then determines the most adequate structural system to ensure the safety and stability of the building. Different structural options must be considered to determine the most economical solution based on the materials available and the soil condition. This result is normally achieved by

1. Idealizing the building into a structural model of load-bearing frames and elements
2. Estimating the different types of loads acting on the building
3. Performing the structural analysis using computer or manual calculations to determine the maximum moments, shear, torsional forces, axial loads, and other forces
4. Proportioning the different structural elements and calculating the reinforcement needed
5. Producing structural drawings and specifications with enough details to enable the contractor to construct the building properly

1.11 ACCURACY OF CALCULATIONS

In the design of concrete structures, exact calculations to determine the size of the concrete elements are not needed. Calculators and computers can give an answer to many figures after the decimal point. For a practical size of a beam, slab, or column, each dimension should be approximated to the nearest 1 or $^1/_2$ inch. Moreover, the steel bars available in the market are limited to specific diameters and areas, as shown in Table A.12 (Appendix A). The designer should choose a group of bars from the table with an area equal to or greater than the area obtained from calculations. Also, the design equations in this book based on the ACI Code are approximate. Therefore, for a practical and economical design, it is adequate to use four figures (or the full number with no fractions if it is greater than four figures) for the calculation of forces, stresses, moments, or dimensions such as length or width of section. For strains, use five or six figures because strains are very small quantities measured in a millionth of an inch (for example, a strain of 0.000358 in./in.). Stresses are obtained by multiplying the strains by the modulus of elasticity of the material, which has a high magnitude (for example, 29,000,000 lb/in.^{2}) for steel. Any figures less than five or six figures in strains will produce quite a change in stresses.

Examples

For forces, use 28.45 K, 2845 lb, 567.8 K (four figures).
For force/length, use 2.451 K/ft or 2451 lb/ft.
For length or width, use 14.63 in., 1.219 ft (or 1.22 ft).
For areas, use 7.537 in.^{2}, and for volumes, use 48.72 in.^{3}.
For strains, use 0.002078.

1.12 CONCRETE HIGH-RISE BUILDINGS

High-rise buildings are becoming the dominant feature of many U.S. cities; a great number of these buildings are designed and constructed in structural concrete.

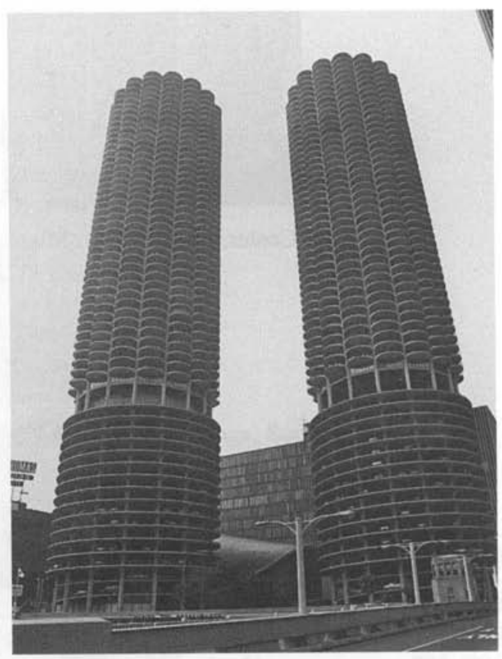
Although at the beginning of the century the properties of concrete and joint behavior of steel and concrete were not fully understood, a 16-story building, the Ingalls Building, was constructed in Cincinnati in 1902 with a total height of 210 ft (64 m). In 1922, the Medical Arts Building, with a height of 230 ft (70 m), was constructed in Dallas, Texas. The design of concrete buildings was based on elastic theory concepts and a high factor of safety, resulting in large concrete sections in beams and columns. After extensive research, high-strength concrete and high-strength steel were allowed in the design of reinforced concrete members. Consequently, small concrete sections as well as savings in materials were achieved, and new concepts of structural design were possible.

To visualize how high concrete buildings can be built, some structural concrete skyscrapers are listed in Table 1.3. The CN Tower is the world's tallest free-standing concrete structure.

The reader should realize that most concrete buildings are relatively low and range from one to five stories. Skyscrapers and high-rise buildings constitute less than 10% of all concrete buildings.

Photos of some different concrete buildings and structures are shown.

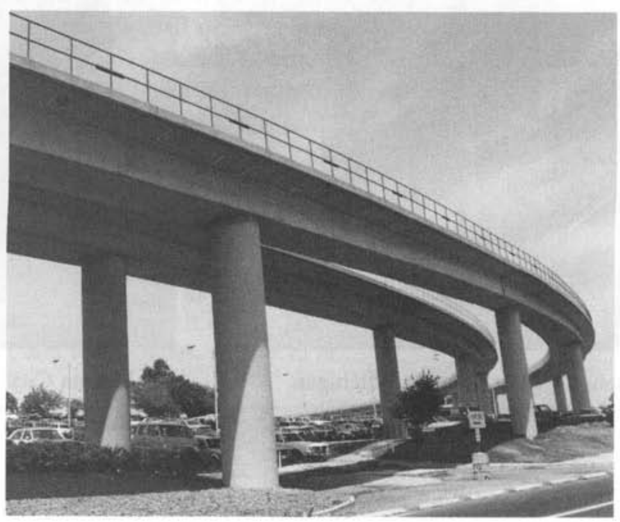
Renaissance Center, Detroit, Michigan.



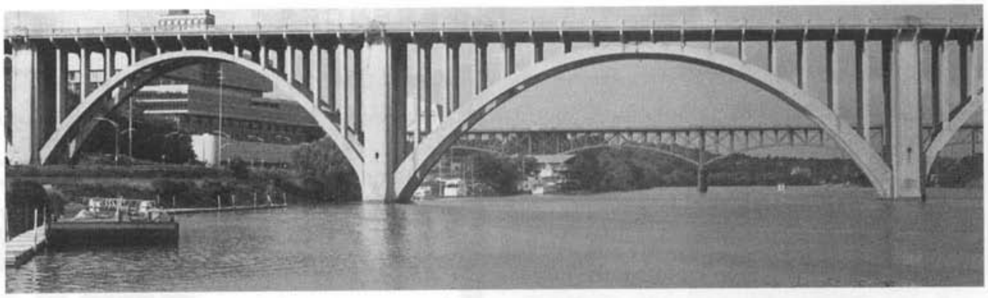
Marina City Towers, Chicago, Illinois.

City Center, Minneapolis, Minnesota.

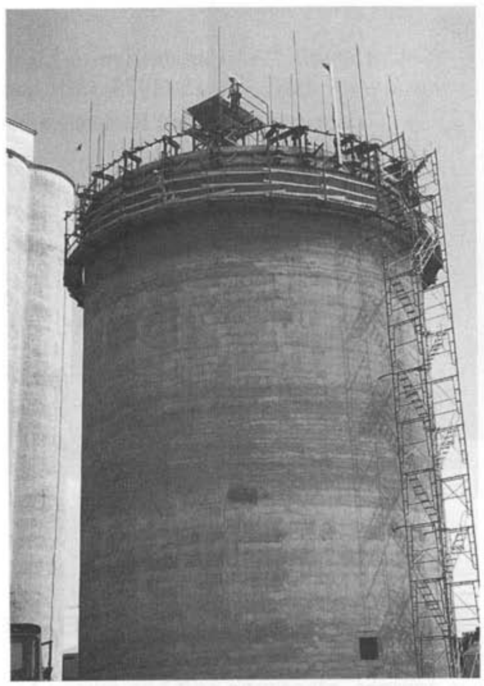
CN Tower, Toronto, Canada (height 1465 ft, or 447 m).



Concrete bridge for the city transit system, Washington, DC.



Concrete bridge, Knoxville, Tennessee.



Reinforced concrete grain silo using the slip form system, Brookings, South Dakota.

Table 1.3 Examples of Reinforced Concrete Skyscrapers

Year	Structure	Location	Stories	Height, ft (m)
1965	Lake Point Tower	Chicago	70	645 (197)
1969	One Shell Plaza	Houston	52	714 (218)
1975	Peachtree Center Plaza Hotel	Atlanta	71	723 (220)
1976	Water Tower Place	Chicago	74	859 (262)
1976	CN Tower	Toronto	—	1465 (447)
1977	Renaissance Center Westin Hotel	Detroit	73	740 (226)
1983	City Center	Minneapolis	40	528 (158)

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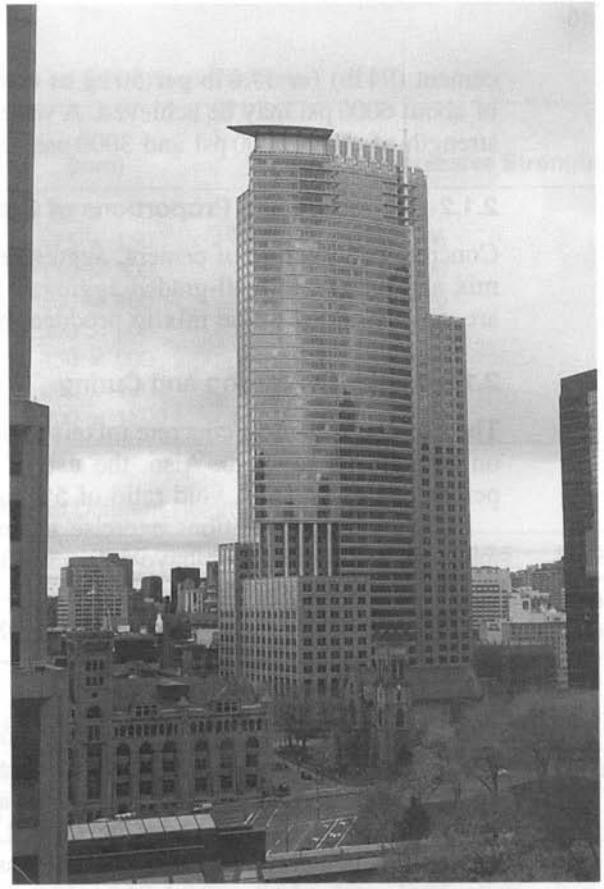
CHAPTER 2

PROPERTIES OF REINFORCED CONCRETE



IBM Building, Montreal, Canada (the highest concrete building in Montreal, with 50 stories).

2.1 FACTORS AFFECTING THE STRENGTH OF CONCRETE

In general, concrete consists of coarse and fine aggregate, cement, water, and—in many cases—different type of admixture. The materials are mixed together until a cement paste is developed, filling most of the voids in the aggregates and producing a uniform dense concrete. The plastic concrete is then placed in a mold and left to set, harden, and develop adequate strength. For the design of concrete mixtures, as well as composition and properties of concrete materials, the reader is referred to Refs. 1–6.

The strength of concrete depends upon many factors and may vary within wide limits with the same production method. The main factors that affect the strength of concrete are described next.

2.1.1 Water–Cement Ratio

The water–cement ratio is one of the most important factors affecting the strength of concrete. For complete hydration of a given amount of cement, a water–cement ratio (by weight) equal to 0.25 is needed. A water–cement ratio of about 0.35 or higher is needed for the concrete to be reasonably workable without additives. This ratio corresponds to 4 gal of water per sack of cement (94 lb) (or 17.8 lb per 50 kg of cement). Based on this cement ratio, a concrete strength of about 6000 psi may be achieved. A water–cement ratio of 0.5 and 0.7 may produce a concrete strength of about 5000 psi and 3000 psi, respectively.

2.1.2 Properties and Proportions of Concrete Constituents

Concrete is a mixture of cement, aggregate, and water. An increase in the cement content in the mix and the use of well-graded aggregate increase the strength of concrete. Special admixtures are usually added to the mix to produce the desired quality and strength of concrete.

2.1.3 Method of Mixing and Curing

The use of mechanical concrete mixers and the proper time of mixing both have favorable effects on strength of concrete. Also, the use of vibrators produces dense concrete with a minimum percentage of voids. A void ratio of 5% may reduce the concrete strength by about 30%.

The curing conditions exercise an important influence on the strength of concrete. Both moisture and temperature have a direct effect on the hydration of cement. The longer the period of moist storage, the greater the strength. If the curing temperature is higher than the initial temperature of casting, the resulting 28-day strength of concrete is reached earlier than 28 days.

2.1.4 Age of the Concrete

The strength of concrete increases appreciably with age, and hydration of cement continues for months. In practice, the strength of concrete is determined from cylinders or cubes tested at the age of 7 days and 28 days. As a practical assumption, concrete at 28 days is 1.5 times as strong as at 7 days: The range varies between 1.3 and 1.7. The British code of practice [2] accepts concrete if the strength at 7 days is not less than two-thirds of the required 28-day strength. For a normal portland cement, the increase of strength with time, relative to 28-day strength, may be assumed as follows:

Age	7 days	14 days	28 days	3 months	6 months	1 year	2 years	5 years
Strength ratio	0.67	0.86	1.0	1.17	1.23	1.27	1.31	1.35

2.1.5 Loading Conditions

The compressive strength of concrete is estimated by testing a cylinder or cube to failure in a few minutes. Under sustained loads for years, the ultimate strength of concrete is reduced by about 30%. Under 1-day sustained loading, concrete may lose about 10% of its compressive strength. Sustained loads and creep effect as well as dynamic and impact effect, if they occur on the structure, should be considered in the design of reinforced concrete members.

2.1.6 Shape and Dimensions of the Tested Specimen

The common sizes of concrete specimens used to predict the compressive strength are either 6-by 12-in. (150- by 300-mm) cylinders or 6-in. (150-mm) cubes. When a given concrete is tested in compression by means of cylinders of like shape but of different sizes, the larger specimens give lower strength indexes. Table 2.1 [4] gives the relative strength for various sizes of cylinders as

Table 2.1 Effect of Size of Compression Specimen on Strength of Concrete

Size of cylinder (in.)	Size of cylinder (mm)	Relative Compressive Strength
2 × 4	50 × 100	1.09
3 × 6	75 × 150	1.06
6 × 12	150 × 300	1.00
8 × 16	200 × 400	0.96
12 × 24	300 × 600	0.91
18 × 36	450 × 900	0.86
24 × 48	600 × 1200	0.84
36 × 72	900 × 1800	0.82

Table 2.2 Strength Correction Factor for Cylinders of Different Height–Diameter Ratios

Ratio	2.0	1.75	1.50	1.25	1.10	1.00	0.75	0.50
Strength correction factor	1.00	0.98	0.96	0.93	0.90	0.87	0.70	0.50
Strength relative to standard cylinder	1.00	1.02	1.04	1.06	1.11	1.18	1.43	2.00

Table 2.3 Relative Strength of Cylinder versus Cube [6]

Compressive Strength											
	(psi)	1000	2200	2900	3500	3800	4900	5300	5900	6400	7300
	(N/mm²)	7.0	15.5	20.0	24.5	27.0	24.5	37.0	41.5	45.0	51.5
Strength Ratio of Cylinder to Cube		0.77	0.76	0.81	0.87	0.91	0.93	0.94	0.95	0.96	0.96

a percentage of the strength of the standard cylinder; the heights of all cylinders are twice the diameters.

Sometimes concrete cylinders of nonstandard shape are tested. The greater the ratio of specimen height to diameter, the lower the strength indicated by the compression test. To compute the equivalent strength of the standard shape, the results must be multiplied by a correction factor. Approximate values of the correction factor are given in Table 2.2, extracted from ASTM C 42/C 42 M-03. The relative strengths of a cylinder and a cube for different compressive strengths are shown in Table 2.3.

2.2 COMPRESSIVE STRENGTH

In designing structural members, it is assumed that the concrete resists compressive stresses and not tensile stresses; therefore, compressive strength is the criterion of quality concrete. The other concrete stresses can be taken as a percentage of the compressive strength, which can be easily and accurately determined from tests. Specimens used to determine compressive strength may be cylindrical, cubical, or prismatic.

Test specimens in the form of a 6-in. (150 mm) or 8-in. (200 mm) cube are used in Great Britain, Germany, and other parts of Europe.

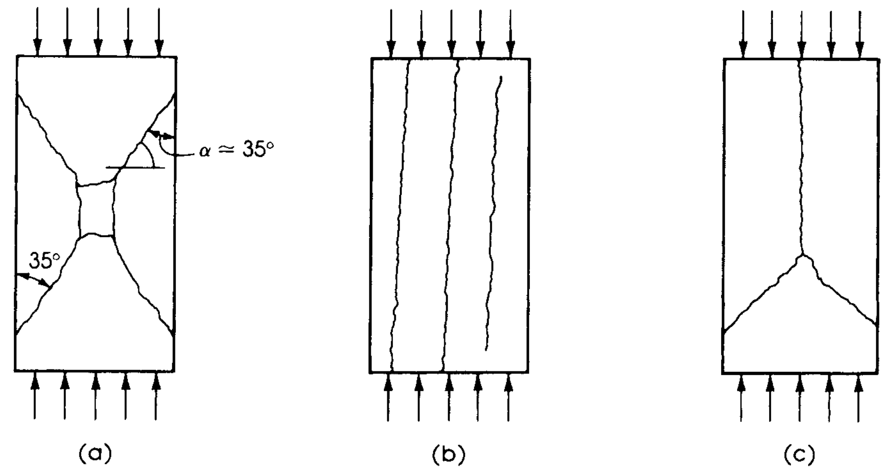


Figure 2.1 Modes of failure of standard concrete cylinders.

Prism specimens are used in France, Russia, and other countries and are usually 70 by 70 by 350 mm or 100 by 100 by 500 mm. They are cast with their longer sides horizontal and are tested, like cubes, in a position normal to the position of cast.

Before testing, the specimens are moist-cured and then tested at the age of 28 days by gradually applying a static load until rupture occurs. The rupture of the concrete specimen may be caused by the applied tensile stress (failure in cohesion), the applied shearing stress (sliding failure), the compressive stress (crushing failure), or combinations of these stresses.

The failure of the concrete specimen can be in one of three modes [5], as shown in Fig. 2.1. First, under axial compression, the specimen may fail in shear, as in Fig. 2.1*a*. Resistance to failure is due to both cohesion and internal friction.

The second type of failure (Fig. 2.1*b*) results in the separation of the specimen into columnar pieces by what is known as splitting, or columnar, fracture. This failure occurs when the strength of concrete is high, and lateral expansion at the end bearing surfaces is relatively unrestrained.

The third type of failure (Fig. 2.1*c*) is seen when a combination of shear and splitting failure occurs.

2.3 STRESS–STRAIN CURVES OF CONCRETE

The performance of a reinforced concrete member under load depends, to a great extent, on the stress–strain relationship of concrete and steel and on the type of stress applied to the member. Stress–strain curves for concrete are obtained by testing a concrete cylinder to rupture at the age of 28 days and recording the strains at different load increments.

Figure 2.2 shows typical stress–strain curves for concretes of different strengths. All curves consist of an initial relatively straight elastic portion, reaching maximum stress at a strain of about 0.002; then rupture occurs at a strain of about 0.003. Concrete having a compressive strength between 3000 and 6000 psi (21 and 42 N/mm^2) may be adopted. High-strength concrete with a compressive strength greater than 6000 psi (6000–15,000 psi) is becoming an important building material for the design of concrete structures.

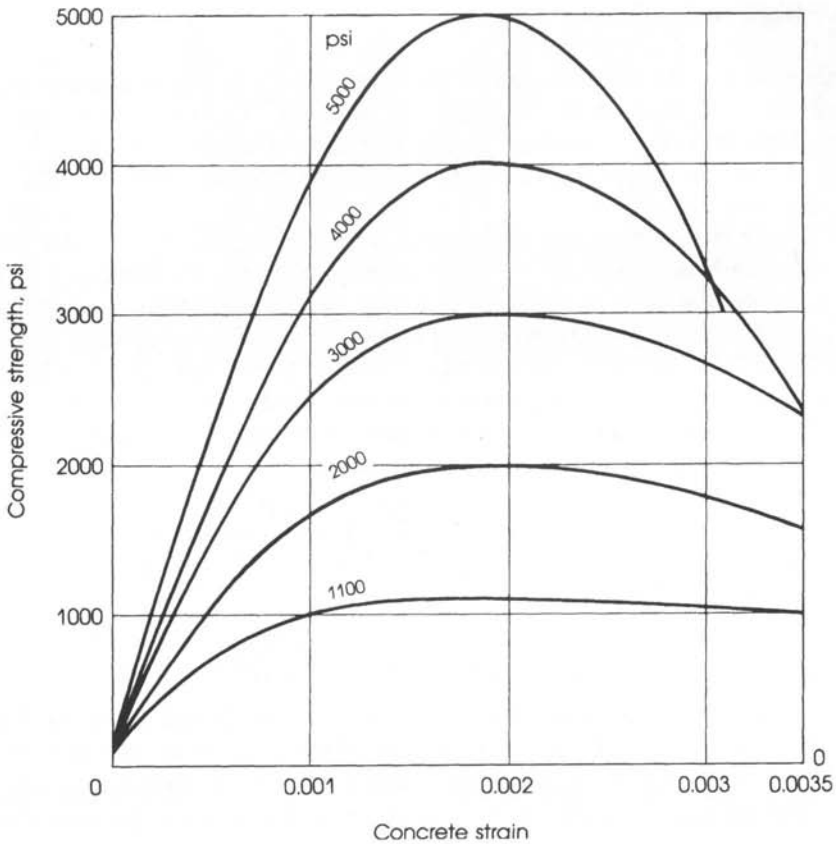


Figure 2.2 Typical stress–strain curves of concrete.

Standard capped cylinders ready for testing.

2.4 TENSILE STRENGTH OF CONCRETE

Concrete is a brittle material, and it cannot resist the high tensile stresses that are important when considering cracking, shear, and torsional problems. The low tensile capacity can be attributed to the high stress concentrations in concrete under load, so that a very high stress is reached in some portions of the specimen, causing microscopic cracks, while the other parts of the specimen are subjected to low stress.

Direct tension tests are not reliable for predicting the tensile strength of concrete, due to minor misalignment and stress concentrations in the gripping devices. An indirect tension test in the form of splitting a 6- by 12-in. (150- by 300-mm) cylinder was suggested by the Brazilian Fernando Carneiro. The test is usually called the *splitting test*. In this test, the concrete cylinder is placed with its axis horizontal in a compression testing machine. The load is applied uniformly along two opposite lines on the surface of the cylinder through two plywood pads, as shown in Fig. 2.3. Considering an element on the vertical diameter and at a distance y from the top fibers, the element is subjected to a compressive stress

$$f_c = \frac{2P}{\pi LD}\left(\frac{D^2}{y(D-y)} - 1\right) \tag{2.1}$$

and a tensile stress

$$f'_{sp} = \frac{2P}{\pi LD} \tag{2.2}$$

where P is the compressive load on the cylinder and D and L are the diameter and length of the cylinder. For a 6- by 12-in. (150- by 300-mm) cylinder and at a distance $y = D/2$, the compression strength is $f_e = 0.0265P$, and the tensile strength is $f'_{sp} = 0.0088P = f_c/3$.

The splitting strength of f'_{sp} can be related to the compressive strength of concrete in that it varies between six and seven times $\sqrt{f'_c}$ for normal concrete and between four and five times $\sqrt{f'_c}$ for lightweight concrete. The direct tensile stress, f'_t, can also be estimated from the

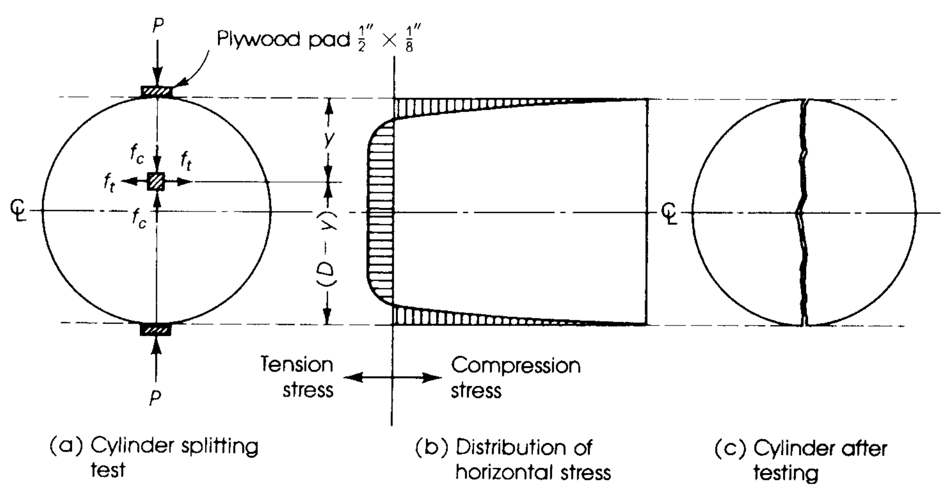
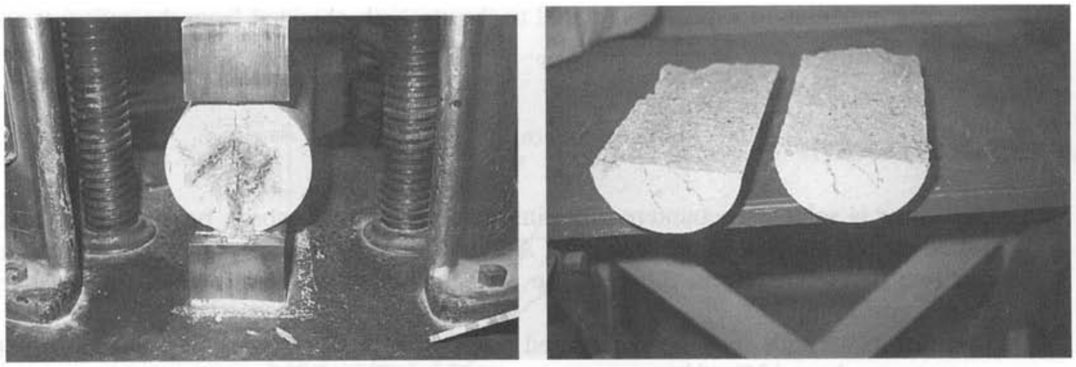


Figure 2.3 Cylinder splitting test [6]: (*a*) configuration of test, (*b*) distribution of horizontal stress, and (*c*) cylinder after testing.



Concrete cylinder splitting test.

split test: Its value varies between 0.5 f'_{sp} and 0.7 f'_{sp}. The smaller of these values applies to higher-strength concrete. The splitting strength, f'_{sp}, can be estimated as 10% of the compressive strength up to $f'_c = 6000$ psi (42 N/mm^2). For higher values of compressive strength, f'_{sp} can be taken as 9% of f'_c.

In general, the tensile strength of concrete ranges from 7% to 11% of its compressive strength, with an average of 10%. The lower the compressive strength, the higher the relative tensile strength.

2.5 FLEXURAL STRENGTH (MODULUS OF RUPTURE) OF CONCRETE

Experiments on concrete beams have shown that ultimate tensile strength in bending is greater than the tensile stress obtained by direct or splitting tests. Flexural strength is expressed in terms of the modulus of rupture of concrete (f_r), which is the maximum tensile stress in concrete in bending. The modulus of rupture can be calculated from the flexural formula used for elastic materials, $f_r = Mc/I$, by testing a plain concrete beam. The beam, 6 by 6 by 28 in. (150 by 150 by 700 mm), is supported on a 24-in. (600-mm) span and loaded to rupture by two loads, 4 in. (100 mm) on either side of the center. A smaller beam of 4 by 4 by 20 in. (100 by 100 by 500 mm) on a 16-in. (400-mm) span may also be used.

The modulus of rupture of concrete ranges between 11% and 23% of the compressive strength. A value of 15% can be assumed for strengths of about 4000 psi (28 N/mm^2). The ACI Code prescribes the value of the modulus of rupture as

$$f_r = 7.5\lambda\sqrt{f'_c}\ \text{(psi)} = 0.62\lambda\sqrt{f'_c}\ (\text{N/mm}^2) \tag{2.3}$$

where

λ is a modification factor for type of concrete (ACI 8.6.1)

$\lambda = 1.0$ Normal-weight concrete

$\lambda = 0.85$ Sand-lightweight concrete

$\lambda = 0.75$ for all-lightweight concrete

Linear interpolation shall be permitted between 0.85 and 1.0 on the basis of volumetric fractions, for concrete containing normal-weight fine aggregate and a blend of lightweight and normal-weight coarse aggregate.

The modulus of rupture as related to the strength obtained from the split test on cylinders may be taken as $f_r = (1.25 \text{ to } 1.50) f'_{sp}$.

2.6 SHEAR STRENGTH

Pure shear is seldom encountered in reinforced concrete members, because it is usually accompanied by the action of normal forces. An element subjected to pure shear breaks transversely into two parts. Therefore, the concrete element must be strong enough to resist the applied shear forces.

Shear strength may be considered as 20% to 30% greater than the tensile strength of concrete, or about 12% of its compressive strength. The ACI Code allows a nominal shear stress of $2\lambda\sqrt{f'_c}$ psi $(0.17\lambda\sqrt{f'_c}$ N/mm$^2)$ on plain concrete sections. For more information, refer to Chapter 8.

2.7 MODULUS OF ELASTICITY OF CONCRETE

One of the most important elastic properties of concrete is its modulus of elasticity, which can be obtained from a compressive test on concrete cylinders. The modulus of elasticity, E_c, can be defined as the change of stress with respect to strain in the elastic range:

$$E_c = \frac{\text{unit stress}}{\text{unit strain}} \tag{2.4}$$

The modulus of elasticity is a measure of stiffness, or the resistance of the material to deformation. In concrete, as in any elastoplastic material, the stress is not proportional to the strain, and the stress–strain relationship is a curved line. The actual stress–strain curve of concrete can be obtained by measuring the strains under increments of loading on a standard cylinder.

The *initial tangent modulus* (Fig. 2.4) is represented by the slope of the tangent to the curve at the origin under elastic deformation. This modulus is of limited value and cannot be determined with accuracy. Geometrically, the tangent modulus of elasticity of concrete, E_c, is the slope of the tangent to the stress–strain curve at a given stress. Under long-time action of load and due to the development of plastic deformation, the stress-to-total-strain ratio becomes a variable nonlinear quantity.

For practical applications, the *secant modulus* can be used. The secant modulus is represented by the slope of a line drawn from the origin to a specific point of stress (B) on the stress–strain curve (Fig. 2.4). Point B is normally located at $f'_c/2$.

The ACI Code section 8.5.1 gives a simple formula for calculating the modulus of elasticity of normal and lightweight concrete considering the secant modulus at a level of stress, f_c, equal to half the ultimate concrete strength, f'_c,

$$E_c = 33w^{1.5}\sqrt{f'_c} \text{ psi}(w \text{ in pcf}) = 0.043\, w^{1.5}\sqrt{f'_c} \text{ N/mm}^2 \tag{2.5}$$

where w = unit weight of concrete (between 90 and 160 lb/ft^3 (pcf) or 1400 to 2600 kg/m^3) and f'_c = ultimate strength of a standard concrete cylinder. For normal-weight concrete, w is approximately 145 pcf (2320 kg/m^3); thus,

$$E_c = 57{,}600\sqrt{f'_c} \text{ psi} = 4780\sqrt{f'_c} \text{ MPa} \tag{2.6}$$

$$E_c = \frac{df_c}{d\varepsilon_c}$$

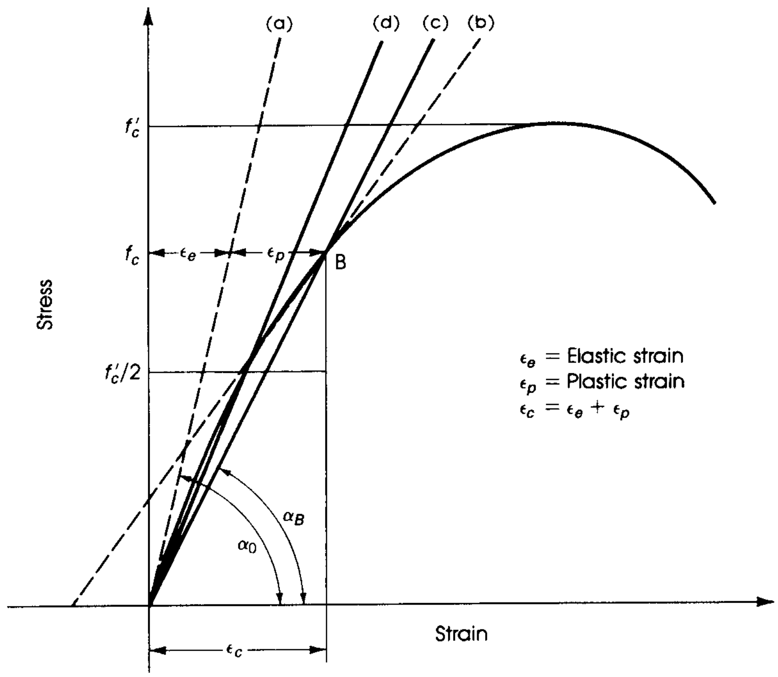


Figure 2.4 Stress–strain curve and modulus of elasticity of concrete. Lines a-d represent (*a*) initial tangent modulus, (*b*) tangent modulus at a stress, f_c, (*c*) secant modulus at a stress, f_c, and (*d*) secant modulus at a stress $f'_c/2$.

The ACI Code allows the use of $E_c = 57{,}000\sqrt{f'_c}$ (psi) $= 4700\sqrt{f'_c}$ MPa. The module of elasticity, E_c, for different values of f'_c are shown in Table A.10.

2.8 POISSON'S RATIO

Poisson's ratio, μ, is the ratio of the transverse to the longitudinal strains under axial stress within the elastic range. This ratio varies between 0.15 and 0.20 for both normal and lightweight concrete. Poisson's ratio is used in structural analysis of flat slabs, tunnels, tanks, arch dams, and other statically indeterminate structures. For isotropic elastic materials, Poisson's ratio is equal to 0.25. An average value of 0.18 can be used for concrete.

2.9 SHEAR MODULUS

The modulus of elasticity of concrete in shear ranges from about 0.4 to 0.6 of the corresponding modulus in compression. From the theory of elasticity, the shear modulus is taken as follows:

$$G_c = \frac{E_c}{2(1+\mu)} \tag{2.7}$$

where μ = Poisson's ratio of concrete. If μ is taken equal to 1/6, then $G_c = 0.43E_c = 24{,}500\sqrt{f'_c}$.

Test on a standard concrete cylinder to determine the modulus of elasticity of concrete.

2.10 MODULAR RATIO

The modular ratio, n, is the ratio of the modulus of elasticity of steel to the modulus of elasticity of concrete: $n = E_s/E_c$.

Because the modulus of elasticity of steel is considered constant and is equal to 29×10^6 psi and $E_c = 33w^{1.5}\sqrt{f_c'}$,

$$n = \frac{29 \times 10^6}{33w^{1.5}\sqrt{f_c'}} \tag{2.8}$$

For normal-weight concrete, $E_c = 57{,}400\sqrt{f_c'}$; hence, n can be taken as

$$n = \frac{500}{\sqrt{f_c'}}(f_c' \text{ in psi}) = \frac{42}{\sqrt{f_c'}}(f_c' \text{ in N/mm}^2) \tag{2.9}$$

The significance and the use of the modular ratio are explained in Chapter 6.

2.11 VOLUME CHANGES OF CONCRETE

Concrete undergoes volume changes during hardening. If it loses moisture by evaporation, it shrinks, but if the concrete hardens in water, it expands. The causes of the volume changes in concrete can be attributed to changes in moisture content, chemical reaction of the cement with water, variation in temperature, and applied loads.

2.11.1 Shrinkage

The change in the volume of drying concrete is not equal to the volume of water removed [6]. The evaporation of free water causes little or no shrinkage. As concrete continues to dry, water evaporates and the volume of the restrained cement paste changes, causing concrete to shrink, probably due to the capillary tension that develops in the water remaining in concrete. Emptying of the capillaries causes a loss of water without shrinkage, but once the absorbed water is removed, shrinkage occurs.

Many factors influence the shrinkage of concrete caused by the variations in moisture conditions [5]:

1. *Cement and water content*. The more cement or water content in the concrete mix, the greater the shrinkage.
2. *Composition and fineness of cement*. High-early-strength and low-heat cements show more shrinkage than normal portland cement. The finer the cement, the greater the expansion under moist conditions.
3. *Type, amount, and gradation of aggregate*. The smaller the size of aggregate particles, the greater the shrinkage. The greater the aggregate content, the smaller the shrinkage [7].
4. *Ambient conditions, moisture, and temperature*. Concrete specimens subjected to moist conditions undergo an expansion of 200 to 300×10^{-6}, but if they are left to dry in air, they shrink. High temperature speeds the evaporation of water and, consequently, increases shrinkage.
5. *Admixtures*. Admixtures that increase the water requirement of concrete increase the shrinkage value.
6. *Size and shape of specimen*. As shrinkage takes place in a reinforced concrete member, tension stresses develop in the concrete, and equal compressive stresses develop in the steel. These stresses are added to those developed by the loading action. Therefore, cracks may develop in concrete when a high percentage of steel is used. Proper distribution of reinforcement, by producing better distribution of tensile stresses in concrete, can reduce differential internal stresses.

The values of final shrinkage for ordinary concrete vary between 200 and 700×10^{-6}. For normal-weight concrete, a value of 300×10^{-6} may be used. The British Code [8] gives a value of 500×10^{-6}, which represents an unrestrained shrinkage of 1.5 mm in a 3 m length of thin, plain concrete sections. If the member is restrained, a tensile stress of about 10 N/mm^2 (1400 psi) arises. If concrete is kept moist for a certain period after setting, shrinkage is reduced; therefore, it is important to cure the concrete for a period of no fewer than 7 days.

Exposure of concrete to wind increases the shrinkage rate on the upwind side. Shrinkage causes an increase in the deflection of structural members, which in turn increases with time. Symmetrical reinforcement in the concrete section may prevent curvature and deflection due to shrinkage.

Generally, concrete shrinks at a high rate during the initial period of hardening, but at later stages the rate diminishes gradually. It can be said that 15% to 30% of the shrinkage value occurs in 2 weeks, 40% to 80% occurs in 1 month, and 70% to 85% occurs in 1 year.

2.11.2 Expansion Due to Rise in Temperature

Concrete expands with increasing temperature and contracts with decreasing temperature. The coefficient of thermal expansion of concrete varies between 4 and 7×10^{-6} per degree Fahrenheit.

An average value of 5.5×10^{-6} per degree Fahrenheit (12×10^{-6} per degree Celsius) can be used for ordinary concrete. The British code [8] suggests a value of 10^{-5} per degree Celsius. This value represents a change of length of 10 mm in a 30-m member subjected to a change in temperature of 33°C. If the member is restrained and unreinforced, a stress of about 7 N/mm^2 (1000 psi) may develop.

In long reinforced concrete structures, expansion joints must be provided at lengths of 100 to 200 ft (30 to 60 m). The width of the expansion joint is about 1 in. (25 mm). Concrete is not a good conductor of heat, whereas steel is a good one. The ability of concrete to carry load is not much affected by temperature.

2.12 CREEP

Concrete is an elastoplastic material, and beginning with small stresses, plastic strains develop in addition to elastic ones. Under sustained load, plastic deformation continues to develop over a period that may last for years. Such deformation increases at a high rate during the first 4 months after application of the load. This slow plastic deformation under constant stress is called *creep*.

Figure 2.5 shows a concrete cylinder that is loaded. The instantaneous deformation is ε_1, which is equal to the stress divided by the modulus of elasticity. If the same stress is kept for a period of time, an additional strain, ε_2, due to creep effect, can be recorded. If load is then released, the elastic strain, ε_1, will be recovered, in addition to some creep strain. The final permanent plastic strain, ε_3, will be left, as shown in Fig. 2.5. In this case, $\varepsilon_3 = (1 - \alpha)\varepsilon_2$, where α is the ratio of the recovered creep strain to the total creep strain. The ratio α ranges between 0.1 and 0.2. The magnitude of creep recovery varies with the previous creep and depends appreciably upon the period of the sustained load. Creep recovery rate will be less if the loading period is increased, probably due to the hardening of concrete while in a deformed condition.

The ultimate magnitude of creep varies between 0.2×10^{-6} and 2×10^{-6} per unit stress (lb/in.^{2}) per unit length. A value of 1×10^{-6} can be used in practice. The ratio of creep strain to elastic strain may be as high as 4.

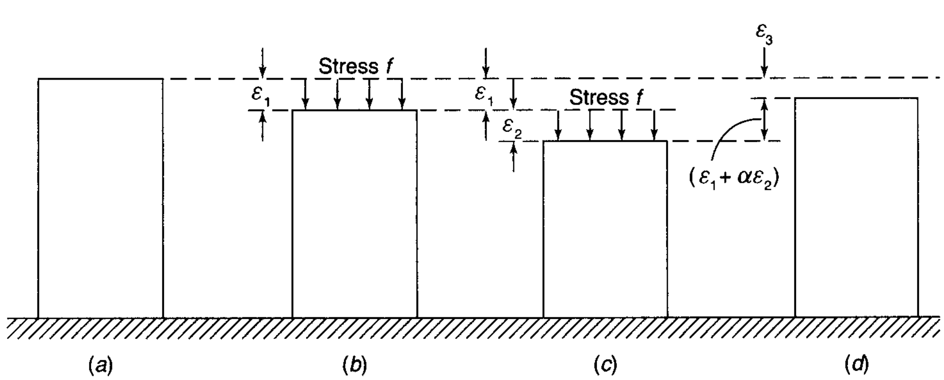


Figure 2.5 Deformation in a loaded concrete cylinder: (*a*) specimen unloaded, (*b*) elastic deformation, (*c*) elastic plus creep deformation, (*d*) permanent deformation after release of load.

Creep takes place in the hardened cement matrix around the strong aggregate. It may be attributed to slippage along planes within the crystal lattice, internal stresses caused by changes in the crystal lattice, and gradual loss of water from the cement gel in the concrete.

The different factors that affect the creep of concrete can be summarized as follows [9]:

1. *Level of stress*. Creep increases with an increase of stress in specimens made from concrete of the same strength and with the same duration of load.
2. *Duration of loading*. Creep increases with the loading period. About 80% of the creep occurs within the first 4 months; 90% occurs after about 2 years.
3. *Strength and age of concrete*. Creep tends to be smaller if concrete is loaded at a late age. Also, creep of 2000 psi (14 N/mm^2)–strength concrete is about 1.41×10^{-6}, whereas that of 4000 psi (28 N/mm^2)–strength concrete is about 0.8×10^{-6} per unit stress and length of time.
4. *Ambient conditions*. Creep is reduced with an increase in the humidity of the ambient air.
5. *Rate of loading*. Creep increases with an increase in the rate of loading when followed by prolonged loading.
6. *Percentage and distribution of steel reinforcement in a reinforced concrete member*. Creep tends to be smaller for higher proportion or better distribution of steel.
7. *Size of the concrete mass*. Creep decreases with an increase in the size of the tested specimen.
8. *Type, fineness, and content of cement*. The amount of cement greatly affects the final creep of concrete, as cement creeps about 15 times as much as concrete.
9. *Water–cement ratio*. Creep increases with an increase in the water–cement ratio.
10. *Type and grading of aggregate*. Well-graded aggregate will produce dense concrete and consequently a reduction in creep.
11. *Type of curing*. High-temperature steam curing of concrete, as well as the proper use of a plasticizer, will reduce the amount of creep.

Creep develops not only in compression, but also in tension, bending, and torsion.

The ratio of the rate of creep in tension to that in compression will be greater than 1 in the first 2 weeks, but this ratio decreases over longer periods [5].

Creep in concrete under compression has been tested by many investigators. Troxell, Raphale, and Davis [10] measured creep strains periodically for up to 20 years and estimated that of the total creep after 20 years, 18% to 35% occurred in 2 weeks, 30% to 70% occurred in 3 months, and 64% to 83% occurred in 1 year.

For normal concrete loaded after 28 days, $C_r = 0.13\sqrt[3]{t}$, where C_r = creep strain per unit stress per unit length. Creep augments the deflection of reinforced concrete beams appreciably with time. In the design of reinforced concrete members, long-term deflection may be critical and has to be considered in proper design. Extensive deformation may influence the stability of the structure.

Sustained loads affect the strength as well as the deformation of concrete. A reduction of up to 30% of the strength of unreinforced concrete may be expected when concrete is subjected to a concentric sustained load for 1 year.

The fatigue strength of concrete is much smaller than its static strength. Repeated loading and unloading cycles in compression lead to a gradual accumulation of plastic deformations. If concrete in compression is subjected to about 2 million cycles, its fatigue limit is about 50% to

60% of the static compression strength. In beams, the fatigue limit of concrete is about 55% of its static strength [11].

2.13 MODELS FOR PREDICTING THE SHRINKAGE AND CREEP OF CONCRETE

2.13.1 The ACI 209 Model

The American Concrete Institute recommend the ACI 209 model [12]. Branson and Christianson [13] first developed this model in 1970. The ACI 209 model was used for many years in the design of concrete structures. The model is simple to use but limited in its accuracy.

Shrinkage calculation. Calculation of shrinkage using the ACI 209 model can be performed if the following parameters and conditions are known: curing method (moist-cured or steam-cured concrete), relative humidity, H, type of cement, specimen shape, ultimate shrinkage strain, ε_{shu}, age of concrete after casting, t, age of the concrete drying commenced, usually taken as the age at the end of moist curing, t_c.

The shrinkage strain is defined as follows:

$$\varepsilon_s(t) = \frac{(t - t_c)}{b + (t - t_c)} K_{\text{ss}} K_{\text{sh}} \varepsilon_{\text{shu}} \tag{2.10}$$

where

t = Age of concrete after casting (days)

t_c = Age of the concrete drying commenced (days)

b = Constant in determining shrinkage strain, depends on curing method according to Table 2.4

K_{ss} = Shape and size correction factor for shrinkage according to the Eq. 2.11

K_{sh} = Relative humidity correction factor for shrinkage according to Eq. 2.12

ε_{shu} (ultimate shrinkage strain) 780×10^{-6} (mm/mm) (for both moist- and steam-cured concrete)

Shape and size correction factor for shrinkage should be calculated as follows:

$$K_{\text{ss}} = 1.14 - 0.0035 \left(\frac{V}{S}\right) \tag{2.11}$$

where

V = volume of the specimen (mm^3)

S = surface of the specimen (mm^2)

Relative humidity correction factor for shrinkage is

$$K_{\text{sh}} = \begin{cases} 1.40 - 0.01H & \text{for } 40\% \leq H \leq 80\% \\ 3.00 - 0.03H & \text{for } 81\% \leq H \leq 100\% \end{cases} \tag{2.12}$$

Table 2.4 Values of Constant b as a Function of Curing Method

Moist-Cured Concrete	Steam-Cured Concrete
$b = 35$	$b = 55$

where

H = Relative humidity (in %)

Creep calculation. The total-load dependent strain at time t, $\varepsilon_{ic}(t, t_0)$ of a concrete member uniaxially loaded at time t_0 with a constant stress σ may be calculated as follows:

$$\varepsilon_{ic}(t, t_0) = \varepsilon_i(t_0) + \varepsilon_c(t, t_0) \tag{2.13}$$

where

$\varepsilon_i(t_0)$ = The initial elastic strain at loading

$\varepsilon_c(t, t_0)$ = The creep strain at time $t \geq t_0$.

$$\varepsilon_i(t_0) = \frac{\sigma}{E_{\text{cmt}_0}} \tag{2.14}$$

$$\varepsilon_c(t, t_0) = \frac{\sigma}{E_{\text{cmt}_0}} C_c(t) \tag{2.15}$$

where

E_{cmt_0} = Modulus of elasticity at age of loading (MPa) as given in Eq. 2.17

$C_c(t)$ = Creep coefficient at time t, as given in Eq. 2.19

Usually, the total-load dependent strain is presented with compliance function, also called creep function, $J(t, t_0)$, which represent the total-load dependent strain at time t produced by a unit constant stress that has been acting since time t_0.

$$J(t, t_0) = \frac{1 + C_c(t)}{E_{\text{cmt}_0}} \tag{2.16}$$

$$E_{\text{cmt}_0} = 0.043(\gamma)^{3/2}\sqrt{f'_c(t_0)} \tag{2.17}$$

where

γ = Concrete unit weight (kg/m^3)

$f'_c(t_0)$ = Mean concrete compressive strength at age of loading (MPa)

$$f'_c(t_0) = f_{\text{cm}_{28}} \frac{t_0}{b + ct_0} \tag{2.18}$$

where

$f_{\text{cm}_{28}}$ = Average 28-day concrete compressive strength (MPa)

b and $c$are constants according to Table 2.5:

Table 2.5 Constants b and c as a Function of Cement Type and Curing Method

Type of Cement	Moist-Cured Concrete		Steam-Cured Concrete	
I	$b = 4$	$c = 0.85$	$b = 1$	$c = 0.95$
III	$b = 2.30$	$c = 0.92$	$b = 0.70$	$c = 0.98$

Table 2.6 Correction Factors

Curing Method	t_0 (days)	H	K_{ca}	K_{ch}	K_{cs}
Moist Cured	$\geq$ 1 day	$\geq$ 40%	N/A	N/A	N/A
	$\geq$ 7 days	$\geq$ 40%	$1.25(t_0)^{-0.118}$	$1.27 - 0.0067H$	$1.14 - 0.0035(V/S)$
Stem Cured	$\geq$ 1 day	$\geq$ 40%	$1.13(t_0)^{-0.095}$	$1.27 - 0.0067H$	$1.14 - 0.0035(V/S)$
	$\geq$ 7 days	$\geq$ 40%	N/A	N/A	N/A

Creep coefficient, $C_c(t)$, can be determined as follows:

$$C_c(t) = \frac{t^{0.60}}{10 + t^{0.60}} C_{cu} K_{ch} K_{ca} K_{cs} \tag{2.19}$$

where

C_{cu} = Ultimate creep coefficient = 2.35
K_{ch} = Relative humidity correction factor for creep determined from Table 2.6
K_{ca} = Age at loading correction factor determined from Table 2.6
K_{cs} = Shape and size correction factor for creep determined from Table 2.6

2.13.2 The B3 Model

The model was developed by Bazant and Baweja [14].

Shrinkage calculation. Required parameters for calculation of shrinkage strain using the B3 model are concrete mean compressive strength at 28 days, curing conditions, cement type, relative humidity, water content in concrete, and specimen shape.

The shrinkage strain can be estimated using the following equation:

$$\varepsilon_s(t) = (\varepsilon_{shu})(K_h)S(t) \tag{2.20}$$

where

ε_{shu} = Ultimate shrinkage strain according to Eq. 2.21
K_h = Humidity function for shrinkage according to Table 2.9
$S(t)$ = Time function for shrinkage according to Eq. 2.22

Ultimate shrinkage strain can be calculated using the following equation:

$$\varepsilon_{shu} = -\alpha_1\alpha_2[0.019(w)^{2.1}(f_{cm_{28}})^{-0.28} + 270] \times 10^{-6} \tag{2.21}$$

where

α_1 = Type of cement correction factor according to Table 2.7
α_2 = Curing condition correction factor according to the Table 2.8
w = Water content (kg/m^3)
$f_{cm_{28}}$ = Mean compressive concrete strength at 28 days (MPa)

Type of cement correction factor α_1 can be determined using Table 2.7.
Curing condition correction factor α_2 can be determined using Table 2.8.
Humidity function for shrinkage, K_h, should be determined according to Table 2.9.

Table 2.7 Correction Factor α_1 as a Function of Cement Type

Type of Cement	α_1
I	1.00
II	0.85
III	1.10

Table 2.8 Correction Factor α_2 as a Function of Type of Curing

Type of Curing	α_2
Steam cured	0.75
Water cured or $H = 100\%$	1.00
Sealed during curing	1.20

Table 2.9 Humidity Function for Shrinkage, K_h

Humidity	K_h
$H \leq 98\%$	$1 - (H/100)^3$
$H = 100\%$	-0.2
$98\% \leq H \leq 100\%$	Linear interpolation

where

H is relative humidity (%)

Time function for shrinkage, $S(t)$, should be calculated according to the following equation:

$$S(t) = \tanh\sqrt{\frac{t - t_c}{T_{\text{sh}}}} \tag{2.22}$$

where

t = Age of concrete after casting (days)
t_c = Age of the concrete drying commenced (days)
T_{sh} = Shrinkage half-time (days) according to the Eq. 2.23

$$T_{\text{sh}} = 0.085(t_c)^{-0.08}(f_{\text{cm}_{28}})^{-0.25}[2K_s(V/S)]^2 \tag{2.23}$$

where K_s = Cross-section shape correction factor according to Table 2.10
K_s can be assumed to be 1 if type of member is not defined.

Creep calculation. The creep function, also called creep compliance, $J(t, t_0)$ is given by Eq. 2.24:

$$J(t, t_0) = q_1 + C_0(t, t_0) + C_d(t, t_0, t_c) \tag{2.24}$$

Table 2.10 Correction Factor, K_s as a Function of Cross-Section Shape

Cross-Section Shape	K_s
Infinite slab	1.00
Infinite cylinder	1.15
Infinite square prism	1.25
Sphere	1.30
Cube	1.55

where

q_1 = The instantaneous strain, given in Eq. 2.25

$C_0(t, t_0)$ = The compliance function for basic creep composed of three terms, an aging viscoelastic term, a nonaging viscoelastic term and an aging flow term given in Eq. 2.27

$C_d(t, t_0, t_c)$ = The compliance function for drying creep, given in Eq. 2.35

$$q_1 = \frac{0.6}{E_{\text{cm}_{28}}} \tag{2.25}$$

where

$E_{\text{cm}_{28}}$ = Modulus of elasticity of concrete at 28 days as given in the following equation:

$$E_{\text{cm}_{28}} = 4735\sqrt{f_{\text{cm}_{28}}} \tag{2.26}$$

The compliance function for basic creep, $C_0(t, t_0)$, should be calculated as follows:

$$C_0(t, t_0) = q_2 Q(t, t_0) + q_3 \ln[1 + (t - t_0)^{0.1}] + q_4 \ln\left(\frac{t}{t_0}\right) \tag{2.27}$$

where

q_2 = Aging viscoelastic compliance parameter

$Q(t, t_0)$ = The binomial integral

q_3 = Nonaging viscoelastic compliance parameter

q_4 = Flow compliance parameter

t_0 = Age of concrete at loading (days)

$$q_2 = 185.4(c)^{0.5}(f_{\text{cm}_{28}})^{-0.9} \times 10^{-6} \tag{2.28}$$

where c is the cement content (kg/m^3).

$$Q(t, t_0) = Q_f(t_0)\left[1 + \frac{Q_f(t_0)^{r(t_0)}}{Z(t, t_0)^{r(t_0)}}\right]^{-1/r(t_0)} \tag{2.29}$$

where

$$Q_f(t_0) = \frac{1}{0.086(t_0)^{2/9} + 1.21(t_0)^{4/9}} \tag{2.30}$$

$$Z(t, t_0) = \frac{\ln[1 + (t - t_0)^{0.1}]}{\sqrt{t_0}} \tag{2.31}$$

$$r(t_0) = 1.7(t_0)^{0.12} + 8 \tag{2.32}$$

$$q_3 = 0.29 q_2 \left(\frac{w}{c}\right)^4 \tag{2.33}$$

$$q_4 = 20.3 \left(\frac{a}{c}\right)^{-0.7} \times 10^{-6} \tag{2.34}$$

The compliance function for drying creep, $C_d(t, t_0, t_c)$, should be calculated as follows:

$$C_d(t, t_0, t_c) = q_5 \sqrt{\exp[-8H(t)] - \exp[-8H(t_0)]} \tag{2.35}$$

where

q_5 = Drying creep compliance parameter that can be calculated from the following equation:

$$q_5 = \frac{0.757 |\varepsilon_{\text{shu}} \times 10^6|^{-0.6}}{f_{\text{cm}_{28}}} \tag{2.36}$$

where

ε_{shu} = Ultimate shrinkage strain, given by Eq. 2.21
$H(t)$ and $H(t_0)$ are spatial averages of pore relative humidity.

$$H(t) = 1 - \left[\left(1 - \frac{H}{100}\right) S(t)\right] \tag{2.37}$$

$$H(t_0) = 1 - \left[\left(1 - \frac{H}{100}\right) S(t_0)\right] \tag{2.38}$$

$S(t)$ is given by Eq. 2.22 and

$$S(t_0) = \tanh \sqrt{\frac{t_0 - t_c}{T_{\text{sh}}}} \tag{2.39}$$

T_{sh} is given by Eq. 2.23.

2.13.3 The GL 2000 Model

The GL 2000 Model was developed by Gardner et. al and is described in Ref. 15.

Shrinkage calculation. Parameters required for calculation of shrinkage strain using the GL 2000 model are mean 28-day concrete compressive strength, $f_{\text{cm}_{28}}$, relative humidity, H, age of concrete at the beginning of shrinkage, t_c, type of cement, and specimen shape.

The shrinkage strain can be calculated using the following equation:

$$\varepsilon_s(t) = \varepsilon_{\text{shu}} \beta(h) \beta(t) \tag{2.40}$$

where

ε_{shu} = Ultimate shrinkage strain according to Eq. 2.41
$\beta(h)$ = Correction term for effect of humidity according to Eq. 2.42
$\beta(t)$ = Correction term for effect of time according to Eq. 2.43

Ultimate shrinkage strain should be calculated from the following equation:

$$\varepsilon_{shu} = (900)K\left(\frac{30}{f_{cm_{28}}}\right)^{1/2} \times 10^{-6} \tag{2.41}$$

where

K = Shrinkage constant, which depends on cement type as shown in Table 2.11
$f_{cm_{28}}$ = Mean 28-day concrete compressive strength (MPa)

Shrinkage constant K can be determined from Table 2.11.

Correction term for effect of humidity, $\beta(h)$, should be calculated as shown:

$$\beta(h) = 1 - 1.18\left(\frac{H}{100}\right)^4 \tag{2.42}$$

where

H = Relative humidity (%)

Correction term for effect of time, $\beta(t)$, should be determined as follows:

$$\beta(t) = \left(\frac{t - t_c}{t - t_c + 0.12(V/S)^2}\right)^{1/2} \tag{2.43}$$

where

t = Age of concrete after casting (days)
t_c = Age of concrete at the beginning of shrinkage (days)
V/S = Volume-to-surface area ratio (mm)

Creep calculation. The creep compliance is composed of two parts: the elastic strain and the creep strain according to the following equation:

$$J(t, t_0) = \frac{1}{E_{cmt_0}} + \frac{\phi(t, t_0)}{E_{cm_{28}}} \tag{2.44}$$

Table 2.11 Shrinkage Constant, K, as a Function of Cement Type

Type of Cement	K
I	1.00
II	0.75
III	1.15

where

E_{cmt_0} = Modulus of elasticity of concrete at loading (MPa)
$E_{\mathrm{cm}_{28}}$ = Modulus of elasticity of concrete at 28 days (MPa)
$\phi(t, t_0)$ = Creep coefficient

$$E_{\mathrm{cmt}_0} = 3500 + 4300\sqrt{f_{\mathrm{cmt}_0}} \tag{2.45}$$

where f_{cmt_0} = Concrete mean compressive strength at loading (MPa), which can be determined as follows:

$$f_{\mathrm{cmt}_0} = f_{\mathrm{cm}_{28}} \frac{t^{3/4}}{a + bt^{3/4}}$$

Coefficients a and b are related to the cement type as shown in Table 2.12.

$$E_{\mathrm{cm}_{28}} = 3500 + 4300\sqrt{f_{\mathrm{cm}_{28}}} \tag{2.46}$$

Creep coefficient, $\phi(t, t_0)$, can be calculated as shown:

$$\phi(t, t_0) = \Phi(t_c)2\left(\frac{(t-t_0)^{0.3}}{(t-t_0)^{0.3}+14}\right) + \left(\frac{7}{t_0}\right)^{0.5}\left(\frac{t-t_0}{t-t_0+7}\right)^{0.5} + 2.5(1-1.086h^2)\left(\frac{t-t_0}{t-t_0+0.12(V/S)^2}\right)^{0.5} \tag{2.47}$$

$$\text{If} \quad t_0 = t_c \quad \text{then } \Phi(t_c) = 1 \tag{2.48}$$

$$\text{If} \quad t_0 > t_c \quad \text{then } \Phi(t_c) = \left[1 - \left(\frac{t_0 - t_c}{t_0 - t_c + 0.12(V/S)^2}\right)^{0.5}\right]^{0.5} \tag{2.49}$$

$h = H/100$ (H = Relative humidity(%))

2.13.4 The CEB 90 Model

The CEB 90 Model was developed by Muller and Hillsdorf [16].

Shrinkage calculation. Parameters required for calculation of shrinkage strain using the CEB 90 model are mean 28-day concrete compressive strength, $f_{\mathrm{cm}_{28}}$, relative humidity, H, age of concrete at the beginning of shrinkage, t_c, type of cement, and specimen shape.

The strain due to shrinkage may be calculated from the following equation:

$$\varepsilon_s(t, t_c) = \varepsilon_{\mathrm{cs}_0}\beta_s(t, t_c) \tag{2.50}$$

Table 2.12 Coefficient *a* and *b* as a Function of Cement Type

Cement Type	*a*	*b*
I	2.8	0.77
II	3.4	0.72
III	1.0	0.92

where

ε_{cs_0} = Notional shrinkage coefficient according to Eq. 2.51
$\beta_s(t, t_c)$ = Coefficient describing development of shrinkage with time according to Eq. 2.54

Notional shrinkage coefficient is

$$\varepsilon_{cs_0} = \varepsilon_s(f_{cm_{28}})\beta_{RH} \tag{2.51}$$

where

$\varepsilon_s(f_{cm_{28}})$ = Concrete strength factor on shrinkage according to Eq. 2.52
β_{RH} = Relative humidity factor on notional shrinkage coefficient according to Table 2.13

Concrete strength factor on shrinkage, $\varepsilon_s(f_{cm_{28}})$, can be calculated as follows:

$$\varepsilon_s(f_{cm_{28}}) = \left[160 + 10(\beta_{sc})\left(9 - \frac{f_{cm_{28}}}{10}\right)\right] \times 10^{-6} \tag{2.52}$$

where

β_{sc} = Coefficient that depends on type of cement according to Table 2.14.
$f_{cm_{28}}$ = Mean 28-day concrete compressive strength (MPa)

Coefficient β_{sc} dependent on humidity, β_{RH}, should be determined according to Table 2.14, where

$$\beta_{arh} = 1 - \left(\frac{H}{100}\right)^3 \tag{2.53}$$

The development of shrinkage with time is given by

$$\beta_s(t - t_c) = \sqrt{\frac{(t - t_c)}{0.56(h_e/4)^2 + (t - t_c)}} \tag{2.54}$$

Table 2.13 Determination of Coefficient β_{RH}

Humidity	β_{RH}
$40\% \leq H < 99\%$	$-1.55 \times \beta_{arh}$
$H \geq 99\%$	0.25

Table 2.14 Coefficient β_{sc}

Type of Cement	**European Type**	**American Type**	β_{sc}
Slow hardening	SL	II	4
Normal/rapid hardening	R	I	5
Rapid hardening, high strength	RS	III	8

where

t = Age of concrete (days)
t_c = Age of concrete at the beginning of shrinkage (days)
h_e = Effective thickness to account for volume/surface ratio (mm)

Effective thickness, h_e, can be determined as follows:

$$h_e = \frac{2A_c}{u} \tag{2.55}$$

where

A_c = Cross-section of the structural member (mm^2)
u = Perimeter of the structural member in contact with the atmosphere (mm)

Creep calculation. Creep compliance represents the total stress dependent strain per unit stress. It can be calculated as

$$J(t, t_0) = \frac{1}{E_{\mathrm{cmt}_0}} + \frac{\phi(t, t_0)}{E_{\mathrm{cm}_{28}}} \tag{2.56}$$

where

E_{cmt_0} = Modulus of elasticity at age of loading (MPa)
$E_{\mathrm{cm}_{28}}$ = Modulus of elasticity at 28 days (MPa)
$\phi(t, t_0)$ = Creep coefficient

$$E_{\mathrm{cmt}_0} = E_{\mathrm{cm}_{28}} \exp\left[0.5S\left(1 - \sqrt{\frac{28}{t}}\right)\right] \tag{2.57}$$

S is the coefficient that depends on cement type and can be determined from Table 2.15.

$$E_{\mathrm{cm}_{28}} = 21500\sqrt[3]{\frac{f_{\mathrm{cm}_{28}}}{10}} \tag{2.58}$$

Creep coefficient, $\phi(t, t_0)$, can be evaluated from the given equation:

$$\phi(t, t_0) = \phi_0 \beta_c(t, t_0) \tag{2.59}$$

Table 2.15 Coefficient S as a Function of Cement Type

Cement Type	European Type	U.S. Type	S
Slow hardening	SL	II	0.38
Normal/rapid hardening	R	I	0.25
Rapid hardening high strength	RS	III	0.20

where

$$\phi_0 = \text{Notional creep coefficient}$$
$$\beta_c(t, t_0) = \text{Equation describing development of creep with time after loading}$$

$$\phi_0 = \phi_{RH}\beta(f_{cm_{28}})\beta(t_0) \tag{2.60}$$

where ϕ_{RH} = Relative humidity factor on the notional creep coefficient, which is given by

$$\phi_{RH} = 1 + \frac{1 - H/100}{0.16\sqrt[3]{h_e/4}} \tag{2.61}$$

$\beta(f_{cm_{28}})$ = Concrete strength factor on the notional creep coefficient, which is given by

$$\beta(f_{cm_{28}}) = \frac{5.3}{\sqrt{f_{cm_{28}}/10}} \tag{2.62}$$

$\beta\ (t_0)$ = Age of concrete at loading factor on the notional creep coefficient, which is given by

$$\beta(t_0) = \frac{1}{0.1 + t_0^{0.2}} \tag{2.63}$$

An equation describing development of creep with time after loading, $\beta_c(t, t_0)$, can be calculated using the following equation:

$$\beta_c(t, t_0) = \left(\frac{t - t_0}{\beta_H + t - t_0}\right)^{0.3} \tag{2.64}$$

$$\beta_H = 1.5h_e[1 + (0.012H)^{18}] + 250 \leq 1500 \text{ days} \tag{2.65}$$

2.13.5 The CEB 90-99 Model

The CEB 90-99 is a modification of the CEB 90 and is described in Ref. 17.

Shrinkage calculation. In this new model, total shrinkage contains of autogenous and drying shrinkage component. In high-performance concrete, autogenous shrinkage is significant and needs to be considered in prediction of shrinkage. This new approach was necessary so that shrinkage of normal as well as high-performance concrete can be predicted with sufficient accuracy [1].

Total shrinkage strain can be calculated using the following equation:

$$\varepsilon_s(t, t_c) = \varepsilon_{as}(t) + \varepsilon_{ds}(t, t_c) \tag{2.66}$$

where

$$\varepsilon_{as}(t) = \text{Autogenous shrinkage at time } t$$
$$\varepsilon_{ds}(t, t_c) = \text{Drying shrinkage at time } t$$

Autogenous shrinkage, $\varepsilon_{as}(t)$, should be calculated as follows:

$$\varepsilon_{as}(t) = \varepsilon_{as_0}(f_{cm_{28}})\beta_{as}(t) \tag{2.67}$$

where

$\varepsilon_{as_0}(f_{cm_{28}})$ = National autogenous shrinkage coefficient according to Eq. 2.68
$\beta_{as}(t)$ = Function to describe the time-development of autogenous shrinkage, from Eq. 2.69

National autogenous shrinkage coefficient, $\varepsilon_{cas_0}(f_{cm})$, can be calculated as follows:

$$\varepsilon_{as_0}(f_{cm_{28}}) = -\alpha_{as}\left(\frac{f_{cm_{28}}/10}{6 + f_{cm_{28}}/10}\right)^{2.5} \times 10^{-6} \tag{2.68}$$

where

α_{as} = Coefficient that depends on type of cement
= 800 for slowly hardening cements
= 700 for normal or rapidly hardening cements
= 600 for rapidly hardening high-strength cements
$f_{cm_{28}}$ = Mean compressive strength of concrete at an age of 28 days (MPa)

Function $\beta_{as}(t)$ should be calculated using the following equation:

$$\beta_{as}(t) = 1 - \exp[-0.2(t)^{0.5}] \tag{2.69}$$

where t = Age of concrete (days)

Drying shrinkage, $\varepsilon_{ds}(t, t_c)$, can be estimated by the following equation:

$$\varepsilon_{ds}(t, t_c) = \varepsilon_{ds_0}(f_{cm_{28}})\beta_{RH}(H)\beta_{ds}(t - t_c) \tag{2.70}$$

where

$\varepsilon_{ds_0}(f_{cm_{28}})$ = Notional drying shrinkage coefficient according to Eq. 2.71
$\beta_{RH}(H)$ = Coefficient to take into account the effect of relative humidity on drying shrinkage according to Eq. 2.72
$\beta_{ds}(t - t_c)$ = Function to describe the time development of drying shrinkage according to Eq. 2.74

Notional drying shrinkage coefficient, $\varepsilon_{ds_0}(f_{cm_{28}})$, may be calculated from the following equation:

$$\varepsilon_{ds_0}(f_{cm_{28}}) = [(220 + 110\alpha_{ds_1})\exp(-\alpha_{ds_2} f_{cm_{28}}/10)] \times 10^{-6} \tag{2.71}$$

where

α_{ds_1} = Coefficient that depends on type of cement
= 3 for slowly hardening cements
= 4 for normal or rapidly hardening cements
= 6 for rapidly hardening high-strength cements
α_{ds_2} = Coefficient that depends on type of cement
= 0.13 for slowly hardening cements
= 0.11 for normal or rapidly hardening cements
= 0.12 for rapidly hardening high-strength cements

Coefficient $\beta_{RH}(H)$ should be calculated as follows:

$$\beta_{RH} = \begin{cases} -1.55\left[1 - \left(\frac{H}{100}\right)^3\right] & \text{for } 40\% \leq H < 99\% \times \beta_{s_1} \\ 0.25 & \text{for } H \geq 99\% \times \beta_{s_1} \end{cases} \tag{2.72}$$

where

H = Ambient relative humidity (%)

β_{s_1} = Coefficient to take into account the self-desiccation in high-performance concrete. It can be determined as follows:

$$\beta_{s_1} = \left(\frac{35}{f_{cm_{28}}}\right)^{0.1} \leq 1.0 \tag{2.73}$$

Function $\beta_{ds}(t - t_c)$ may be estimated as follows:

$$\beta_{ds}(t - t_c) = \left(\frac{(t - t_c)}{0.56(h_e/4)^2 + (t - t_c)}\right)^{0.5} \tag{2.74}$$

where

t_c = Concrete age at the beginning of drying (days)

$h_e = \frac{2A_c}{u}$ = notional size of member (mm), where A_c is the cross-section (mm^2) and u is the perimeter of the member in contact with the atmosphere (mm)

Creep calculation. Total stress-dependent strain per unit stress, also called creep compliance or creep function can be determined as follows:

$$J(t, t_0) = \frac{1}{E_{cmt_0}} + \frac{\phi(t, t_0)}{E_{cm_{28}}} \tag{2.75}$$

where

E_{cmt_0} = Modulus of elasticity at age of loading (MPa)

$E_{cm_{28}}$ = Modulus of elasticity at day 28 (MPa)

$\phi(t, t_0)$ = Creep coefficient

$$E_{cmt_0} = E_{cm_{28}} \exp\left[0.5S\left(1 - \sqrt{\left(\frac{28}{t_0}\right)}\right)\right] \tag{2.76}$$

S is the coefficient that depends on cement type and compressive strength and can be determined from Table 2.16.

$$E_{cm_{28}} = 21500\sqrt[3]{\frac{f_{cm_{28}}}{10}} \tag{2.77}$$

Creep coefficient, $\phi(t, t_0)$, can be evaluated from the given equation:

$$\phi(t, t_0) = \phi_0\beta_c(t, t_0) \tag{2.78}$$

Table 2.16 Coefficient S as a Function of Cement Type and Compressive Strength

$f_{cm_{28}}$ (MPa)	Type of Cement	S
	Rapidly hardening high strength	0.20
≤60	Normal and rapidly hardening	0.25
	Slow hardening	0.38
>60	All types	0.20

where

ϕ_0 = Notional creep coefficient

$\beta_c(t, t_0)$ = Equation describing development of creep with time after loading

$$\phi_0 = \phi_{RH}\beta(f_{cm_{28}})\beta(t_0) \tag{2.79}$$

ϕ_{RH} = Relative humidity factor on the notional creep coefficient

$$\phi_{RH} = \left[1 + \frac{1 - H/100}{0.16\sqrt[3]{h_e/4}}\alpha_1\right]\alpha_2 \tag{2.80}$$

where

$$\alpha_1 = \left[\frac{35}{f_{cm_{28}}}\right]^{0.7} \tag{2.81}$$

$$\alpha_2 = \left[\frac{35}{f_{cm_{28}}}\right]^{0.2} \tag{2.82}$$

$\beta(f_{cm_{28}})$ = Concrete strength factor on the notional creep coefficient,

$$\beta(f_{cm_{28}}) = \frac{5.3}{\sqrt{f_{cm_{28}}/10}} \tag{2.83}$$

$\beta(t_0)$ = Age of concrete at loading factor on the notional creep coefficient

$$\beta(t_0) = \frac{1}{0.1 + t_0^{0.2}} \tag{2.84}$$

where

$$t_0 = t_{0,T}\left[\frac{9}{2 + t_{0,T}^{1.2}} + 1\right]^{\alpha} \geq 0.5 \text{ days} \tag{2.85}$$

t_0 = Age of concrete at loading (days)

$t_{0,T}$ = Age of concrete at loading adjusted according to the concrete temperature;

for T = 20 °C, $t_{0,T}$ corresponds to t_0

α = Coefficient that depends on type of cement

= −1 for slowly hardening cement

= 0 for normal or rapidly hardening cement

= 1 for rapidly hardening high-strength cement

An equation describing development of creep with time after loading, $\beta_c(t, t_0)$, can be calculated using the following equation:

$$\beta_c(t, t_0) = \left(\frac{t - t_0}{\beta_H + t - t_0}\right)^{0.3} \tag{2.86}$$

$$\beta_H = 1.5h_e[1 + (0.012H)^{18}] + 250\alpha_3 \leq 1500\alpha_3 \tag{2.87}$$

$$\alpha_3 = \left[\frac{35}{f_{\mathrm{cm}_{28}}}\right]^{0.5} \tag{2.88}$$

2.13.6 The AASHTO Model

Shrinkage calculation. Parameters required for calculation of shrinkage strain using the AASHTO model are: curing method (moist-cured or steam-cured concrete), 28-day concrete compressive strength, $f_{\mathrm{cm}_{28}}$, relative humidity, H, drying time of concrete, t, type of cement, and specimen shape.

The strain due to shrinkage may be calculated from the following equation:

- For moist-cured concrete:

$$\varepsilon_{sh} = -k_s k_h \left(\frac{t}{35.0 + t}\right) 0.51 \times 10^{-3} \tag{2.89}$$

- For steam-cured concrete:

$$\varepsilon_{sh} = -k_s k_h \left(\frac{t}{55.0 + t}\right) 0.56 \times 10^{-3} \tag{2.90}$$

where

t = drying time (day)
k_s = size factor for shrinkage specified in Eq. 2.91
k_h = humidity factor for shrinkage specified in Eq. 2.92

Size factor for shrinkage should be calculated as follows:

$$k_s = \left[\frac{\dfrac{t}{26e^{0.36(V/S)} + t}}{\dfrac{t}{45 + t}}\right]\left[\frac{1064 - 94(V/S)}{923}\right] \tag{2.91}$$

where

V = Volume of the specimen (in.^{3})
S = Surface of the specimen (in.^{2})

Humidity factor for shrinkage is:

$$k_h = \frac{140 - H}{70} \quad \text{for } H < 80\%$$
$$k_h = \frac{3(100 - H)}{70} \quad \text{for } H \geq 80\% \tag{2.92}$$

where

H = Relative humidity (%)

Creep calculation. The creep compliance represents the total stress dependent strain per unit stress. It can be calculated as:

$$J(t, t_0) = \frac{1}{E_c} + \frac{\psi(t, t_0)}{E_c} \tag{2.93}$$

where

$\Psi(t, t_0)$ = Creep coefficient as given in Eq. 2.94

E_c = Modulus of elasticity at 28 days (ksi) as given in Eq. 2.97

The creep coefficient may be calculated from the following equation:

$$\psi(t, t_0) = 3.5k_c k_f \left(1.58 - \frac{H}{120}\right) t_0^{-0.118} \frac{(t - t_0)^{0.6}}{10.0 + (t - t_0)^{0.6}} \tag{2.94}$$

where

t = Maturity of concrete (day)

t_0 = Age of concrete when load is initially applied (day)

H = Relative humidity (%)

k_f = Factor for the effect of concrete strength as given in Eq. 2.95

k_c = Factor for the effect of the volume-to-surface ratio of the component as given in Eq. 2.96

The factor for the effect of concrete strength should be calculated as follows:

$$k_f = \frac{1}{0.67 + \frac{f_{cm_{28}}}{9}} \tag{2.95}$$

where

$f_{cm_{28}}$ = Specified concrete compressive strength at 28 days (Ksi)

The factor for the effect of the volume-to-surface ratio of the component should be calculated as follows:

$$k_c = \left[\frac{\frac{t}{26e^{0.36(V/S)} + t}}{\frac{t}{45 + t}}\right]\left[\frac{1.80 + 1.77e^{-0.54(V/S)}}{2.587}\right] \tag{2.96}$$

where

V = Volume of the specimen (in.^{3})

S = Surface of the specimen (in.^{2})

The modulus of elasticity at 28 days should be calculated as follows:

$$E_c = 33000\omega_c^{1.5}\sqrt{f_c'} \tag{2.97}$$

where

ω_c = Concrete unit weight (Kcf)

f_c' = Specified concrete compressive strength at 28 days (Ksi)

Example 2.1

Calculate shrinkage strain and creep compliance for the concrete specimen given below. Use the ACI 209 model.

Given factors:

Humidity = 75%
$h_e = 2V/S = 2A_c/u = 76\,\text{mm}$
$f_{\text{cm}_{28}} = 45.2$ MPa
$w = 207.92\,\text{kg/m}^3$
$w/c = 0.46$
$a/c = 3.73$
$t = 35$ days
$t_0 = 28$ days
$t_c = 8$ days
$\gamma = 2405\,\text{kg/m}^3$
Cement type III
Moist-cured concrete

Solution

Shrinkage calculation

$$\varepsilon_s(t) = \frac{(t - t_c}{b + (t - t_c)} K_{\text{ss}} K_{\text{sh}} \varepsilon_{\text{shu}}$$

$$\varepsilon_{\text{shu}} = 780 \times 10^{-6}\ \text{mm/mm}$$

According to Table 2.4, $b = 35$

$$V/S = 38\ \text{mm}$$

$$K_{\text{ss}} = 1.14 - 0.0035\left(\frac{V}{S}\right) = 1.14 - 0.0035(38) = 1.007$$

For $H = 75\%$,

$$K_{\text{sh}} = 1.40 - 0.01H = 1.40 - 0.01(75) = 0.65$$

$$\varepsilon_s(t) = \frac{(t - t_c)}{b + (t - t_c)} K_{\text{ss}} K_{\text{sh}} \varepsilon_{\text{shu}}$$

$$= \frac{(35 - 8)}{35 + (35 - 8)}(1.007)(0.65)(780 \times 10^{-6}) = 222.3 \times 10^{-6}\ \text{mm/mm}$$

Creep calculation

$$J(t, t_0) = \frac{1 + C_c(t)}{E_{\text{cmt}_0}} \qquad \text{(Eq. 2.7)}$$

Determination of E_{cmt_0}

$$b = 2.30,\ c = 0.92\ \text{(Table 2.5)}$$

$$f'_c(t_0) = f_{\text{cm}_{28}} \frac{t_0}{b + ct_0} = 45.2 \frac{28}{2.3 + 0.92 \times 28} = 45.1\ \text{MPa}$$

$$E_{\text{cmt}_0} = 0.043(\gamma)^{3/2}\sqrt{f'_c(t_0)} = 0.043(2405)^{3/2}\sqrt{45.1} = 34058.8\ \text{MPa}$$

Determination of $C_c(t)$

$$C_{cu} = 2.35$$

$$K_{\text{ch}} = 1.27 - 0.0067(H) = 1.27 - 0.0067(75) = 0.767$$

$$K_{\text{ca}} = 1.25(t_0)^{-0.118} = 1.25(28)^{-0.118} = 0.844$$

$$K_{\text{cs}} = 1.14 - 0.0035(V/S) = 1.14 - 0.0035(38) = 1.007$$

$$C_c(t) = \frac{t^{0.60}}{10 + t^{0.60}} C_{\text{cu}} K_{\text{ch}} K_{\text{ca}} K_{\text{cs}} = \frac{35^{0.60}}{10 + 35^{0.60}} 2.35 \times 0.767 \times 0.844 \times 1.00 = 0.702$$

$$J(t, t_0) = \frac{1 + C_c(t)}{E_{\text{cmt}_0}} = \frac{1 + 0.702}{34058.8} = 49.9 \times 10^{-6} \frac{1}{\text{MPa}}$$

Example 2.2

Using the B3 model, calculate shrinkage strain and creep function for the specimen given in Example 2.1.

Solution

Shrinkage calculation

$$\varepsilon_s(t) = (\varepsilon_{\text{shu}})(K_{\text{h}})S(t)$$

Determination of ε_{shu}

$$\alpha_1 = 1.10 \text{ (Table 2.7)}$$

$$\alpha_2 = 1.0 \text{ (Table 2.8)}$$

$$\varepsilon_{\text{shu}} = \alpha_1 \alpha_2 [0.019(w)^{2.1} (f_{\text{cm}_{28}})^{-0.28} + 270] \times 10^{-6}$$

$$= (1.10)(1.0)[0.019(207.92)^{2.1}(45.2)^{-0.28} + 270] \times 10^{-6} = 827 \times 10^{-6} \text{ mm/mm}$$

Determination of K_h

According to the Table 2.6, for $H = 75\%$

$$K_h = 1 - \left(\frac{H}{100}\right)^3 = 1 - \left(\frac{75}{100}\right)^3 = 0.578$$

Determination of $S(t)$

$K_s = 1.0$, since the type of member is not defined

$$T_{\text{sh}} = 0.085(t_c)^{-0.08} (f_{\text{cm}_{28}})^{-0.25} [2K_s(V/S)]^2$$

$$= 0.085(8)^{-0.08}(45.2)^{-0.25}[2(1.0)(38)]^2 = 160.3$$

$$S(t) = \tanh\sqrt{\frac{t - t_c}{T_{\text{sh}}}} = \tanh\sqrt{\frac{35 - 8}{160.3}} = 0.389$$

$$\varepsilon_s(t) = (\varepsilon_{\text{shu}})(K_h)S(t) = (827 \times 10^{-6})(0.578)(0.389) = 185.9 \times 10^{-6} \text{ mm/mm}$$

Creep calculation

$$J(t, t_0) = q_1 + C_0(t, t_0) + C_d(t, t_0, t_c) \tag{2.15}$$

Determination of q_1

$$E_{\text{cm}_{28}} = 4735\sqrt{f_{\text{cm}_{28}}} = 4735\sqrt{45.2} = 31833.9 \text{ MPa}$$

$$q_1 = \frac{0.6}{E_{\text{cm}_{28}}} = \frac{0.6}{31833.9} = 18.85 \times 10^{-6}\frac{1}{\text{MPa}}$$

Calculation of $C_0(t, t_0)$

$$c = \frac{w}{w/c} = \frac{207.92}{0.46} = 452 \text{ kg/m}^3$$

$$q_2 = 185.4(c)^{0.5}(f_{\text{cm}_{28}})^{-0.9} \times 10^{-6} = 185.4(452)^{0.5}(45.2)^{-0.9} \times 10^{-6}$$

$$= 127.6 \times 10^{-6}$$

$$Q_f(t_0) = \frac{1}{0.086(t_0)^{2/9} + 1.21(t_0)^{4/9}} = \frac{1}{0.086(28)^{2/9} + 1.21(28)^{4/9}} = 0.182$$

$$Z(t, t_0) = \frac{\ln[1 + (t - t_0)^{0.1}]}{\sqrt{t_0}} = \frac{\ln[1 + (35 - 28)^{0.1}]}{\sqrt{28}} = 0.150$$

$$r(t_0) = 1.7(t_0)^{0.12} + 8 = 1.7(28)^{0.12} + 8 = 10.54$$

$$Q(t, t_0) = Q_f(t_0)\left[1 + \frac{Q_f(t_0)^{r(t_0)}}{Z(t, t_0)^{r(t_0)}}\right]^{-1/r(t_0)} = 0.182\left[1 + \frac{0.182^{10.54}}{0.150^{10.54}}\right]^{-1/10.54} = 0.148$$

$$q_3 = 0.29q_2\left(\frac{w}{c}\right)^4 = 0.29(127.6 \times 10^{-6})(0.46)^4 = 1.66 \times 10^{-6}$$

$$q_4 = 20.3\left(\frac{a}{c}\right)^{-0.7} \times 10^{-6} = 20.3(3.73)^{-0.7} \times 10^{-6} = 8.08 \times 10^{-6}$$

$$C_0(t, t_0) = q_2 Q(t, t_0) + q_3 \ln[1 + (t - t_0)^{0.1}] + q_4 \ln\left(\frac{t}{t_0}\right)$$

$$= (127.6 \times 10^{-6})(0.148) + (1.66 \times 10^{-6}) \ln[1 + (35 - 28)^{0.1}] + (8.08 \times 10^{-6}) \ln\left(\frac{35}{28}\right)$$

$$= 22.01 \times 10^{-6}\frac{1}{\text{MPa}}$$

Calculation of $C_d(t, t_0, t_c)$:

$$q_5 = \frac{0.757|\varepsilon_{\text{shu}} \times 10^6|^{-0.6}}{f_{\text{cm}_{28}}} = \frac{0.757|827 \times 10^{-6} \times 10^6|^{-0.6}}{45.2} = 297.5 \times 10^{-6}$$

$$S(t) = 0.389$$

$$S(t_0) = \tanh\sqrt{\frac{t_0 - t_c}{T_{\text{sh}}}} = \tanh\sqrt{\frac{28 - 8}{160.3}} = 0.339$$

$$H(t) = 1 - \left[\left(1 - \frac{H}{100}\right)S(t)\right] = 1 - \left(1 - \frac{75}{100}\right)0.389] = 0.903$$

$$H(t_0) = 1 - \left[\left(1 - \frac{H}{100}\right)S(t_0)\right] = 1 - \left[\left(1 - \frac{75}{100}\right)0.339\right] = 0.915$$

$$C_d(t, t_0, t_c) = q_5\sqrt{\exp[-8H(t)] - \exp[-8H(t_0)]}$$
$$= (297 \times 10^{-6})\sqrt{\exp[-8 \times 0.903] - \exp[-8 \times 0.915]} = 2.43 \times 10^{-6}\frac{1}{\text{MPa}}$$

$$J(t, t_0) = q_1 + C_0(t, t_0) + C_d(t, t_0, t_c)$$
$$= (18.85 \times 10^{-6}) + (22.01 \times 10^{-6}) + (2.43 \times 10^{-6}) = 43.3 \times 10^{-6}\frac{1}{\text{MPa}}$$

Example 2.3

Using the GL 2000 model, calculate the shrinkage strain and creep function for the specimen given in Example 2.1.

Solution

Shrinkage calculation

$$\varepsilon_s(t) = \varepsilon_{\text{shu}}\beta(h)\beta(t)$$

Calculation of ε_{shu}

$K = 1.15$ (Table 2.11)

$$\varepsilon_{\text{shu}} = (900)K\left(\frac{30}{f_{\text{cm}_{28}}}\right)^{1/2} \times 10^{-6} = (900)(1.15)\left(\frac{30}{45.2}\right)^{1/2} \times 10^{-6} = 843.2 \times 10^{-6} \text{ mm/mm}$$

Calculation of $\beta(h)$

$$\beta(h) = 1 - 1.18\left(\frac{H}{100}\right)^4 = 1 - 1.18\left(\frac{75}{100}\right)^4 = 0.627$$

Calculation of $\beta(t)$:

$$\beta(t) = \left(\frac{t - t_c}{t - t_c + 0.12(V/S)^2}\right)^{1/2} = \left(\frac{35 - 8}{35 - 8 + 0.12(38)^2}\right)^{1/2} = 0.367$$

$$\varepsilon_s(t) = \varepsilon_{\text{shu}}\beta(h)\beta(t) = (843.2 \times 10^{-6})(0.627)(0.367) = 194 \times 10^{-6} \text{ mm/mm}$$

Creep calculation

$$J(t, t_0) = \frac{1}{E_{\text{cmt}_0}} + \frac{\phi(t, t_0)}{E_{\text{cm}_{28}}}$$

Calculation of E_{cmt_0} and $E_{\text{cm}_{28}}$

$$t_0 = 28 \text{ days} \Rightarrow E_{\text{cmt}_0} = E_{\text{cm}_{28}}$$

$$E_{\text{cm}_{28}} = 3500 + 4300\sqrt{f_{\text{cm}_{28}}} = 3500 + 4300\sqrt{45.2} = 32409.3 \text{ MPa}$$

Calculation of $\phi(t, t_0)$

$t_0 = 28 > t_c = 8$ days

$$\Phi(t_c) = \left[1 - \left(\frac{t_0 - t_c}{t_0 - t_c + 0.12(V/S)^2}\right)^{0.5}\right]^{0.5} = \left[1 - \left(\frac{28 - 8}{28 - 8 + 0.12(38)^2}\right)^{0.5}\right]^{0.5} = 0.824$$

$$h = H/100 = 75/100 = 0.75$$

$$\phi(t, t_0) = \Phi(t_c)\left[2\left(\frac{(t-t_0)^{0.3}}{(t-t_0)^{0.3}+14}\right) + \left(\frac{7}{t_0}\right)^{0.5}\left(\frac{t-t_0}{t-t_0+7}\right)^{0.5}\right.$$
$$\left. + 2.5(1-1.086h^2)\left(\frac{t-t_0}{t-t_0+0.12(V/S)^2}\right)^{0.5}\right]$$
$$= 0.824\left[2\left(\frac{(35-28)^{0.3}}{(35-28)^{0.3}+14}\right) + \left(\frac{7}{28}\right)^{0.5}\left(\frac{35-28}{35-28+7}\right)^{0.5}\right.$$
$$\left. + 2.5(1-1.086(0.75)^2)\left(\frac{35-28}{35-28+0.12(38)^2}\right)^{0.5}\right] = 0.636$$

$$J(t, t_0) = \frac{1}{E_{\text{cmt}_0}} + \frac{\phi(t, t_0)}{E_{\text{cm}_{28}}} = \frac{1}{32409.3} + \frac{0.636}{32409.3} = 50.5 \times 10^{-6}\frac{1}{\text{MPa}}$$

Example 2.4

Using the CEB 90 model, calculate shrinkage strain and creep function for the specimen given in Example 2.1.

Solution

Shrinkage calculation

$$\varepsilon_s(t, t_c) = (\varepsilon_{\text{cs}_0})\beta_s(t, t_c)$$

Calculation of $\varepsilon_{\text{cs}_0}$

$$\beta_{\text{sc}} = 8$$

$$\varepsilon_s(f_{\text{cm}_{28}}) = \left[160 + 10(\beta_{\text{sc}})\left(9 - \frac{f_{\text{cm}_{28}}}{10}\right)\right] \times 10^{-6}$$
$$= \left[160 + 10(8)\left(9 - \frac{45.2}{10}\right)\right] \times 10^{-6} = 518.4 \times 10^{-6} \text{ mm/mm}$$

For $H = 75\%$,

$$\beta_{\text{RH}} = -1.55\beta_{\text{arh}}$$

$$\beta_{\text{arh}} = 1 - (H/100)^3 = 1 - (75/100)^3 = 0.578$$

$$\beta_{\text{RH}} = -1.55\beta_{\text{arh}} = -1.55 \times 0.578 = -0.896$$

$$\varepsilon_{\text{cs}_0} = \varepsilon_s(f_{\text{cm}_{28}})(\beta_{\text{RH}}) = (518.4 \times 10^{-6})(-0.896) = -464.2 \times 10^{-6} \text{ mm/mm}$$

Calculation of $\beta_s(t - t_c)$

$$h_e = \frac{2A_c}{u} = 76 \text{ mm}$$

$$\beta_s(t - t_c) = \sqrt{\frac{(t-t_c)}{0.56(h_e/4)^2 + (t-t_c)}} = \sqrt{\frac{(35-8)}{0.56(76/4)^2 + (35-8)}} = 0.343$$

$$\varepsilon_s(t, t_c) = (\varepsilon_{\text{cs}_0})\beta_s(t - t_c) = (-464.2 \times 10^{-6})(0.343) = -159.3 \times 10^{-6} \text{ mm/mm}$$

Creep calculation

$$J(t, t_0) = \frac{1}{E_{\text{cmt}_0}} + \frac{\phi(t, t_0)}{E_{\text{cm}_{28}}}$$

Calculation of E_{cmt_0} and $E_{\text{cm}_{28}}$

$$t_0 = 28 \text{ days} \Rightarrow E_{\text{cmt}_0} = E_{\text{cm}_{28}}$$

$$E_{\text{cm}_{28}} = 21500\sqrt[3]{\frac{f_{\text{cm}_{28}}}{10}} = 21500\sqrt[3]{\frac{45.2}{10}} = 35548 \text{ MPa}$$

Calculation of $\phi(t, t_0)$

$$\phi_{\text{RH}} = 1 + \frac{1 - H/100}{0.16\sqrt[3]{h_e/4}} = 1 + \frac{1 - 75/100}{0.16\sqrt[3]{76/4}} = 1.586$$

$$\beta(f_{\text{cm}_{28}}) = \frac{5.3}{\sqrt{f_{\text{cm}_{28}}/10}} = \frac{5.3}{\sqrt{45.2/10}} = 2.49$$

$$\beta(t_0) = \frac{1}{0.1 + t_0^{0.02}} = \frac{1}{0.1 + 28^{0.2}} = 0.488$$

$$\phi_0 = \phi_{\text{RH}}\beta(f_{\text{cm}_{28}})\beta(t_0) = (1.586)(2.49)(0.488) = 1.927$$

$$\beta_H = 1.5h_e[1 + (0.012H)^{18}] + 250 = 1.5(76)[1 + (0.012 \times 75)^{18}] + 250$$

$$= 379 \leq 1500 \text{ days}$$

$$\beta_c(t, t_0) = \left(\frac{t - t_0}{\beta_H + t - t_0}\right)^{0.3} = \left(\frac{35 - 28}{379 + 35 - 28}\right)^{0.3} = 0.3$$

$$\phi(t, t_0) = \phi_0\beta_c(t, t_0) = 1.927 \times 0.3 = 0.578$$

$$J(t, t_0) = \frac{1}{E_{\text{cmt}_0}} + \frac{\phi(t, t_0)}{E_{\text{cm}_{28}}} = \frac{1}{35548} + \frac{0.578}{35548} = 44.4 \times 10^{-6}\frac{1}{\text{MPa}}$$

Example 2.5

Using the new CEB 90–99 model to calculate shrinkage strain and creep function for the specimen given in Example 2.1.

Solution

Shrinkage calculation

$$\varepsilon_s(t, t_c) = \varepsilon_{\text{as}}(t) + \varepsilon_{\text{ds}}(t, t_c)$$

Calculation of $\varepsilon_{\text{as}}(t)$

$$\alpha_{\text{as}} = 600 \text{ for rapidly hardening high-strength cements}$$

$$\varepsilon_{\text{as}_0}(f_{\text{cm}_{28}}) = -\alpha_{\text{as}}\left(\frac{f_{\text{cm}_{28}}/10}{6 + f_{\text{cm}_{28}}/10}\right)^{2.5} \times 10^{-6}$$

$$= -600\left(\frac{45.2/10}{6 + 45.2/10}\right)^{2.5} \times 10^{-6} = -72.6 \times 10^{-6} \text{ mm/mm}$$

$$\beta_{\text{as}}(t) = 1 - \exp(-0.2(t)^{0.5}) = 1 - \exp(-0.2(35)^{0.5}) = 0.694$$

$$\varepsilon_{\text{as}}(t) = \varepsilon_{\text{as}_0}(f_{\text{cm}_{28}})\beta_{\text{as}}(t) = (-72.6 \times 10^{-6})(0.694) = -50.4 \times 10^{-6} \text{ mm/mm}$$

Calculation of $\varepsilon_{ds}(t, t_c)$

$$\alpha_{ds_1} = 6 \text{ for rapidly hardening high-strength cements}$$

$$\alpha_{ds_2} = 0.12 \text{ for rapidly hardening high-strength cements}$$

$$\varepsilon_{ds0}(f_{cm_{28}}) = [(220 + 110\alpha_{ds_1})\exp(-\alpha_{ds_2} f_{cm_{28}}/10)] \times 10^{-6}$$

$$= [(220 + 110 \times 6)\exp(-0.12 \times 45.2/10)] \times 10^{-6} = 511.6 \times 10^{-6} \text{ mm/mm}$$

$$\beta_{s1} = \left(\frac{35}{f_{cm_{28}}}\right)^{0.1} = \left(\frac{35}{45.2}\right)^{0.1} = 0.97 \le 1.0$$

For 40% < H = 75% < 99% (0.97) = 96.5%,

$$\beta_{RH} = -1.55\left[1 - \left(\frac{H}{100}\right)^3\right] = -1.55\left[1 - \left(\frac{75}{100}\right)^3\right] = -0.896$$

$$\beta_{ds}(t - t_c) = \left(\frac{(t - t_c)}{0.56(h/4)^2 + (t - t_c)}\right)^{0.5}$$

$$= \left(\frac{(35 - 8)}{0.56(76/100)^2 + (35 - 8)}\right)^{0.5} = 0.343$$

$$\varepsilon_{ds}(t, t_c) = \varepsilon_{ds0}(f_{cm_{28}})\beta_{RH}(H)\beta_{ds}(t - t_c)$$

$$= (511.6 \times 10^{-6})(-0.896)(0.343) = -157.2 \times 10^{-6} \text{ mm/mm}$$

$$\varepsilon_s(t, t_c) = \varepsilon_{as}(t) + \varepsilon_{ds}(t, t_c) = (-50.4 \times 10^{-6}) + (-157.2 \times 10^{-6}) = -207.6 \times 10^{-6} \text{ mm/mm}$$

Creep calculation

$$J(t, t_0) = \frac{1}{E_{cmt_0}} + \frac{\phi(t, t_0)}{E_{cm_{28}}}$$

Calculation of E_{cmt_0} and $E_{cm_{28}}$

$$t_0 = 28 \text{ days} \Rightarrow E_{cmt_0} = E_{cm_{28}}$$

$$E_{cm_{28}} = 21{,}500\sqrt[3]{\frac{f_{cm_{28}}}{10}} = 21{,}500\sqrt[3]{\frac{45.2}{10}} = 35{,}548 \text{ MPa}$$

Calculation of $\phi(t, t_0)$

$$\alpha_1 = \left[\frac{35}{f_{cm_{28}}}\right]^{0.7} = \left[\frac{35}{45.2}\right]^{0.7} = 0.836$$

$$\alpha_2 = \left[\frac{35}{f_{cm_{28}}}\right]^{0.2} = \left[\frac{35}{45.2}\right]^{0.2} = 0.950$$

$$\phi_{RH} = \left[1 + \frac{1 - H/100}{0.16\sqrt[3]{h_e/4}}\alpha_1\right]\alpha_2 = \left[1 + \frac{1 - 75/100}{0.16\sqrt[3]{76/4}}0.836\right]0.950 = 1.419$$

$$\beta(f_{cm_{28}}) = \frac{5.3}{\sqrt{f_{cm_{28}}/10}} = \frac{5.3}{\sqrt{45.2/10}} = 2.49$$

$$t_0 = t_{0.T}\left[\frac{9}{2 + t_{0.T}^{1.2}} + 1\right]^{\alpha} = 28\left[\frac{9}{2 + 28^{1.2}} + 1\right] = 32.5 \ge 0.5 \text{ days}$$

$$\beta(t_0) = \frac{1}{0.1 + t_0^{0.2}} = \frac{1}{0.1 + 32.5^{0.2}} = 0.475$$

$$\phi_0 = \phi_{RH}\beta(f_{cm_{28}})\beta(t_0) = 1.415 \times 2.49 \times 0.475 = 1.674$$

$$\alpha_3 = \left[\frac{35}{f_{cm_{28}}}\right]^{0.5} = \left[\frac{35}{45.2}\right]^{0.5} = 0.880$$

$$\beta_H = 1.5h_e[1 + (0.012H)^{18}] + 250\alpha_3$$

$$= 1.5 \times 76 \times [1 + (0.012 \times 75)^{18}] + 250 \times 0.88 = 351 \le 1500 \times 0.880 = 1320$$

$$\beta_c(t, t_0) = \left(\frac{t - t_0}{\beta_H + t - t_0}\right)^{0.3} = \left(\frac{35 - 28}{351 + 35 - 28}\right)^{0.3} = 0.307$$

$$\phi(t, t_0) = \phi_0\beta_c(t, t_0) = 1.674 \times 0.307 = 0.514$$

$$J(t, t_0) = \frac{1}{E_{cmt_0}} + \frac{\phi(t, t_0)}{E_{cm_{28}}} = \frac{1}{35{,}548} + \frac{0.514}{35{,}548} = 42.6 \times 10^{-6}\frac{1}{\text{MPa}}$$

Example 2.6

Using the AASHTO model, calculate shrinkage strain and creep function for the specimen given in Example 2.1.

Solution

Shrinkage calculation

For moist-cured concrete, ε_{sh} should be taken as:

$$\varepsilon_{sh} = -K_s K_h \left(\frac{t}{35.0 + t}\right) 0.51 \times 10^{-3}$$

Determination of K_s:

$$V/S = 38 \text{ mm} = 1.5 \text{ in.}$$

$$K_s = \left[\frac{\dfrac{t}{26e^{0.36(V/S)} + t}}{\dfrac{t}{45 + t}}\right]\left[\frac{1064 - 94(V/S)}{923}\right]$$

$$K_s = \left[\frac{\dfrac{t}{26e^{0.36(1.5)} + t}}{\dfrac{t}{45 + t}}\right]\left[\frac{1064 - 94(1.5)}{923}\right] = 1$$

Determination of K_h:
For $H = 75\%$,

$$K_h = \left[\frac{140 - H}{70}\right] = \frac{140 - 75}{70} = 0.93$$

Calculation of ε_{sh}:

$$\varepsilon_{sh} = -1 \times 0.93 \times \left(\frac{35}{35.0 + 35}\right) 0.51 \times 10^{-3} = -237.15 \times 10^{-6} \text{ in/in.}$$

Creep calculation

The creep coefficient should be taken as:

$$\psi(t,t_0) = 3.5K_cK_f\left(1.58 - \frac{H}{120}\right)t_0^{-0.118}\frac{(t-t_0)^{0.6}}{10.0+(t-t_0)^{0.6}}$$

Determination of k_c:

$$k_c = \left[\frac{\dfrac{t}{26e^{0.36(V/S)}+t}}{\dfrac{t}{45+t}}\right]\left[\frac{1.80-1.77e^{-0.54(V/S)}}{2.587}\right]$$

$$k_c = \left[\frac{\dfrac{t}{26e^{0.36(1.5)}+t}}{\dfrac{t}{45+t}}\right]\left[\frac{1.80+1.77e^{-0.54\times 1.5}}{2.587}\right] = 1$$

Determination of k_f:

$$f_{cm_{28}} = 45.2 \text{ MPa} = 6.55 \text{ Ksi}$$

$$k_f = \left[\frac{1}{0.67+\dfrac{f_{cm_{28}}}{9}}\right] = \frac{1}{0.67+\dfrac{6.55}{9}} = 0.715$$

Calculation of $\psi\ (t,t_0)$:

$$\psi(t,t_0) = 3.5\times 1\times 0.715\left(1.58-\frac{75}{120}\right)\times 28^{-0.118}\times\frac{(35-28)^{0.6}}{10.0+(35-28)^{0.6}}$$

$$\psi(t,t_0) = 0.3923$$

Determination of E_c:

$$\omega_c = 2405 \text{ Kg/m}^3 = 0.15 \text{ Kcf}$$

$$E_c = 33000\omega_c^{1.5}\sqrt{f_c'}$$

$$E_c = 33000\times 0.15^{1.5}\sqrt{6.55} = 4906.5 \text{ Ksi}$$

Calculation of $J(t,t_0)$:

$$J(t,t_0) = \frac{1}{E_c}+\frac{\psi(t,t_0)}{E_c}$$

$$J(t,t_0) = \frac{1}{4906.5}+\frac{0.3923}{4906.5} = 284\times 10^{-6}\frac{1}{\text{Ksi}}$$

$$J(t,t_0) = 284\times 10^{-6}\frac{1}{\text{Ksi}} = 41.2\times 10^{-6}\frac{1}{\text{MPa}}$$

2.14 UNIT WEIGHT OF CONCRETE

The unit weight, w, of hardened normal concrete ordinarily used in buildings and similar structures depends on the concrete mix, maximum size and grading of aggregates, water–cement ratio, and strength of concrete. The following values of the unit weight of concrete may be used:

1. Unit weight of plain concrete using maximum aggregate size of $\frac{3}{4}$ in. (20 mm) varies between 145 and 150 lb/ft^3 (2320 to 2400 kg/m^3). For concrete of strength less than 4000 psi (280 kg/cm^2), a value of 145 lb/ft^3 (2320 kg/m^3) can be used, whereas for higher-strength concretes, w can be assumed to be equal to 150 lb/ft^3 (2400 kg/m^3).
2. Unit weight of plain concrete of maximum aggregate size of 4 to 6 in. (100 to 150 mm) varies between 150 and 160 lb/ft^3 (2400 to 2560 kg/m^3). An average value of 155 lb/ft^3 may be used.
3. Unit weight of reinforced concrete, using about 0.7% to 1.5% of steel in the concrete section, may be taken as 150 lb/ft^3 (2400 kg/m^3). For higher percentages of steel, the unit weight, w, can be assumed to be 155 lb/ft^3 (2500 kg/m^3).
4. Unit weight of lightweight concrete used for fireproofing, masonry, or insulation purposes varies between 20 and 90 lb/ft^3 (320 and 1440 kg/m^3). Concrete of upper values of 90 pcf or greater may be used for load-bearing concrete members.

The unit weight of heavy concrete varies between 200 and 270 lb/ft^3 (3200 and 4300 kg/m^3). Heavy concrete made with natural barite aggregate of $1\frac{1}{2}$ in. maximum size (38 mm) weighs about 225 lb/ft^3 (3600 kg/m^3). Iron ore sand and steel-punchings aggregate produce a unit weight of 270 lb/ft^3 (4320 kg/m^3). [18].

2.15 FIRE RESISTANCE

Fire resistance of a material is its ability to resist fire for a certain time without serious loss of strength, distortion, or collapse [19]. In the case of concrete, fire resistance depends on the thickness, type of construction, type and size of aggregates, and cement content. It is important to consider the effect of fire on tall buildings more than on low or single-story buildings, because occupants need more time to escape.

Reinforced concrete is a much better fire-resistant material than steel. Steelwork heats rapidly, and its strength drops appreciably in a short time. Concrete itself has low thermal conductivity. The effect of temperatures below 250°C is small on concrete, but definite loss is expected at higher temperatures.

2.16 HIGH-PERFORMANCE CONCRETE

High-performance concrete may be assumed to imply that the concrete exhibits combined properties of strength, toughness, energy absorption, durability, stiffness, and a relatively higher ductility than normal concrete. This improvement in concrete quality may be achieved by using a new generation of additives and superplasticizers, which improves the workability of concrete and, consequently, its strength. Also, the use of active microfillers such as silica fume, fly ash, and polymer improves the strength, porosity, and durability of concrete. The addition of different

Casting and finishing precast concrete wall panels.

types of fiber to the concrete mix enhances many of its properties, including ductility, strength, toughness, and many other properties.

Because it is difficult to set a limit to measure high-performance concrete, one approach is to define a lower-bound limit based on the shape of its stress-strain response in tension [20]. If the stress-strain relationship curve shows a quasi strain-hardening behavior—or, in other words, a postcracking strength larger than the cracking strength with an elastic-plastic behavior—then high performance is achieved [20]. In this behavior, multicracking stage is reached with high energy-absorption capacity. Substantial progress has been made recently in understanding the behavior and practical application of high-performance concrete.

2.17 LIGHTWEIGHT CONCRETE

Lightweight concrete is a concrete that has been made lighter than conventional normal-weight concrete and, consequently, it has a relatively lower density. Basically, reducing the density requires the inclusion of air in the concrete composition. This, however, can be achieved in four distinct ways:

1. By omitting the finer sizes from the aggregate grading, thereby creating what is called *no-fines* concrete. It is a mixture of cement, water, and coarse aggregate only ($\frac{3}{4}-\frac{3}{8}$), mixed to produce concrete with many uniformly distributed voids.
2. By replacing the gravel or crushed rock aggregate with a hollow cellular or porous aggregate, which includes air in the mix. This type is called *lightweight aggregate concrete*. Lightweight aggregate may be natural, such as pumice, pozzolans, and volcanic slags; artificial (from industrial by-products), such as furnace clinker and foamed slag; or industrially produced, such as perlite, vermiculite, expanded clay, shale, and slate.
3. By creating gas bubbles in a cement slurry, which, when it sets, leaves a spongelike structure. This type is called *aerated concrete*.
4. By forming air cells in the slurry by chemical reaction or by vigorous mixing of the slurry with a preformed stable foam, which is produced by using special foam concentrate in a high-speed mixer. This type is called *cellular concrete*.

Structural lightweight concrete has a unit weight that ranges from 90 to 115 lb/ft^3, compared with 145 lb/ft^3 for normal-weight concrete. It is used in the design of floor slabs in buildings and other structural members where high-strength concrete is not required. Structural lightweight concrete can be produced with a compressive strength of 2500 to 5000 psi for practical applications.

2.18 FIBROUS CONCRETE

Fibrous concrete is made primarily of concrete constituents and discrete reinforcing fibers. The brittle nature of concrete and its low flexural tensile strength are major reasons for the growing interest in the performance of fibers in concrete technology. Various types of fibers—mainly steel, glass, and organic polymers—have been used in fibrous concrete. Generally, the length and diameter of the fibers do not exceed 3 in. (75 mm) and 0.04 in. (1 mm), respectively. The addition of fibers to concrete improves its mechanical properties, such as ductility, toughness, shear, flexural strength, impact resistance, and crack control. A convenient numerical parameter describing a fiber is its aspect ratio, which is the fiber length divided by an equivalent fiber diameter. Typical aspect ratios range from about 30 to 150, with the most common ratio being about 100. More details on fibrous concrete are given in [21].

2.19 STEEL REINFORCEMENT

Reinforcement, usually in the form of steel bars, is placed in the concrete member, mainly in the tension zone, to resist the tensile forces resulting from external load on the member. Reinforcement is also used to increase the member's compression resistance. Steel costs more than concrete, but it has a yield strength about 10 times the compressive strength of concrete. The function and behavior of both steel and concrete in a reinforced concrete member are discussed in Chapter 3.

Longitudinal bars taking either tensile or compression forces in a concrete member are called *main reinforcement*. Additional reinforcement in slabs, in a direction perpendicular to the main reinforcement, is called *secondary*, or *distribution, reinforcement*. In reinforced concrete beams, another type of steel reinforcement is used, transverse to the direction of the main steel and bent in a box or U shape. These are called *stirrups*. Similar reinforcements are used in columns, where they are called *ties*. Refer to Figure 8.8 and Figure 10.3.

2.19.1 Types of Steel Reinforcement

Different types of steel reinforcement are used in various reinforced concrete members. These types can be classified as follows:

Round bars. Round bars are used most widely for reinforced concrete. Round bars are available in a large range of diameters, from $\frac{1}{4}$ (6 mm) to $1\frac{3}{8}$ (36 mm), plus two special types, $1\frac{3}{4}$ (45 mm) and $2\frac{1}{4}$ (57 mm). Round bars, depending on their surfaces, are either plain or deformed bars. Plain bars are used mainly for secondary reinforcement or in stirrups and ties. Deformed bars have projections or deformations on the surface for the purpose of improving the bond with concrete and reducing the width of cracks opening in the tension zone.

The diameter of a plain bar can be measured easily, but for a deformed bar, a nominal diameter is used that is the diameter of a circular surface with the same area as the section of the deformed bar. Requirements of surface projections on bars are specified by ASTM Specification A 305, or A 615. The bar sizes are designated by numbers 3 through 11, corresponding to the diameter in one-eighths of an inch. For instance, a no. 7 bar has a nominal diameter of $\frac{7}{8}$ in. and a no. 4 bar has a nominal diameter of $\frac{1}{2}$ in. The two largest sizes are designated no. 14 and no. 18, respectively. American standard bar marks are shown on the steel reinforcement to indicate the initial of the producing mill, the bar size, and the type of steel (Fig. 2.6). The grade of the reinforcement is indicated on the bars by either the continuous-line system or the number system. In the first system, one longitudinal line is added to the bar, in addition to the main ribs, to indicate the high-strength grade of 60 ksi (420 N/mm^2), according to ASTM Specification A 617. If only the main ribs are shown on the bar, without any additional lines, the steel is of the ordinary grade according to ASTM A 615 for the structural grade ($f_y = 40$ ksi, or 280 N/mm^2). In the number system, the yield strength of the high-strength grades is marked clearly on every bar. For ordinary grades, no strength marks are indicated. The two types are shown in Fig. 2.6.

Welded fabrics and mats. Welded fabrics and mats consist of a series of longitudinal and transverse cold-drawn steel wires, generally at right angles and welded together at all points of intersection. Steel reinforcement may be built up into three-dimensional cages before being placed in the forms.

Prestressed concrete wires and strands. Prestressed concrete wires and strands use special high-strength steel (see Chapter 20). High-tensile steel wires of diameters 0.192 in. (5 mm) and 0.276 in. (7 mm) are used to form the prestressing cables by winding six steel wires around a seventh wire of slightly larger diameter. The ultimate strength of prestressed strands is 250 ksi or 270 ksi.

2.19.2 Grades and Strength

Different grades of steel are used in reinforced concrete. Limitations on the minimum yield strength, ultimate strength, and elongation are explained in ASTM specifications for reinforcing steel bars (Table 2.17). The properties and grades of metric reinforcing steel are shown in Tables 2.18 and 2.19.

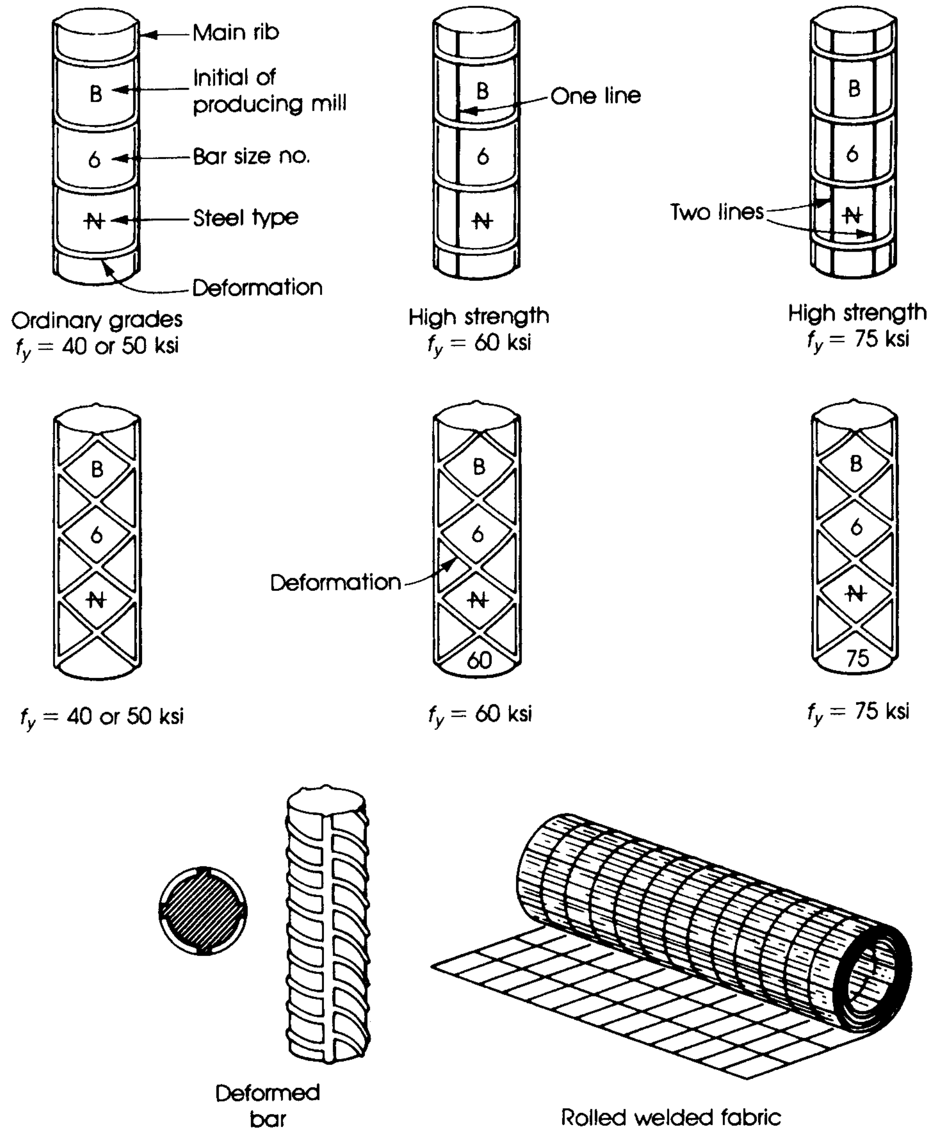


Figure 2.6 Some types of deformed bars and American standard bar marks.

2.19.3 Stress–Strain Curves

The most important factor affecting the mechanical properties and stress–strain curve of the steel is its chemical composition. The introduction of carbon and alloying additives in steel increases its strength but reduces its ductility. Commercial steel rarely contains more than 1.2% carbon; the proportion of carbon used in structural steels varies between 0.2% and 0.3%.

Two other properties are of interest in the design of reinforced concrete structures; the first is the modulus of elasticity, E_s. It has been shown that the modulus of elasticity is constant for all types of steel. The ACI Code has adopted a value of $E_s = 29 \times 10^6$ psi (2.0×10^5 MPa).

Table 2.17 Grade of ASTM Reinforcing Steel Bars

	Minimum Yield Strength f_y		Ultimate Strength f_{su}	
Steel	**ksi**	**MPa**	**ksi**	**MPa**
Billet steel				
Grade 40	40	276	70	483
60	60	414	90	621
75	75	518	100	690
Rail steel				
Grade 50	50	345	80	551
60	60	414	90	621
Deformed wire				
Reinforcing	75	518	85	586
Fabric	70	483	80	551
Cold-drawn wire				
Reinforcing	70	483	80	551
Fabric	65	448	75	518
Fabric	56	386	70	483

Table 2.18 ASTM 615 M (Metric) for Reinforcing Steel Bars

Bar No.	Diameter (mm)	Area (mm^2)	Weight (kg/m)
10 M	11.3	100	0.785
15 M	16.0	200	1.570
20 M	19.5	300	2.355
25 M	25.2	500	3.925
30 M	29.9	700	5.495
35 M	35.7	1000	7.850
45 M	43.7	1500	11.770
55 M	56.4	2500	19.600

Table 2.19 ASTM Metric Specifications

		Grade	
ASTM	**Bar size no.**	**MPa**	**ksi**
A615 M	10, 15, 20	300	43.5
Billet steel	10–55	400	58.0
	35, 45, 55	500	72.5
A616 M	10–35	350	50.75
Rail steel	10–35	400	58.0
A617 M	10–35	300	43.5
Axle steel	10–35	400	58.0
A706	10–55	400	58.0
Low alloy			

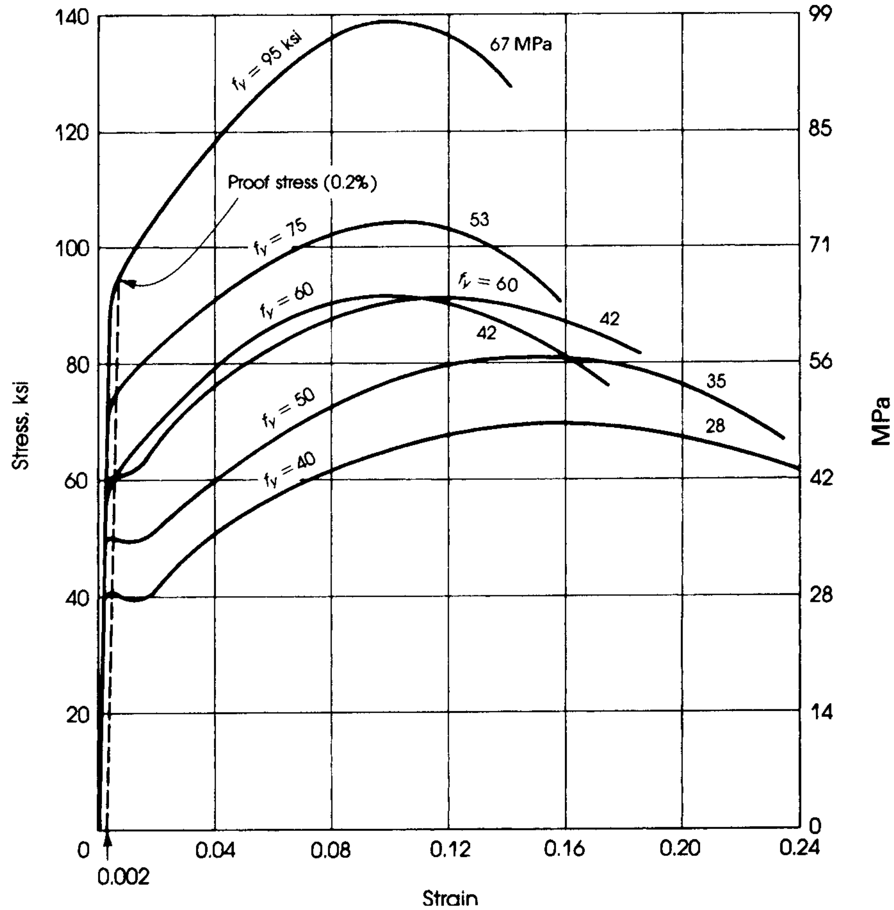


Figure 2.7 Typical stress-strain curves for some reinforcing steel bars of different grades. Note that 60-ksi steel may or may not show a definite yield point.

The modulus of elasticity is the slope of the stress-strain curve in the elastic range up to the proportional limit; E_s = stress/strain. Second is the yield strength, f_y. Typical stress—strain curves for some steel bars are shown in Fig. 2.7. In high-tensile steel, a definite yield point may not show on the stress-strain curve. In this case, ultimate strength is reached gradually under an increase of stress (Fig. 2.7). The yield strength or proof stress is considered the stress that leaves a residual strain of 0.2% on the release of load, or a total strain of 0.5% to 0.6% under load.

SUMMARY

Section 2.1

The main factors that affect the strength of concrete are the water—cement ratio, properties and proportions of materials, age of concrete, loading conditions, and shape of tested specimen.

$$f'_c \text{ (cylinder)} = 0.85 f'_c \text{ (cube)} = 1.10 f'_c \text{ (prism)}$$

Sections 2.2–2.6

1. The usual specimen used to determine the compressive strength of concrete at 28 days is a 6- by 12-in. (150- by 300-mm) cylinder. Compressive strength between 3000 and 6000 psi is usually specified for reinforced concrete structures. Maximum stress, f'_c, is reached at an estimated strain of 0.002, whereas rupture occurs at a strain of about 0.003.
2. Tensile strength of concrete is measured indirectly by a splitting test performed on a standard cylinder using formula $f'_{sp} = 2P/\pi LD$. Tensile strength of concrete is approximately $0.1 f'_c$.
3. Flexural strength (modulus of rupture, f_r) of concrete is calculated by testing a 6- by 6- by 28-in. plain concrete beam, $f_r = 7.5\lambda\sqrt{f'_c}$ (psi), where λ is a modification factor related to unit weight of concrete.
4. Nominal shear stress is $2\lambda\sqrt{f'_c}$ (psi).

Sections 2.7–2.9

The modulus of elasticity of concrete, E_c for unit weight w between 90 and 160 pcf, is $E_c = 33w^{1.5}\sqrt{f'_c}$ (psi) $= 0.043w^{1.5}\sqrt{f'_c}$ MPa.

For normal-weight concrete, $w = 145$ pcf.

$$E_c = 57{,}600\sqrt{f'_c} \quad \text{or} \quad E_c = 57{,}000\sqrt{f'_c} = 4700\sqrt{f'_c}\ \text{MPa}$$

The shear modulus of concrete is $G_c = E_c/2(1 + \mu) = 0.43\ E_c$ for a Poisson's ratio $\mu = \frac{1}{6}$. Poisson's ratio, μ, varies between 0.15 and 0.20, with an average value of 0.18.

Section 2.10

Modular ratio is $n = E_s/E_c = 500/\sqrt{f'_c}$, where f'_c is in psi.

Section 2.11

1. Values of shrinkage for normal concrete fall between 200×10^{-6} and 700×10^{-6}. An average value of 300×10^{-6} may be used.
2. The coefficient of expansion of concrete falls between 4×10^{-6} and 7×10^{-6}/°F.

Section 2.12–2.13

The ultimate magnitude of creep varies between 0.2×10^{-6} and 2×10^{-6} per unit stress per unit length. An average value of 1×10^{-6} may be adopted in practical problems. Of the ultimate (20-year) creep, 18% to 35% occurs in 2 weeks, 30% to 70% occurs in 3 months, and 64% to 83% occurs in 1 year.

Section 2.14

The unit weight of normal concrete is 145 pcf for plain concrete and 150 pcf for reinforced concrete.

Section 2.15

Reinforced concrete is a much better fire-resistant material than steel. Concrete itself has a low thermal conductivity. An increase in concrete cover in structural members such as walls, columns, beams, and floor slabs will increase the fire resistance of these members.

Sections 2.16–2.18

1. High-performance concrete implies that concrete exhibits properties of strength, toughness, energy absorption, durability, stiffness, and ductility higher than normal concrete.
2. Concrete is made lighter than normal-weight concrete by inclusion of air in the concrete composition. Types of lightweight concrete are no-fines concrete, lightweight aggregate concrete, aerated concrete, and cellular concrete.
3. Fibrous concrete is made of concrete constituents and discrete reinforcing fibers such as steel, glass, and organic polymers.

Section 2.19

The grade of steel mainly used is grade 60 ($f_y = 60$ ksi). The modulus of elasticity of steel is $E_s = 29 \times 10^6$ psi (2×10^{-5} MPa).

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PROBLEMS

2.1 Explain the modulus of elasticity of concrete in compression and the shear modulus.

2.2 Determine the modulus of elasticity of concrete by the ACI formula for a concrete cylinder that has a unit weight of 120 pcf (1920 kg/m^3) and a compressive strength of 3000 psi (21 MPa).

2.3 Estimate the modulus of elasticity and the shear modulus of a concrete specimen with a dry density of 150 pcf (2400 kg/m^3) and compressive strength of 4500 psi (31 MPa) using Poisson's ratio, $\mu = 0.18$.

2.4 What is meant by the modular ratio and Poisson's ratio? Give approximate values for concrete.

2.5 What factors influence the shrinkage of concrete?

2.6 What factors influence the creep of concrete?

2.7 What are the types and grades of the steel reinforcement used in reinforced concrete?

2.8 On the stress-strain diagram of a steel bar, show and explain the following: proportional limit, yield stress, ultimate stress, yield strain, and modulus of elasticity.

2.9 Calculate the modulus of elasticity of concrete, E_c, for the following types of concrete:

$$E_c = 33W^{1.5}\sqrt{f'_c} \text{ (ft)},$$

$$E_c = 0.043W^{1.5}\sqrt{f'_c} \text{ (SI)}$$

Density	**Strength f'_c**
160 pcf	5000 psi
145 pcf	4000 psi
125 pcf	2500 psi
2400 kg/m^3	35 MPa
2300 kg/m^3	30 MPa
2100 kg/m^3	25 MPa

2.10 Determine the modular ratio, n, and the modulus of rupture for each case of Problem 2.9. Tabulate your results.

$$f_r = 7.5\lambda\sqrt{f'_c} \text{ (psi)} \quad f_r = 0.62\lambda\sqrt{f'_c} \text{ (MPa)}$$

2.11 A standard normal 6 × 12-in. concrete cylinder was tested to failure, and the following loads and strains were recorded.

Load, kips	Strain x 10^{-4}	Load, kips	Strain x 10^{-4}
0.0	0.0	72	10.0
12	1.2	84	13.6
24	2.0	96	18.0
36	3.2	108	30.0
48	5.2	95	39.0
60	7.2	82	42.0

a. Draw the stress–strain diagram of concrete and determine the maximum stress and corresponding strain.

b. Determine the initial modulus and secant modulus.

c. Calculate the modulus of elasticity of concrete using the ACI formula for normal-weight concrete and compare results.

$$E_c = 57{,}000\sqrt{f_c'} \text{ psi}$$

$$E_c = 4730\sqrt{f_c'} \text{ MPa}$$

CHAPTER 3

FLEXURAL ANALYSIS OF REINFORCED CONCRETE BEAMS

Apartment building, Fort Lauderdale, Florida.

3.1 INTRODUCTION

The analysis and design of a structural member may be regarded as the process of selecting the proper materials and determining the member dimensions such that the design strength is equal or greater than the required strength. The required strength is determined by multiplying the actual applied loads, the dead load, the assumed live load, and other loads, such as wind, seismic, earth pressure, fluid pressure, snow, and rain loads, by load factors. These loads develop external forces such as bending moments, shear, torsion, or axial forces depending on how these loads are applied to the structure.

In proportioning reinforced concrete structural members, three main items can be investigated:

1. The safety of the structure, which is maintained by providing adequate internal design strength.
2. Deflection of the structural member under service loads. The maximum value of deflection must be limited and is usually specified as a factor of the span, to preserve the appearance of the structure.
3. Control of cracking conditions under service loads. Visible cracks spoil the appearance of the structure and also permit humidity to penetrate the concrete, causing corrosion of steel and consequently weakening the reinforced concrete member. The ACI Code implicitly limits crack widths to 0.016 in. (0.40 mm) for interior members and 0.013 in. (0.33 mm) for exterior members. Control of cracking is achieved by adopting and limiting the spacing of the tension bars (see Chapter 6).

It is worth mentioning that the strength design approach was first permitted in the United States in 1956 and in Britain in 1957. The latest ACI Code emphasizes the strength concept based on specified strain limits on steel and concrete that develop tension-controlled, compression-controlled, or transition conditions.

3.2 ASSUMPTIONS

Reinforced concrete sections are heterogeneous (nonhomogeneous), because they are made of two different materials, concrete and steel. Therefore, proportioning structural members by ultimate-strength design is based on the following assumptions:

1. Strain in concrete is the same as in reinforcing bars at the same level, provided that the bond between the steel and concrete is adequate.
2. Strain in concrete is linearly proportional to the distance from the neutral axis.
3. The modulus of elasticity of all grades of steel is taken as $E_s = 29 \times 10^6$ lb/in.^{2} (200,000 MPa or N/mm^2). The stress in the elastic range is equal to the strain multiplied by E_s.
4. Plane cross-sections continue to be plane after bending.
5. Tensile strength of concrete is neglected because (1) concrete's tensile strength is about 10% of its compressive strength, (2) cracked concrete is assumed to be not effective, and (3) before cracking, the entire concrete section is effective in resisting the external moment.
6. The method of elastic analysis, assuming an ideal behavior at all levels of stress, is not valid. At high stresses, nonelastic behavior is assumed, which is in close agreement with the actual behavior of concrete and steel.
7. At failure the maximum strain at the extreme compression fibers is assumed equal to 0.003 by the ACI Code provision.
8. For design strength, the shape of the compressive concrete stress distribution may be assumed to be rectangular, parabolic, or trapezoidal. In this text, a rectangular shape will be assumed (ACI Code, Section 10.2).

3.3 BEHAVIOR OF A SIMPLY SUPPORTED REINFORCED CONCRETE BEAM LOADED TO FAILURE

Concrete being weakest in tension, a concrete beam under an assumed working load will definitely crack at the tension side, and the beam will collapse if tensile reinforcement is not provided. Concrete cracks occur at a loading stage when its maximum tensile stress reaches the modulus of rupture of concrete. Therefore, steel bars are used to increase the moment capacity of the beam; the steel bars resist the tensile force, and the concrete resists the compressive force.

To study the behavior of a reinforced concrete beam under increasing load, let us examine how two beams were tested to failure. Details of the beams are shown in Fig. 3.1. Both beams had a section of 4.5 in. by 8 in. (110 mm by 200 mm), reinforced only on the tension side by two no. 5 bars. They were made of the same concrete mix. Beam 1 had no stirrups, whereas beam 2 was provided with no. 3 stirrups spaced at 3 in. The loading system and testing procedure were the same for both beams. To determine the compressive strength of the concrete and its modulus of elasticity, E_c, a standard concrete cylinder was tested, and strain was measured at different load increments. The following observations were noted at different distinguishable stages of loading.

Stage 1. At zero external load, each beam carried its own weight in addition to that of the loading system, which consisted of an I-beam and some plates. Both beams behaved similarly at this stage. At any section, the entire concrete section, in addition to the steel reinforcement, resisted

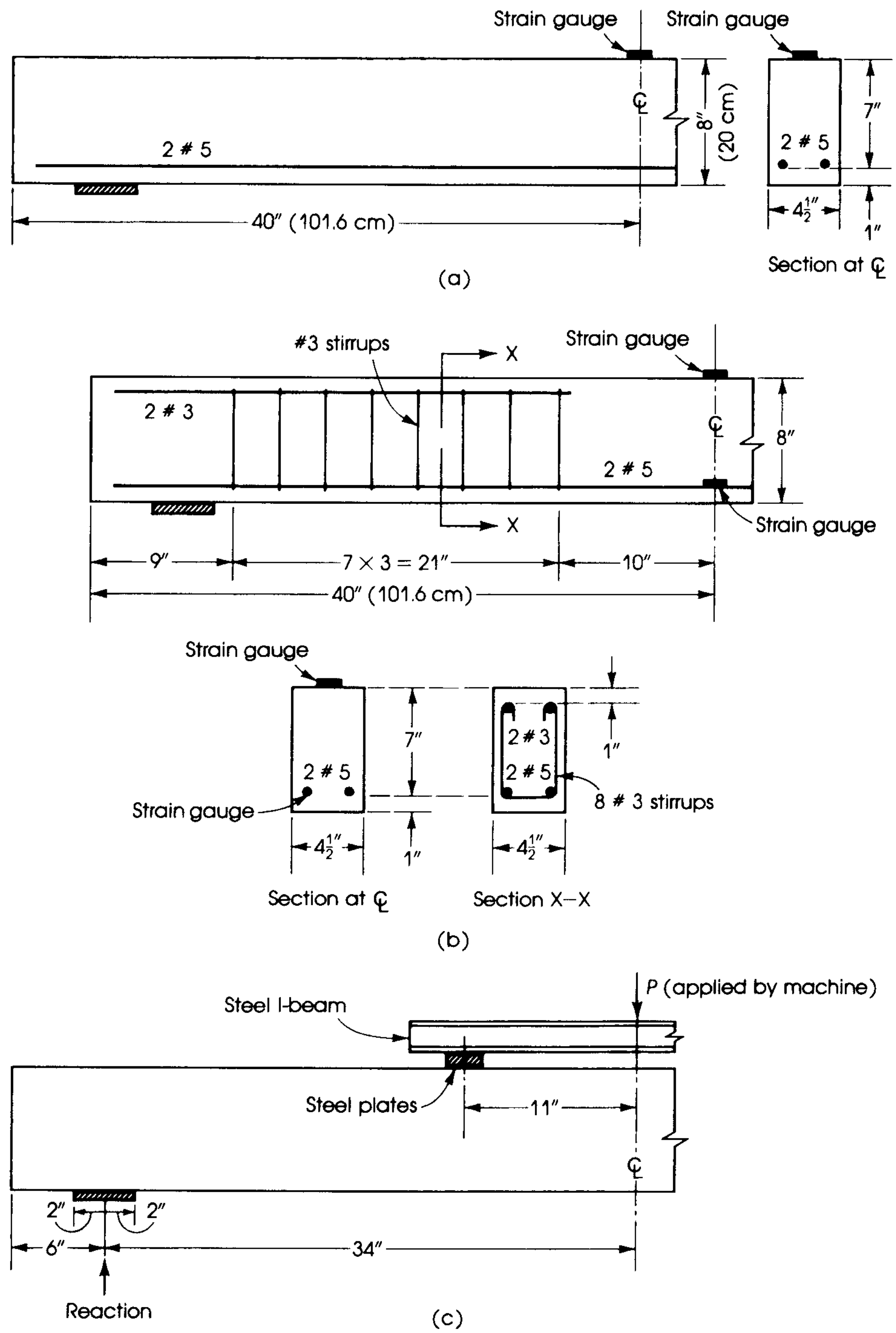
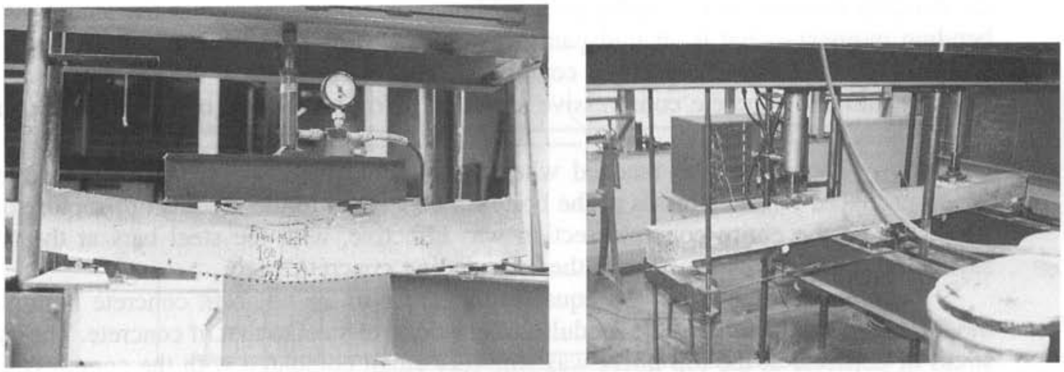
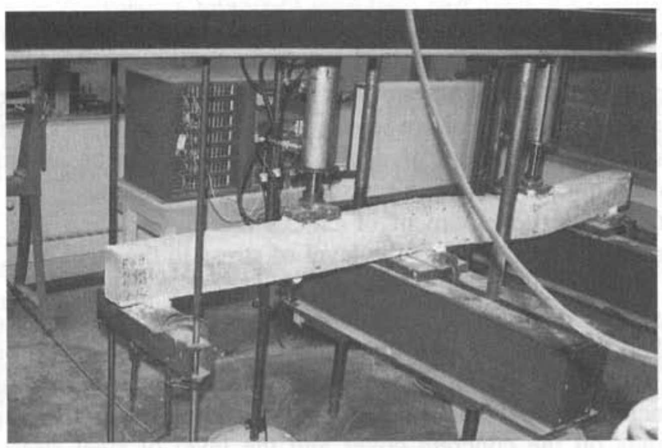


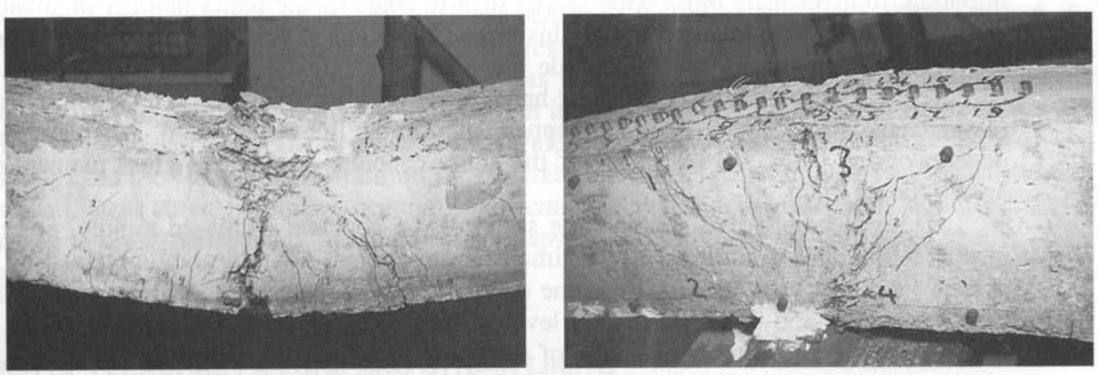
Figure 3.1 Details of tested beams: (*a*) beam 1, (*b*) beam 2, and (*c*) loading system. All beams are symmetrical about the centerline.



Test on a simply supported beam and a two-span continuous beam loaded to failure.



Two-span continuous reinforced concrete beam loaded to failure.



Failure conditions at the positive- and negative-moment sections in a continuous reinforced concrete beam.

the bending moment and shearing forces. Maximum stress occurred at the section of maximum bending moment—that is, at midspan. Maximum tension stress at the bottom fibers was much less than the modulus of rupture of concrete. Compressive stress at the top fibers was much less than the ultimate concrete compressive stress, f_c'. No cracks were observed at this stage.

Stage 2. This stage was reached when the external load, P, was increased from 0 to P_1, which produced tensile stresses at the bottom fibers equal to the modulus of rupture of concrete. At this stage the entire concrete section was effective, with the steel bars at the tension side sustaining a strain equal to that of the surrounding concrete.

Stress in the steel bars was equal to the stress in the adjacent concrete multiplied by the modular ratio, n, the ratio of the modulus of elasticity of steel to that of concrete. The compressive stress of concrete at the top fibers was still very small compared with the compressive strength, f_c'. The behavior of beams was elastic within this stage of loading.

Stage 3. When the load was increased beyond P_1, tensile stresses in concrete at the tension zone increased until they were greater than the modulus of rupture, f_r, and cracks developed. The neutral axis shifted upward, and cracks extended close to the level of the shifted neutral axis. Concrete in the tension zone lost its tensile strength, and the steel bars started to work effectively and to resist the entire tensile force. Between cracks, the concrete bottom fibers had tensile stresses, but they were of negligible value. It can be assumed that concrete below the neutral axis did not participate in resisting external moments.

In general, the development of cracks and the spacing and maximum width of cracks depend on many factors, such as the level of stress in the steel bars, distribution of steel bars in the section, concrete cover, and grade of steel used.

At this stage, the deflection of the beams increased clearly, because the moment of inertia of the cracked section was less than that of the uncracked section. Cracks started about the midspan of the beam, but other parts along the length of the beam did not crack. When load was again increased, new cracks developed, extending toward the supports. The spacing of these cracks depends on the concrete cover and the level of steel stress. The width of cracks also increased. One or two of the central cracks were most affected by the load, and their crack widths increased appreciably, whereas the other crack widths increased much less. It is more important to investigate those wide cracks than to consider the larger number of small cracks.

If the load were released within this stage of loading, it would be observed that permanent fine cracks of no significant magnitude were left. On reloading, cracks would open quickly, because the tensile strength of concrete had already been lost. Therefore, it can be stated that the second stage, once passed, does not happen again in the life of the beam. When cracks develop under working loads, the resistance of the entire concrete section and gross moment of inertia are no longer valid.

At high compressive stresses, the strain of the concrete increased rapidly, and the stress of concrete at any strain level was estimated from a stress—strain graph obtained by testing a standard cylinder to failure for the same concrete. As for the steel, the stresses were still below the yield stress, and the stress at any level of strain was obtained by multiplying the strain of steel, ε_s, by the modulus of elasticity of steel, E_s.

Stage 4. In beam 1, at a load value of 9500 lb (42.75 kN), shear stress at a distance of about the depth of the beam from the support increased and caused diagonal cracks at approximately 45° from horizontal in the direction of principal stresses resulting from the combined action of bending moment and shearing force. The diagonal crack extended downward to the level of

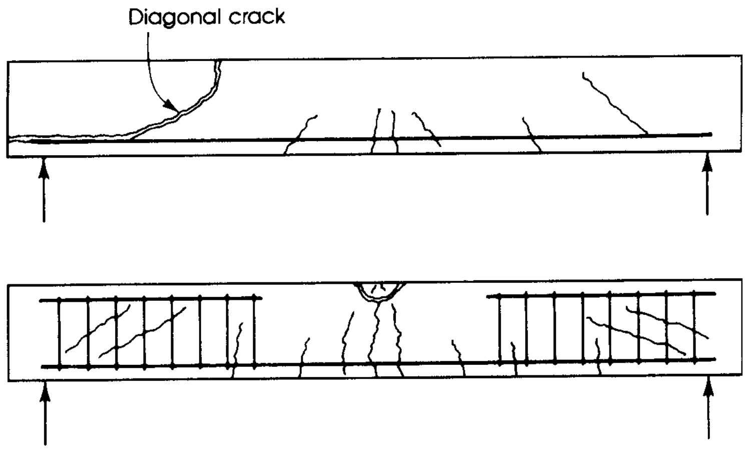


Figure 3.2 Shape of beam 1 at shear failure (*top*) and beam 2 at bending moment failure (*bottom*).

the steel bars and then extended horizontally at that level toward the support. When the crack, which had been widening gradually, reached the end of the beam, a concrete piece broke off and failure occurred suddenly (Fig. 3.2). The failure load was 13,600 lb (61.2 kN). Stresses in concrete and steel at the midspan section did not reach their failure stresses. (The shear behavior of beams is discussed in Chapter 8.)

In beam 2, at a load of 11,000 lb (49.5 kN), a diagonal crack developed similar to that of beam 1; then other parallel diagonal cracks appeared, and the stirrups started to take an effective part in resisting the principal stresses. Cracks did not extend along the horizontal main steel bars, as in beam 1. On increasing the load, diagonal cracks on the other end of the beam developed at a load of 13,250 lb (59.6 kN). Failure did not occur at this stage because of the presence of stirrups.

Stage 5. When the load on beam 2 was further increased, strains increased rapidly until the maximum carrying capacity of the beam was reached at ultimate load, $P_u = 16{,}200$ lb (72.9 kN).

In beam 2, the amount of steel reinforcement used was relatively small. When reached, the yield strain can be considered equal to yield stress divided by the modulus of elasticity of steel, $\varepsilon_y = f_y/E_s$; the strain in the concrete, ε_c, was less than the strain at maximum compressive stress, f'_c. The steel bars yielded, and the strain in steel increased to about 12 times that of the yield strain without increase in load. Cracks widened sharply, deflection of the beam increased greatly, and the compressive strain on the concrete increased. After another very small increase of load, steel strain hardening occurred, and concrete reached its maximum strain, ε'_c, and it started to crush under load; then the beam collapsed. Figure 3.2 shows the failure shapes of the two beams.

3.4 TYPES OF FLEXURAL FAILURE AND STRAIN LIMITS

3.4.1 Flexural Failure

Three types of flexural failure of a structural member can be expected depending on the percentage of steel used in the section.

1. Steel may reach its yield strength before the concrete reaches its maximum strength, Fig. 3.3a. In this case, the failure is due to the yielding of steel reaching a high strain equal to or greater than 0.005. The section contains a relatively small amount of steel and is called a tension-controlled section.
2. Steel may reach its yield strength at the same time as concrete reaches its ultimate strength, Fig. 3.3*b*. The section is called a balanced section.
3. Concrete may fail before the yield of steel, Fig. 3.3*c*, due to the presence of a high percentage of steel in the section. In this case, the concrete strength and its maximum strain of 0.003 are reached, but the steel stress is less than the yield strength, that is, f_s is less than f_y. The strain in the steel is equal to or less than 0.002. This section is called a compression-controlled section.

It can be assumed that concrete fails in compression when the concrete strain reaches 0.003. A range of 0.0025 to 0.004 has been obtained from tests and the ACI Code assumes a strain of 0.003.

In beams designed as tension-controlled sections, steel yields before the crushing of concrete. Cracks widen extensively, giving warning before the concrete crushes and the structure collapses. The ACI Code adopts this type of design. In beams designed as balanced or compression-controlled sections, the concrete fails suddenly, and the beam collapses immediately without warning. The ACI Code does not allow this type of design.

3.4.2 Strain Limits for Tension and Tension-Controlled Sections

The design provisions for both reinforced and prestressed concrete members are based on the concept of tension or compression-controlled sections, ACI Code, Section 10.3. Both are defined in terms of net tensile strain (NTS), (ε_t, in the extreme tension steel at nominal strength, exclusive of prestress strain. Moreover, two other conditions may develop: (1) the balanced strain condition and (2) the transition region condition. These four conditions are defined as follows:

1. Compression-controlled sections are those sections in which the net tensile strain, NTS, in the extreme tension steel at nominal strength is equal to or less than the compression-controlled strain limit at the time when concrete in compression reaches its assumed strain limit of 0.003, ($\varepsilon_c = 0.003$). For grade 60 steel, ($f_y = 60$ ksi), the compression-controlled strain limit may be taken as a net strain of 0.002, Fig. 3.4*a*. This case occurs mainly in columns subjected to axial forces and moments.
2. Tension-controlled sections are those sections in which the NTS, ε_t, is equal to or greater than 0.005 just as the concrete in the compression reaches its assumed strain limit of 0.003, Fig. 3.4*c*.
3. Sections in which the NTS in the extreme tension steel lies between the compression-controlled strain limit (0.002 for $f_y = 60$ ksi) and the tension-controlled strain limit of 0.005 constitute the transition region, Fig. 3.4*b*.
4. The balanced strain condition develops in the section when the tension steel, with the first yield, reaches a strain corresponding to its yield strength, f_y or $\varepsilon_s = f_y/E_s$, just as the maximum strain in concrete at the extreme compression fibers reaches 0.003, Fig. 3.5.

In addition to the above four conditions, Section 10.3.5 of the ACI Code indicates that the net tensile strain, ε_t, at nominal strength, within the transition region, shall not be less than 0.004 for reinforced concrete flexural members without or with an axial load less than 0.10 $f'_c A_g$, where A_g = gross area of the concrete section.

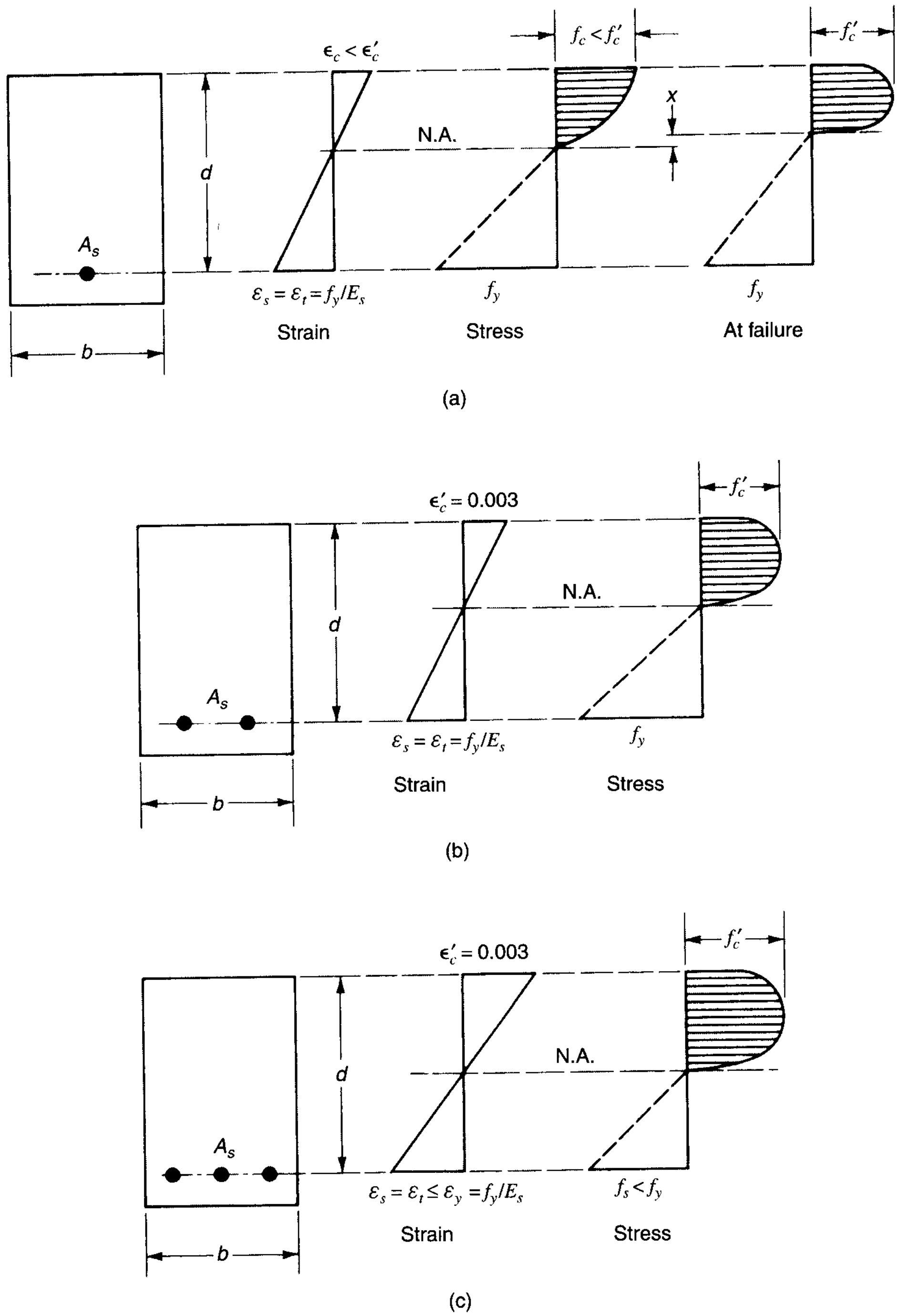


Figure 3.3 Stress and strain diagrams for (*a*) tension-controlled, (*b*) balanced, and (*c*) compression-controlled sections.

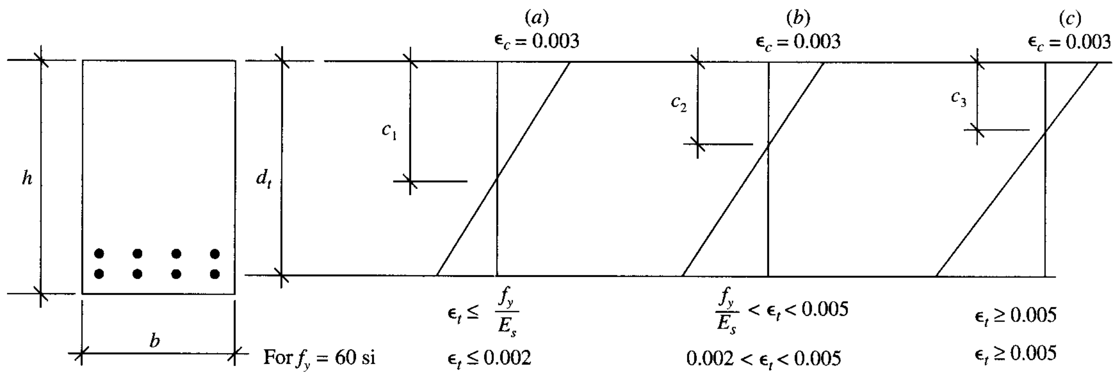


Figure 3.4 Strain limit distribution, $c_1 > c_2 > c_3$: (*a*) compression-controlled section, (*b*) transition region, and (*c*) tension-controlled section.

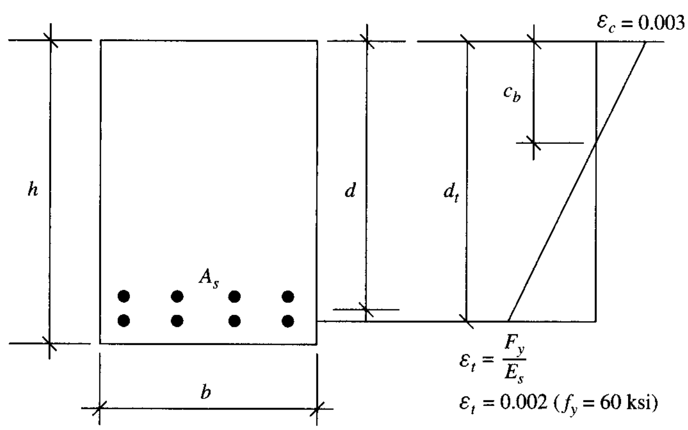


Figure 3.5 Balanced strain section (occurs at first yield or at distance d_t).

Note that d_t in Fig. 3.4, is the distance from the extreme concrete compression fiber to the extreme tension steel, while the effective depth, d, equals the distance from the extreme concrete compression fiber to the centroid of the tension reinforcement, Fig. 3.5. These cases are summarized in Table 3.1.

Table 3.1 Strain Limits of Fig. 3.4

Section Condition	Concrete Strain	Steel Strain	Notes (f_y = 60 ksi)
Compression-controlled	0.003	$\varepsilon_t \leq f_y/E_s$	$\varepsilon_t \leq 0.002$
Tension-controlled	0.003	$\varepsilon_t \geq 0.005$	$\varepsilon_t \geq 0.005$
Transition region	0.003	$f_y/E_s < \varepsilon_t < 0.005$	$0.002 < \varepsilon_t < 0.005$
Balanced strain	0.003	$\varepsilon_s = f_y/E_s$	$\varepsilon_s = 0.002$
Transition region (flexure)	0.003	$0.004 \leq \varepsilon_t < 0.005$	$0.004 \leq \varepsilon_t < 0.005$

3.5 LOAD FACTORS

The types of loads and the safety provisions were explained earlier in Sections 1.7 and 1.8.

For the design of structural members, the factored design load is obtained by multiplying the dead load by a load factor and the specified live load by another load factor. The magnitude of the load factor must be adequate to limit the probability of sudden failure and to permit an economical structural design. The choice of a proper load factor or, in general, a proper factor of safety depends mainly on the importance of the structure (whether a courthouse or a warehouse), the degree of warning needed prior to collapse, the importance of each structural member (whether a beam or column), the expectation of overload, the accuracy of artisanry, and the accuracy of calculations.

Based on historical studies of various structures, experience, and the principles of probability, the ACI Code adopts a load factor of 1.2 for dead loads and 1.6 for live loads. The dead load factor is smaller, because the dead load can be computed with a greater degree of certainty than the live load. Moreover, the choice of factors reflects the degree of the economical design as well as the degree of safety and serviceability of the structure. It is also based on the fact that the performance of the structure under actual loads must be satisfactorily within specific limits.

If the required strength is denoted by U (ACI Code, Section 9.2), and those due to wind and seismic forces are W and E, respectively, according to the ACI Code, the required strength U, shall be the most critical of the following factors (based on the ASCE 7-05):

1. In the case of dead, live, and wind loads,

$$U = 1.4D \tag{3.1a}$$

$$U = 1.2D + 1.6L \tag{3.1b}$$

$$U = 1.2D + 1.0L + 1.6W \tag{3.1c}$$

$$U = 0.9D + 1.6W \tag{3.1d}$$

2. In the case of dead, live, and seismic (earthquake) forces, E,

$$U = 1.2D + 1.0(L + E) \tag{3.2a}$$

$$U = 0.9D + 1.0E \tag{3.2b}$$

3. When the earth pressure load, H, is included,

$$U = 1.2D + 1.6(L + H) \tag{3.3a}$$

$$U = 0.9D + 1.6(W + H) \tag{3.3b}$$

$$U = 0.9D + 1.0E + 1.6H \tag{3.3c}$$

4. When pressure loads from fluids, F, are included,

$$U = 1.4(D + F) \tag{3.4a}$$

$$U = 1.2(D + F) + 1.6(L + H) \tag{3.4b}$$

5. For load combination due to roof live load, L_r, rain load, R, snow load, S, the effect of temperature T (including the effect of creep, shrinkage, and differential settlement) in addition to the above loads,

$$U = 1.2(D + F + T) + 1.6(L + H) + 0.5(L_r \text{ or } S \text{ or } R) \quad (3.5a)$$

$$U = 1.2D + 1.6(L_r \text{ or } S \text{ or } R) + (1.0L \text{ or } 0.8W) \quad (3.5b)$$

$$U = 1.2D + 1.6W + 1.0L + 0.5(L_r \text{ or } S \text{ or } R) \quad (3.5c)$$

$$U = 1.2D + 1.0E + 1.0L + 0.2S \quad (3.5d)$$

It is to noted that

1. The load factor L in Eqs. 3.1c, 3.2a, and 3.5a, b, c, and d shall be permitted to be reduced to $0.5L$, except for garages, areas occupied as places of public assembly, and all areas where the live load, L, is greater than 100 pounds per square foot (psf).
2. When the wind load, W, is not reduced by a directionality factor, it is permitted to use $1.3W$ in place of $1.6W$ in Eqs. 3.1d and 3.3b.
3. If the service level of the seismic load E is used, $1.4E$ shall be used in place of $1.0E$ in Eqs. 3.2a and b and 3.3c.
4. If the structural action due to H counteracts that due to W or E, the load factor of H shall be set to 0.
5. In a flood zone area, the flood load or load combinations of ASCE shall be used.
6. Impact effects shall be included with the live load, L.

The ACI Code does not specify a value for impact, but AASHTO specifications give a simple factor for impact, I, as a percentage of the live load, L, as follows:

$$I = 50/(125 + S) \leq 30\% \quad (3.6)$$

where I = percentage of impact, S = part of the span loaded, and live load including impact $= L(1 + I)$.

When a better estimation is known from experiments or experience, the adjusted value shall be used.

The above equations indicate that the dead load factor is 1.2, whereas the live load factor is 1.6. These values are less than those specified by the 1999 ACI Code of 1.4 for the dead load and 1.7 for the live load. The new factors are based on the ASCE specifications ASCE 7-05.

For applied concentrated dead and live loads, P_D, P_L, the factored concentrated load $P_U = 1.2P_D + 1.6P_L$; also $M_U = 1.2\ M_D + 1.6\ M_L$, where M_D and M_L are the service dead-load and live-load moments, respectively.

3.6 STRENGTH-REDUCTION FACTOR ϕ

The nominal strength of a section, say M_n, for flexural members, calculated in accordance with the requirements of the ACI Code provisions must be multiplied by the strength reduction factor, ϕ, which is always less than 1. The strength reduction factor has several purposes:

1. To allow for the probability of under-strength sections due to variations in dimensions, material properties, and inaccuracies in the design equations
2. To reflect the importance of the member in the structure
3. To reflect the degree of ductility and required reliability under the applied loads

The ACI Code, Section 9.3, specifies the following values to be used:

For tension-controlled sections,	$\phi = 0.90$
For compression-controlled section	
a. with spiral reinforcement,	$\phi = 0.75$
b. other reinforced members,	$\phi = 0.65$
For plain concrete,	$\phi = 0.60$
For shear and torsion,	$\phi = 0.75$
For bearing on concrete,	$\phi = 0.65$
For strut and tie models,	$\phi = 0.75$

A higher ϕ factor is used for tension-controlled sections than for compression-controlled sections, because the latter sections have less ductility and they are more sensitive to variations in concrete strength. Also, spirally reinforced compression members have a ϕ value of 0.75 compared to 0.65 for tied compression members; this variation reflects the greater ductility behavior of spirally reinforced concrete members under the applied loads. In the ACI Code provisions, the ϕ factor is based on the behavior of the cross-section at nominal strength, (P_n, M_n), defined in terms of the NTS, ε_t, in the extreme tensile strains, as given in Table 3.1. For tension-controlled members, $\phi = 0.9$. For compression-controlled members, $\phi = 0.75$ (with spiral reinforcement) and $\phi = 0.65$ for other members.

For the transition region, ϕ may be determined by linear interpolation between 0.65 (or 0.75) and 0.9. Figure 3.6*a* shows the variation of ϕ for grade 60 steel. The linear equations are as follows:

$$\phi = 0.75 + (\varepsilon_t - 0.002)(50) \text{ (for spiral members)} \tag{3.7}$$

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) \text{ (for other members)} \tag{3.8}$$

Alternatively, ϕ may be determined in the transition region, as a function of (c/d_t) for grade 60 steel as follows:

$$\phi = 0.75 + 0.15\left[\frac{1}{c/d_t} - \frac{5}{3}\right] \text{ (for spiral members)} \tag{3.9}$$

$$\phi = 0.65 + 0.25\left[\frac{1}{c/d_t} - \frac{5}{3}\right] \text{ (for other members)} \tag{3.10}$$

where c = the depth of the neutral axis at nominal strength (c_2 in Fig. 3.4). At the limit strain of 0.002 for grade 60 steel and from the triangles of Fig. 3.4*a*, $c/d_t = 0.003/(0.002 + 0.003) = 0.6$. Similarly, at a strain, $\varepsilon_t = 0.005$, $c/d_t = 0.003/(0.005 + 0.003) = 0.375$. Both values are shown in Fig. 3.6.

For reinforced concrete flexural members, the NTS, ε_t, should be equal to or greater than 0.004 (ACI Code, Section 10.3). In this case,

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) = 0.82 \tag{3.11}$$

Figure 3.6*b* shows the range of ϕ for flexural members. For grade 60 steel, the range varies between 0.9 for $\varepsilon_t \geq 0.005$ and 0.82 for $\varepsilon_t = 0.004$. Other values of ϕ can be obtained from Eq. 3.11 or by interpolation.

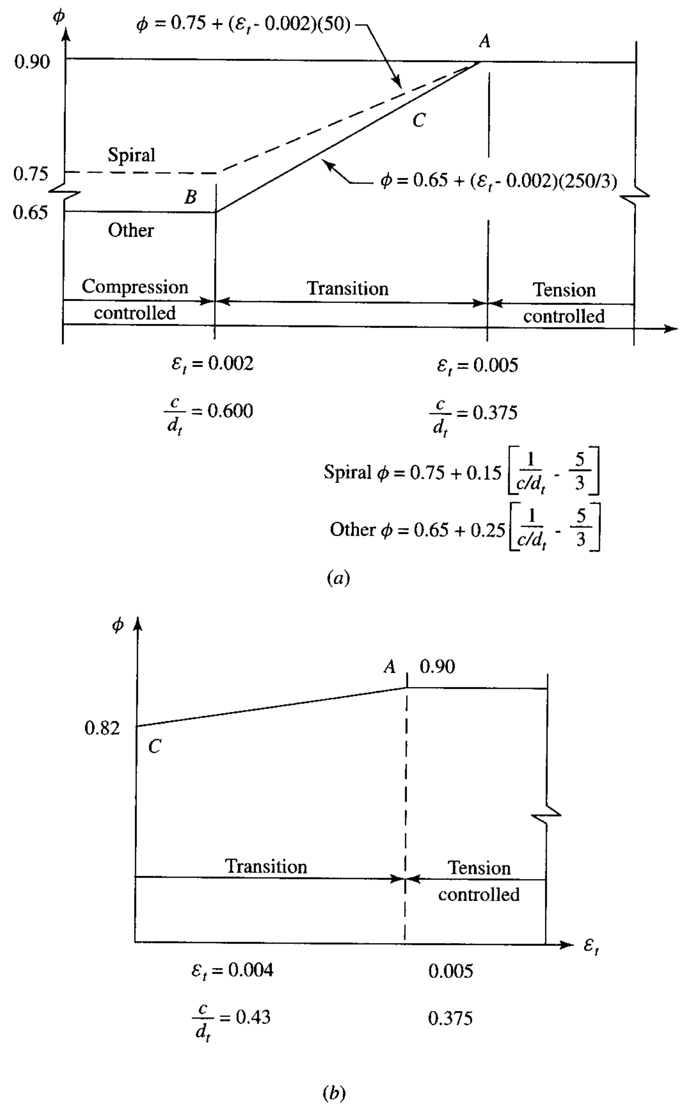


Figure 3.6 (*a*) Variation of ϕ, with the net tensile strain for grade 60 steel and for prestressed steel, [1]; (*b*) variation of ϕ and strain limit in flexural member with f_y = 60 ksi.

3.7 SIGNIFICANCE OF ANALYSIS AND DESIGN EXPRESSIONS

Two approaches for the investigations of a reinforced concrete member will be used in this book:

Analysis of a section implies that the dimensions and steel used in the section (in addition to concrete strength and steel yield strength) are given, and it is required to calculate the internal design moment capacity of the section so that it can be compared with the applied external required moment.

Design of a section implies that the external required moment is known from structural analysis, and it is required to compute the dimensions of an adequate concrete section and the amount of steel reinforcement. Concrete strength and yield strength of steel used are given.

3.8 EQUIVALENT COMPRESSIVE STRESS DISTRIBUTION

The distribution of compressive concrete stresses at failure may be assumed to be a rectangle, trapezoid, parabola, or any other shape that is in good agreement with test results.

When a beam is about to fail, the steel will yield first if the section is under-reinforced, and in this case the steel is equal to the yield stress. If the section is over-reinforced, concrete crushes first and the strain is assumed to be equal to 0.003, which agrees with many tests of beams and columns. A compressive force, C, develops in the compression zone and a tension force, T, develops in the tension zone at the level of the steel bars. The position of force T is known, because its line of application coincides with the center of gravity of the steel bars. The position of compressive force C is not known unless the compressive volume is known and its center of gravity is located. If that is done, the moment arm, which is the vertical distance between C and T, will consequently be known.

In Fig. 3.7, if concrete fails, $\varepsilon_c = 0.003$, and if steel yields, as in the case of a balanced section, $f_s = f_y$.

The compression force, C, is represented by the volume of the stress block, which has the nonuniform shape of stress over the rectangular hatched area of bc. This volume may be considered equal to $C = bc(\alpha_1 f'_c)$, where $\alpha_1 f'_c$ is an assumed average stress of the nonuniform stress block.

The position of compression force C is at a distance z from the top fibers, which can be considered as a fraction of the distance c (the distance from the top fibers to the neutral axis), and z can be assumed to be equal to $\alpha_2 c$, where $\alpha_2 < 1$. The values of α_1 and α_2 have been estimated from many tests, and their values, as suggested by Mattock, Kriz, and Hognestad [3], are as follows:

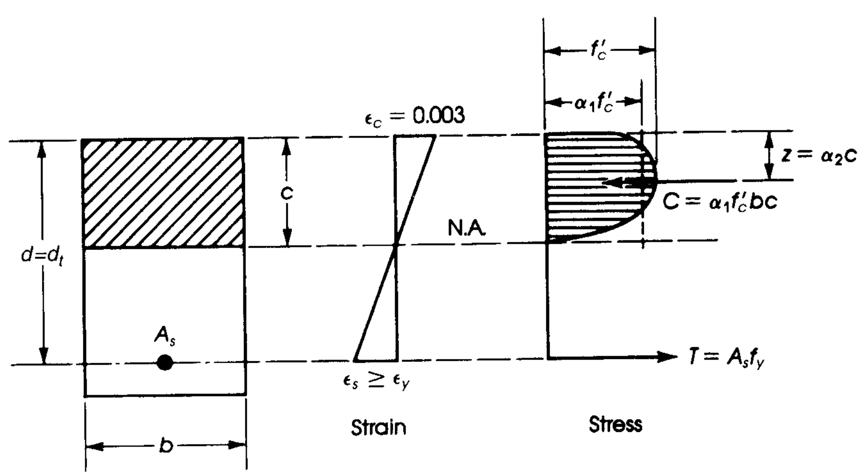


Figure 3.7 Ultimate forces in a rectangular section.

$\alpha_1 = 0.72$ for $f'_c \leq 4000$ psi (27.6 MPa); it decreases linearly by 0.04 for every 1000 psi (6.9 MPa) greater than 4000 psi

$\alpha_2 = 0.425$ for $f'_c < 4000$ psi (27.6 MPa); it decreases linearly by 0.025 for every 1000 psi greater than 4000 psi

The decrease in the value of α_1 and α_2 is related to the fact that high-strength concretes show more brittleness than low-strength concretes [2].

To derive a simple rational approach for calculations of the internal forces of a section, the ACI Code adopted an equivalent rectangular concrete stress distribution, which was first proposed by C. S. Whitney and checked by Mattock and others [3]. A concrete stress of $0.85\ f'_c$ is assumed to be uniformly distributed over an equivalent compression zone bounded by the edges of the cross-section and a line parallel to the neutral axis at a distance $a = \beta_1 c$ from the fiber of maximum compressive strain, where c is the distance between the top of the compressive section and the neutral axis (Fig. 3.8). The fraction β_1 is 0.85 for concrete strengths $f'_c \leq 4000$ psi (27.6 MPa) and is reduced linearly at a rate of 0.05 for each 1000 psi (6.9 MPa) of stress greater than 4000 psi (Fig. 3.9), with a minimum value of 0.65.

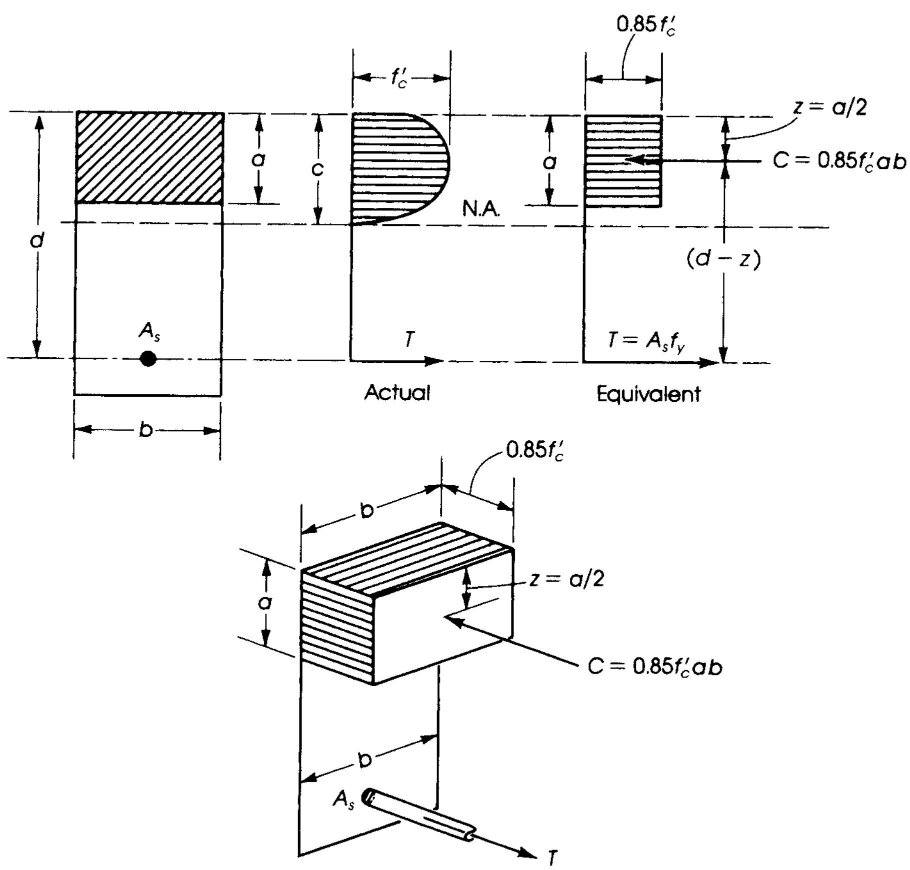


Figure 3.8 Actual and equivalent stress distributions at failure.

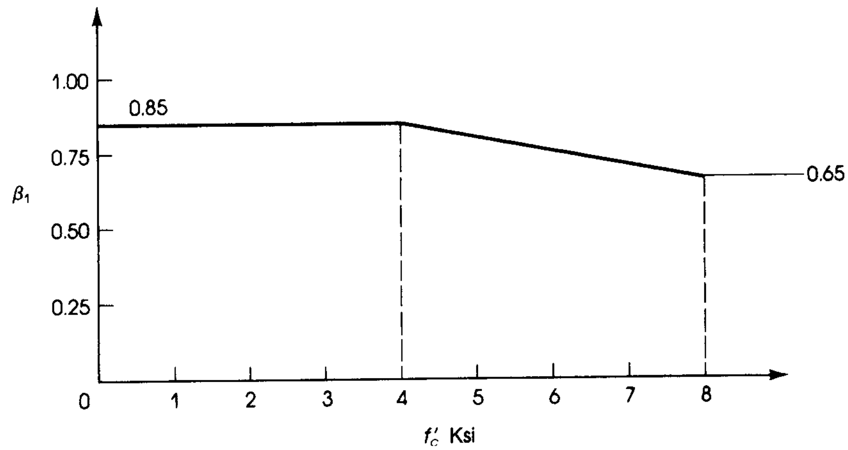


Figure 3.9 Values of β_1 for different compressive strengths of concrete, f'_c.

The preceding discussion applies in general to any section, and it is not confined to a rectangular shape. In the rectangular section, the area of the compressive zone is equal to ba, and every unit area is acted on by a uniform stress equal to $0.85f'_c$, giving a total stress volume equal to $0.85f'_c ab$, which corresponds to the compressive force, C. For any other shape, the force C is equal to the area of the compressive zone multiplied by a constant stress equal to $0.85f'_c$.

For example, in the section shown in Fig. 3.10, the force C is equal to the shaded area of the cross-section multiplied by $0.85f'_c$:

$$C = 0.85f'_c(6 \times 3 + 10 \times 2) = 32.3f'_c \text{ lb}$$

The position of the force C is at a distance z from the top fibers, at the position of the resultant force of all small-element forces of the section. As in the case when the stress is uniform and equals $0.85f'_c$, the resultant force C is located at the center of gravity of the compressive zone, which has a depth of a.

In this example, z is calculated by taking moments about the top fibers:

$$z = \frac{\left(6 \times 3 \times \frac{3}{2}\right) + 10 \times 2(1+3)}{6 \times 3 + 10 \times 2} = \frac{107}{38} = 2.82 \text{ in.}$$

3.9 SINGLY REINFORCED RECTANGULAR SECTION IN BENDING

We explained previously that a balanced condition is achieved when steel yields at the same time as the concrete fails, and that failure usually happens suddenly. This implies that the yield strain in the steel is reached ($\varepsilon_y = f_y/E_s$) and that the concrete has reached its maximum strain of 0.003. The percentage of reinforcement used to produce a balanced condition is called the *balanced steel ratio*, ρ_b. This value is equal to the area of steel, A_s, divided by the effective cross-section, bd:

$$\rho_b = \frac{A_s(\text{balanced})}{bd}$$

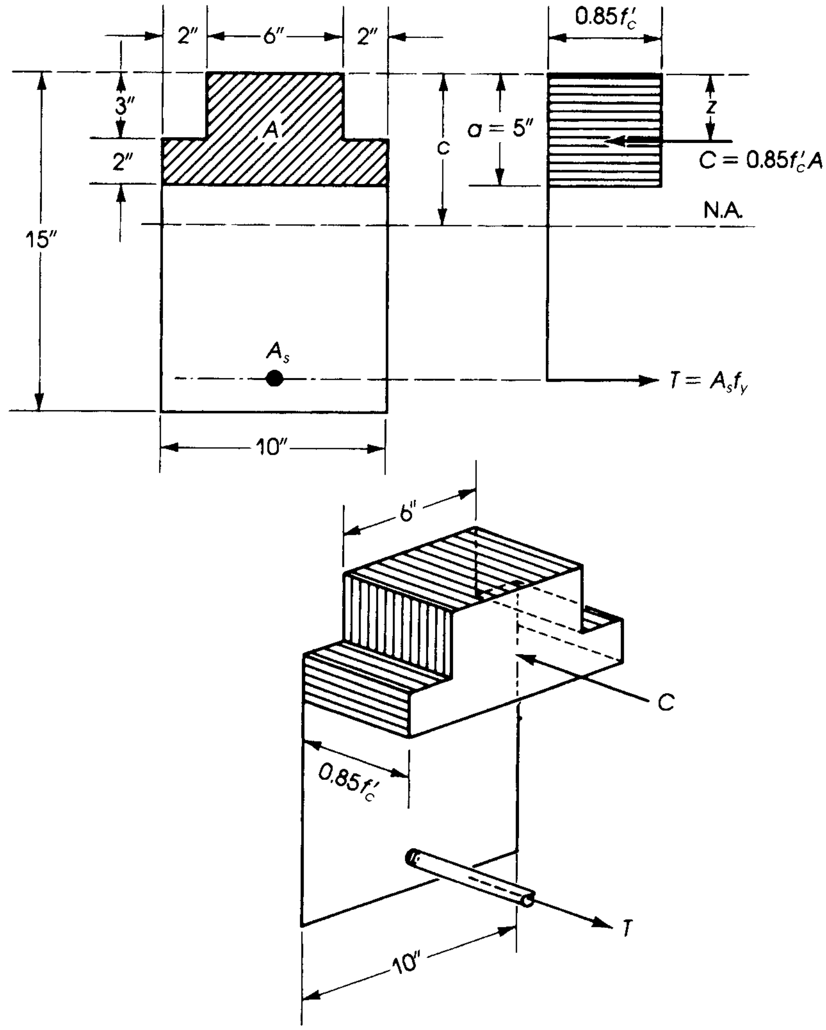


Figure 3.10 Ultimate forces in a nonrectangular section.

where

b = width of the compression face of the member

d = distance from the extreme compression fiber to the centroid of the longitudinal tension reinforcement

Two basic equations for the analysis and design of structural members are the two equations of equilibrium that are valid for any load and any section:

1. The compression force should be equal to the tension force; otherwise, a section will have linear displacement plus rotation:

$$C = T \tag{3.12}$$

2. The internal nominal bending moment, M_n, is equal to either the compressive force, C, multiplied by its arm or the tension force, T, multiplied by the same arm:

$$M_n = C(d - z) = T(d - z)$$

$$(M_u = \phi M_n \text{ after reduction by the factor } \phi) \tag{3.13}$$

The use of these equations can be explained by considering the case of a rectangular section with tension reinforcement (Fig. 3.8). The section may be balanced, under-reinforced, or over-reinforced, depending on the percentage of steel reinforcement used.

3.9.1 The Balanced Section

Let us consider the case of a balanced section, which implies that at ultimate load the strain in concrete equals 0.003 and that of steel equals the first yield stress at distance d_t divided by the modulus of elasticity of steel, f_y/E_s. This case is explained by the following steps.

Step 1. From the strain diagram of Fig. 3.11,

$$\frac{c_b}{d - c_b} = \frac{0.003}{f_y/E_s}$$

From triangular relationships (where is c_b is c for a balanced section) and by adding the numerator to the denominator,

$$\frac{c_b}{d} = \frac{0.003}{0.003 + f_y/E_s}$$

Substituting $E_s = 29 \times 10^3$ ksi,

$$c_b = \left(\frac{87}{87 + f_y}\right) d \tag{3.14}$$

where f_y is in ksi.

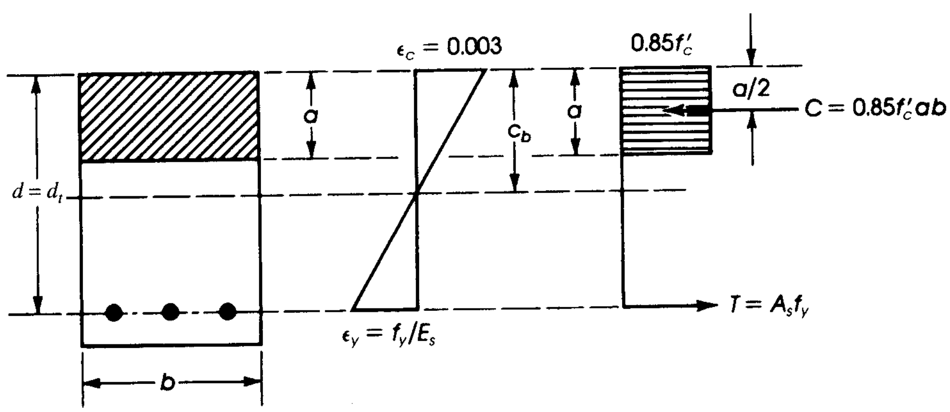


Figure 3.11 Rectangular balanced section.

Step 2. From the equilibrium equation,

$$C = T$$

$$0.85 f'_c ab = A_s f_y \tag{3.15}$$

$$a = \frac{A_s f_y}{0.85 f'_c b} \tag{3.16}$$

Here a is the depth of the compressive block, equal to $\beta_1 c$, where $\beta_1 = 0.85$ for $f'_c \leq 4000$ psi (27.6 MPa) and decreases linearly by 0.05 per 1000 psi (6.9 MPa) for higher concrete strengths (Fig. 3.9). Because the balanced steel reinforcement ratio is used,

$$\rho_b = \frac{A_s(\text{balanced})}{bd} = \frac{A_{sb}}{bd} \tag{3.17}$$

and substituting the value of A_{sb} in Eq. 3.15,

$$0.85 f'_c ab = f_y \rho_b bd$$

Therefore,

$$\rho_b = \frac{0.85 f'_c}{f_y d} a = \frac{0.85 f'_c}{f_y d} (\beta_1 c_b)$$

Substituting the value of c_b from Eq. 3.14, the general equation of the balanced steel ratio becomes

$$\rho_b = 0.85 \beta_1 \frac{f'_c}{f_y} \left(\frac{87}{87 + f_y} \right) \tag{3.18}$$

Step 3. The internal nominal moment, M_n, is calculated by multiplying either C or T by the distance between them:

$$M_n = C(d - z) = T(d - z) \tag{3.13}$$

For a rectangular section, the distance $z = a/2$ as the line of application of the force C lies at the center of gravity of the area ab, where

$$a = \frac{A_s f_y}{0.85 f'_c b}$$

$$M_n = C\left(d - \frac{a}{2}\right) = T\left(d - \frac{a}{2}\right)$$

For a balanced or an under-reinforced section, $T = A_s f_y$. Then

$$M_n = A_s f_y \left(d - \frac{a}{2}\right) \tag{3.19}$$

To get the usable design moment ϕM_n, the previously calculated M_n must be reduced by the capacity reduction factor, ϕ,

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right) = \phi A_s f_y \left(d - \frac{A_s f_y}{1.7 f'_c b}\right) \tag{3.19a}$$

Equation 3.19a can be written in terms of the steel percentage ρ:

$$\rho = \frac{A_s}{bd} \qquad A_s = \rho bd$$

$$\phi M_n = \phi f_y \rho bd \left(d - \frac{\rho bd f_y}{1.7 f_c' b}\right) = \phi \rho f_y bd^2 \left(1 - \frac{\rho f_y}{1.7 f_c'}\right) \tag{3.20}$$

Equation 3.20 can be written as

$$\phi M_n = R_u bd^2 \tag{3.21}$$

where

$$R_u = \phi \rho f_y \left(1 - \frac{\rho f_y}{1.7 f_c'}\right) \tag{3.22}$$

The ratio of the equivalent compressive stress block depth, a, to the effective depth of the section, d, can be found from Eq. 3.15:

$$0.85 f_c' ab = \rho bd f_y \tag{3.23}$$

$$\frac{a}{d} = \frac{\rho f_y}{0.85 f_c'}$$

3.9.2 Upper Limit of Steel Percentage

The upper limit or the maximum steel percentage, $\rho_{\max}$, that can be used in a singly reinforced concrete section in bending is based on the net tensile strain in the tension steel, the balanced steel ratio, and the grade of steel used. The relationship between the steel percentage, ρ, in the section, and the net tensile strain, ε_t, is as follows:

$$\varepsilon_t = \left(\frac{0.003 + f_y/E_s}{\rho/\rho_b}\right) - 0.003 \tag{3.24}$$

For $f_y = 60$ ksi, and assuming $f_y/E_s = 0.002$,

$$\varepsilon_t = \left(\frac{0.005}{\rho/\rho_b}\right) - 0.003 \tag{3.25}$$

These expressions are obtained by referring to Fig. 3.12. For a balanced section,

$$c_b = \frac{a_b}{\beta_1} = \frac{A_{\text{sb}} f_y}{0.85 f_c' b\beta_1} = \frac{\rho_b f_y d}{0.85 f_c' \beta_1}$$

Similarly, for any steel ratio, ρ,

$$c = \frac{\rho f_y d}{0.85 f_c' \beta_1} \quad \text{and} \quad \frac{c}{c_b} = \frac{\rho}{\rho_b}$$

Divide both sides by d to get

$$\frac{c}{d} = \left(\frac{\rho}{\rho_b}\right)\left(\frac{c_b}{d}\right) \tag{3.26}$$

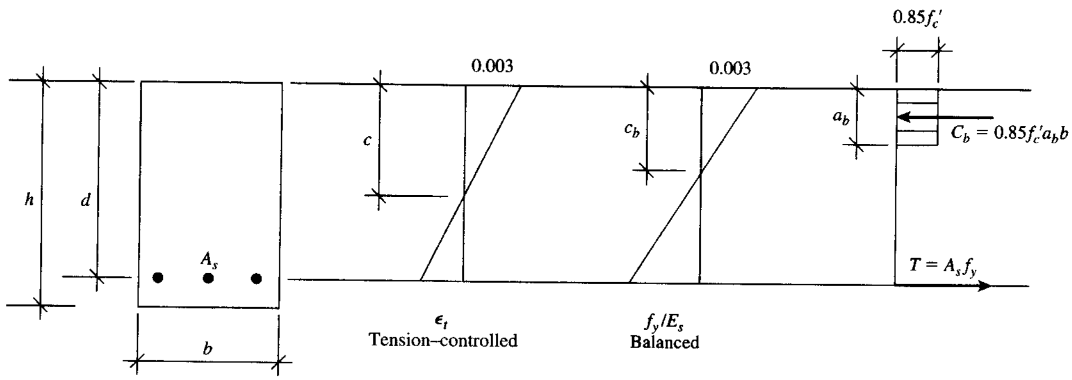


Figure 3.12 Strains in tension-controlled and balanced conditions.

From the triangles of the strain diagrams,

$$\frac{c}{d} = \frac{0.003}{0.003 + \varepsilon_t}$$
$$\varepsilon_t = \frac{0.003}{(c/d)} - 0.003 \tag{3.27}$$

Similarly,

$$\frac{c_b}{d} = \frac{0.003}{0.003 + f_y/E_s} \tag{3.28}$$

From Eqs. 3.26 and 3.28,

$$\frac{c}{d} = \left(\frac{\rho}{\rho_b}\right)\left(\frac{c_b}{d}\right) = \left(\frac{\rho}{\rho_b}\right)\left(\frac{0.003}{0.003 + f_y/E_s}\right)$$

Substitute this value in Eq. 3.27 to get

$$\varepsilon_t = \frac{0.003}{(c/d)} - 0.003 = \left[\frac{0.003 + f_y/E_s}{\rho/\rho_b}\right] - 0.003 \tag{3.24}$$

For grade 60 steel, $f_y = 60$ ksi, $E_s = 29{,}000$ ksi, and $f_y/E_s = 0.00207$, then

$$\varepsilon_t = \left(\frac{0.00507}{\rho/\rho_b}\right) - 0.003 \tag{3.25}$$

To determine the upper limit or the maximum steel percentage, ρ, in a singly reinforced concrete section, refer to Fig. 3.6. It can be seen that concrete sections subjected to flexure or axial load and bending moment may lie in compression-controlled, transition, or tension-controlled zones. When $\varepsilon_t \leq 0.002$ (or $c/d_t \geq 0.6$), compression controls, whereas when $\varepsilon_t \geq 0.005$ (or $c/d_t \leq 0.375$), tension controls. The transition zone occurs when $0.002 < \varepsilon_t < 0.005$ or $0.6 > c/d_t > 0.375$.

For members subjected to flexure, the relationship between the steel ratio, ρ, was given in Eq. 3.24:

$$\varepsilon_t + 0.003 = \frac{0.003 + f_y/E_s}{\rho/\rho_b} \tag{3.24}$$

or

$$\frac{\rho}{\rho_b} = \frac{0.003 + f_y/E_s}{0.003 + \varepsilon_t} \tag{3.29}$$

For $f_y = 60$ ksi and $E_s = 29{,}000$ ksi, f_y/E_s may be assumed to be 0.00207.

$$\frac{\rho}{\rho_b} = \frac{0.00507}{0.003 + \varepsilon_t} \tag{3.30}$$

The limit for tension to control is $\varepsilon_t \geq 0.005$. For $\varepsilon_t = 0.005$, Eq. 3.30 becomes

$$\frac{\rho}{\rho_b} = \frac{0.005}{0.008} = \frac{5}{8} = 0.625 \tag{3.30a}$$

or $\rho \leq 0.63375\rho_b$ for tension-controlled sections if $\varepsilon_t = 0.00507 = f_y/E_s$. Both values can be used for practical analysis and design. The small increase in ρ will slightly increase the moment capacity of the section. For example, if $f_c' = 4$ ksi and $f_y = 60$ ksi, $\rho_b = 0.0285$ and $\rho \leq 0.01806$ for tension to control (as in the case of flexural members). The ϕ factor in this case is 0.9. This value is less than $\rho_{\max} = 0.75\rho_b = 0.0214$ allowed by the ACI Code for flexural members when $\phi = 0.9$ can be used.

Design of beams and other flexural members can be simplified using the limit of $\varepsilon_t = 0.005$.

$$\frac{\rho}{\rho_b} = \frac{0.003 + f_y/E_s}{0.008} \tag{3.31}$$

In this case, $\rho = \rho_{\max}$ = upper limit for tension-controlled sections.

$$\rho_{\max} = \left(\frac{0.003 + f_y/E_s}{0.008}\right)\rho_b \tag{3.31a}$$

Note that when ρ used $\leq \rho_{\max}$, tension controls and $\phi = 0.9$. When $\rho > \rho_{\max}$, section will be in the transition region with $\phi < 0.9$.
And for $f_y = 60$ ksi and $f_y/E_s = 0.00207$,

$$\frac{\rho_{\max}}{\rho_b} = 0.63375 \tag{3.32}$$

This steel ratio will provide adequate ductility before beam failure.

Similarly,

$$\text{for } f_y = 40 \text{ ksi}, \quad \rho_{\max} = 0.5474\rho_b \tag{3.32a}$$

$$\text{for } f_y = 50 \text{ ksi}, \quad \rho_{\max} = 0.5905\rho_b \tag{3.32b}$$

$$\text{for } f_y = 75 \text{ ksi}, \quad \rho_{\max} = 0.6983\rho_b \tag{3.32c}$$

It was established that $\phi M_n = R_u bd^2$ (Eq. 3.21), where $R_u = \phi\rho f_y(1 - \rho f_y/1.7f_c')$ (Eq. 3.22). Once f_c' and f_y are known, then ρ_b, ρ, R_u, and bd^2 can be calculated. For example, for $f_c' = 4$ ksi, $f_y = 60$ ksi, $\phi = 0.9$, $\varepsilon_t = 0.005$, and one row of bars in the section,

$$\rho_b = 0.0285 \quad \rho = 0.01806 \quad R_u = 820 \text{ psi}$$

Note that for one row of bars in the section, it can be assumed that $d = d_t = h - 2.5$ in., whereas for two rows of bars, $d = h - 3.5$ in., and $d_t = h - 2.5$ in. $= d + 1.0$ in. (Refer to Figs. 3.4 and 3.5 and Section 4.3.3.)

Table 3.2 Values of ρ_{max} and $R_u = M_u/bd^2$ for Flexural Tension-Controlled Sections with One Row of Bars, $\varepsilon_t = 0.005$

f'_c (ksi)	f_y (ksi)	ρ_b	$\rho_{max} = 0.63375\,\rho_b$	R_u (psi) (Eq. 3.22)
3	60	0.0214	0.01356	615
4	60	0.0285	0.01806	820
5	60	0.0335	0.02123	975
6	60	0.0377	0.02389	1109

Table 3.2 gives the values of ρ, ρ_b, and $R_u = M_u/bd^2$ for flexural tension-controlled sections with one row of bars.

For reinforced concrete flexural members with $\rho > \rho_{max}$, ε_t will be less than 0.005. Section 10.3 of the ACI Code specifies that ε_t should not be less than 0.004 in the transition region to maintain adequate ductility and warning before failure.

For this limitation of $\varepsilon_t = 0.004$, the general equation (3.29) becomes

$$\frac{\rho}{\rho_b} = \frac{0.003 + f_y/E_s}{0.007} \tag{3.33}$$

For $f_y = 60$ ksi,

$$\frac{\rho}{\rho_b} = \frac{0.003 + 0.00207}{0.007} = 0.724 \tag{3.34}$$

and the limit in the transition region is

$$\rho_{max\,t} = 0.724\,\rho_b \tag{3.34a}$$

Note that the t here refers to the transition region. In this case, limit of ϕ is

$$\phi_t = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) = 0.817 < 0.9 \tag{3.35}$$

For $f_y = 60$ ksi and $f'_c = 4$ ksi, $\rho_b = 0.0285$, $\rho_{max\,t} = 0.02063$, $R_n = 1012$ psi (from Eq.3.22, and $R_u = \phi R_n = 0.817(1012) = 826$ psi.

This steel ratio in Eq. 3.33 is the upper limit ($\rho_{max\,t}$) for a singly reinforced concrete section in the transition region with $\phi < 0.9$.

It can be noticed that the aforementioned $R_u = 826$ psi calculated for $\varepsilon_t = 0.004$, is very close to $R_u = 820$ psi for $\rho_{max} = 0.63375\rho_b$ and $\phi = 0.9$. Therefore, adding reinforcement beyond ρ_{max} (for $\varepsilon_t = 0.005$, Table 3.2) reduces ϕ because of the reduced ductility resulting in little or nonsubstantial gain in design strength. Adding compression reinforcement in the section is a better solution to increase the design moment, keeping the section in the tension-controlled region with $\phi = 0.9$. (Refer to Section 3.14.)

Table 3.3 gives the values of ρ_t(limit), ρ_b, and R_u for flexural members in the transition region for $f_y = 60$ ksi and $\varepsilon_t = 0.004$ and one row of bars. In this case $\phi = 0.817$ (Eq. 3.35) and $\rho/\rho_b = 0.724$. It is clear that for $f_y = 60$ ksi, the design R_u in both cases, when $\varepsilon_t = 0.005$ with $\phi = 0.9$ and when $\varepsilon_{max} = 0.004$ with $\phi = 0.816$, are quite close.

Table 3.3 Values of ρ_t and R_u for Sections in the Transition Region with $\varepsilon_t = 0.004$, $f_y = 60$ ksi, and One Row of Bars ($\phi = 0.817$)

f'_c (ksi)	ρ_b	ρ_t (limit)	R_u (psi)
3	0.0214	0.0155	617
4	0.0285	0.0206	822
5	0.0335	0.0243	980
6	0.0377	0.0273	1116

Example 3.1

For the section shown in Fig. 3.13, calculate

a. The balanced steel reinforcement

b. The maximum reinforcement area allowed by the ACI Code for tension-controlled section and in the transition region

c. The position of the neutral axis and the depth of the equivalent compressive stress block for the tension-controlled section in *b*.

Given: $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

a. $\rho_b = 0.85\,\beta_1 \frac{f'_c}{f_y}\left(\frac{87}{87 + f_y}\right)$

Because $f'_c = 4000$ psi, $\beta_1 = 0.85$:

$$\rho_b = (0.85)^2\left(\frac{4}{60}\right)\left(\frac{87}{87 + 60}\right) = 0.0285$$

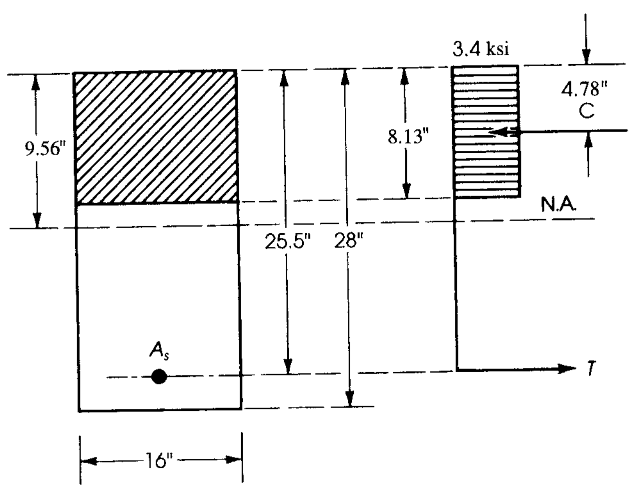


Figure 3.13 Example 3.1.

The area of steel reinforcement to provide a balanced condition is

$$A_{sb} = \rho_b bd = 0.0285 \times 16 \times 25.5 = 11.63 \text{ in.}^2$$

b. For a tension-controlled section, $\rho_{max} = 0.63375\ \rho_b = 0.63375 \times 0.0285 = 0.01806$ or, from Eq. 3.32,

$$A_{s\,max} = \rho_{max} bd = 0.01806 \times 16 \times 25.5 = 7.37 \text{ in.}^2 \text{ for } \phi = 0.9.$$

For the transition region, $\rho_{max\,t} = 0.724\ \rho_b = 0.0206$. For the case of $\varepsilon_t = 0.004$, $A_{s\,max\,t} = 0.0206(16 \times 25.5) = 8.41 \text{ in.}^2$ for $\phi = 0.817$

c. The depth of the equivalent compressive block using $A_{s\,max}$ is

$$a_{max} = \frac{A_{s\,max} f_y}{0.85 f'_c b} = \frac{7.37 \times 60}{0.85 \times 4 \times 16} = 8.13 \text{ in.}$$

The distance from the top fibers to the neutral axis is $c = \alpha/\beta_1$. Because $f'_c = 4000$ psi, $\beta_1 = 0.85$; thus,

$$c = \frac{8.13}{0.85} = 9.56 \text{ in.}$$

or $c/d = 0.375$ and $c = 0.375(25.5) = 9.56$ in.

Example 3.2

Determine the design moment strength and the position of the neutral axis of the rectangular section shown in Fig. 3.14 if the reinforcement used is three no. 9 bars. Given: $f'_c = 3$ ksi and $f_y = 60$ ksi.

Solution

1. The area of three no. 9 bars is 3.0 in.^{2}

$$\rho = \frac{A_s}{bd} = \frac{3.0}{21 \times 12} = 0.0119$$

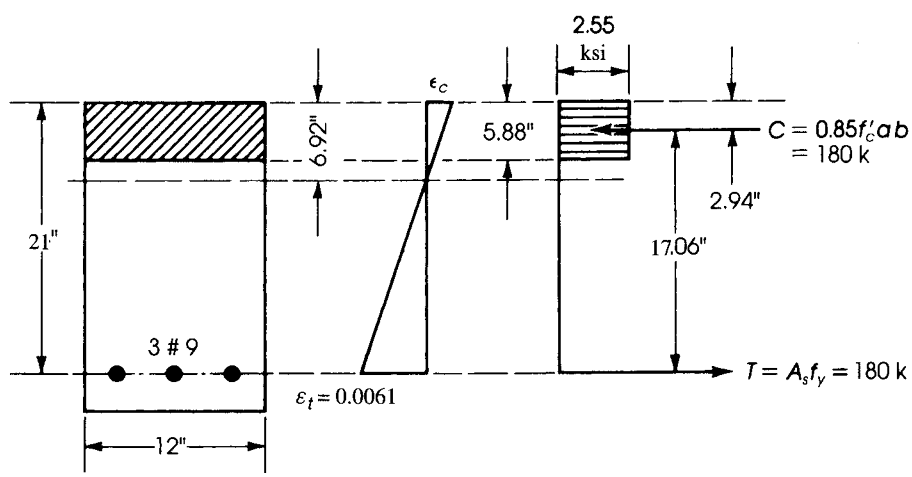


Figure 3.14 Example 3.2.

2. $\rho_{max} = 0.01356 > \rho$, tension-controlled section, $\phi = 0.9$ or check ε_t:

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{3(60)}{0.85 \times 3 \times 12} = 5.88 \text{ in.}$$

$$c = \frac{a}{0.85} = 6.92 \text{ in.}$$

$$d_t = d = 21 \text{ in.}$$

$$\varepsilon_t = \left(\frac{21 - 6.92}{6.92}\right) 0.003$$

$$= 0.0061 > 0.005, \quad \phi = 0.9$$

$$\text{or } \frac{c}{d_t} = 0.33 < 0.375 \quad \text{(o.k.)}$$

3. $\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right)$

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{3.0 \times 60}{0.85 \times 3 \times 12} = 5.88 \text{ in.}$$

$$\phi M_n = 0.9 \times 3.0 \times 60 \left(21 - \frac{5.88}{2}\right) = 2926 \text{ K·in.} = 243.8 \text{ K·ft}$$

Discussion

In this example, the section is tension-controlled, which implies that the steel will yield before the concrete reaches its ultimate strain. A simple check can be made from the strain diagram (Fig. 3.14). From similar triangles,

$$\frac{\varepsilon_c}{\varepsilon_y} = \frac{c}{(d - c)} \text{ and } \varepsilon_y = \frac{f_y}{E_s} = \frac{60}{29000} = 0.00207$$

$$\varepsilon_c = \frac{6.92}{(21 - 6.92)} \times 0.00207 = 0.00102$$

which is much less than 0.003. Therefore, steel yields before concrete reaches its limiting strain of 0.003.

Example 3.3

Repeat Example 3.2 using three no. 10 bars as the tension steel (Fig. 3.15).

Solution

1. Check ε_t:

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{3.81(60)}{0.85 \times 3 \times 12} = 7.47 \text{ in.}$$

$$c = \frac{a}{0.85} = 8.79 \text{ in.} \quad d_t = d = 21 \text{ in.} \quad \frac{c}{d_t} = 0.419 > 0.375$$

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) 0.003 = \left(\frac{21 - 8.79}{8.79}\right) 0.003 = 0.004168$$

This value is less than 0.005 but greater than 0.004 (transition region), $\phi < 0.9$.

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) = 0.831$$

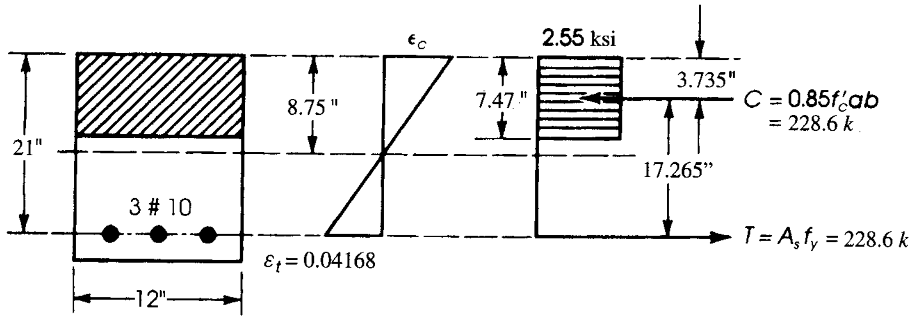


Figure 3.15 Example 3.3.

2. Calculate ϕM_n:

$$\phi M_n = 0.831(3.81)(60)\left[21 - \frac{7.47}{2}\right] = 3278 \text{ K·in.} = 273 \text{ K·ft}$$

Discussion

For a tension-controlled section, $\varepsilon_t = 0.005$ and $\rho = 0.63375\ \rho_b = 0.01356$ (Table 3.2), $\phi = 0.9$.

$$A_s \max = 0.01356(12 \times 21) = 3.417 \text{ in.}^2 < 3.81 \text{ in.}^2$$

$$a = \frac{3.417 \times 60}{0.85 \times 3 \times 12} = 6.7 \text{ in.}$$

$$\phi M_n = 0.9 \times 3.417 \times 60\left(21 - \frac{6.7}{2}\right) = 271.4 \text{ K·ft}$$

which is close to the above ϕM_n. This is a somewhat conservative approach.

3.10 LOWER LIMIT OR MINIMUM PERCENTAGE OF STEEL

If the factored moment applied on a beam is very small and the dimensions of the section are specified (as is sometimes required architecturally) and are larger than needed to resist the factored moment, the calculation may show that very small or no steel reinforcement is required. In this case, the maximum tensile stress due to bending moment may be equal to or less than the modulus of rupture of concrete: $f_r = \lambda 7.5\sqrt{f_c'}$. If no reinforcement is provided, sudden failure will be expected when the first crack occurs, thus giving no warning. The ACI Code, 10.5, specifies a minimum steel area, A_s,

$$A_{s\,\min} = \left(\frac{3\sqrt{f_c'}}{f_y}\right) b_w d \geq \left(\frac{200}{f_y}\right) b_w d$$

or the minimum steel ratio, $\rho_{\min} = (3\sqrt{f_c'}/f_y) \geq 200/f_y$, where the units of f_c' and f_y are in psi. This ρ ratio represents the lower limit. The first term of the preceding equation was specified to accommodate a concrete strength higher than 5 ksi. The two minimum ratios are equal when $f'_c = 4440$ psi. This indicates that

$$\rho_{\min} = \frac{200}{f_y} \text{ when } f'_c < 4500 \text{ psi}$$

$$\rho_{\min} = \frac{3\sqrt{f'_c}}{f_y} \text{ when } f'_c \geq 4500 \text{ psi}$$

For example, if $f_y = 60$ ksi, $\rho_{\min} = 0.00333$ when $f'_c < 4500$ psi, whereas $\rho_{\min} = 0.00353$ when $f'_c = 5000$ psi and 0.00387 when $f'_c = 6000$ psi.

In the case of a rectangular section, use $b = b_w$ in the preceding expressions. For statically determinate T-sections with the flange in tension, as in the case of cantilever beams, the value of $A_{s\min}$ should be equal to or greater than the *smaller* of (a) and (b):

(a) $$A_{s\min} = \left(\frac{6\sqrt{f'_c}}{f_y}\right) b_w d$$

(b) $$A_{s\min} = \left(\frac{3\sqrt{f'_c}}{f_y}\right) b_w d \geq \left(\frac{200}{f_y}\right) b_w d$$

where b_w and b are the width of the beam web and flange, respectively, and f'_c and f_y are in psi. For example, if $b = 48$ in., $b_w = 16$ in., $f'_c = 4000$ psi, and $f_y = 60{,}000$ psi, than $A_{s\min} = 2.02\text{ in.}^2$ in (a) controls, which is smaller than the value of $A_{s\min}$ in (b) (3.2 in.^2).

3.11 ADEQUACY OF SECTIONS

A given section is said to be *adequate* if the internal moment strength of the section is equal to or greater than the externally applied factored moment, M_u, or $\phi M_n \geq M_u$. The procedure can be summarized as follows:

1. Calculate the external applied factored moment, M_u.

$$M_u = 1.2M_D + 1.6M_L$$

2. Calculate ϕM_n for the basic singly reinforced section:
   a. Check that $\rho_{\min} \leq \rho \leq \rho_{\max}$.
   b. Calculate $a = A_s f_y/(0.85 f'_c b)$ and check ε_t for ϕ.
   c. Calculate $\phi M_n = \phi\, A_s f_y(d - a/2)$.
3. If $\phi M_n \geq M_u$, then the section is adequate; Fig. 3.16 shows a typical tension-controlled section.

Example 3.4

An 8-ft-span cantilever beam has a rectangular section and reinforcement as shown in Fig. 3.17. The beam carries a dead load, including its own weight, of 1.5 K/ft and a live load of 0.9 K/ft. Using $f'_c = 4$ ksi and $f_y = 60$ ksi, check if the beam is safe to carry the above loads.

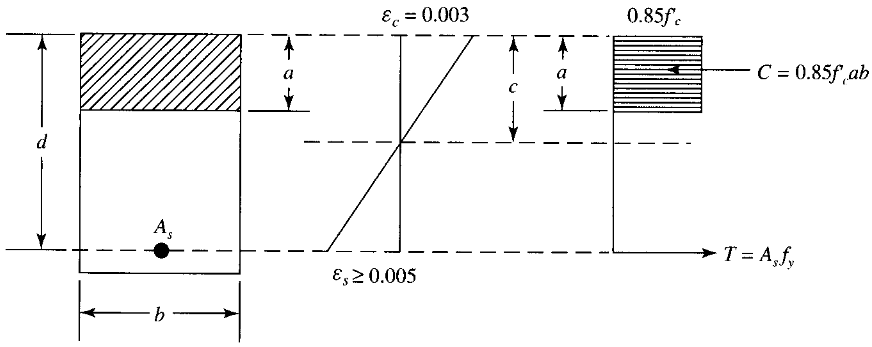


Figure 3.16 Tension-controlled rectangular section.

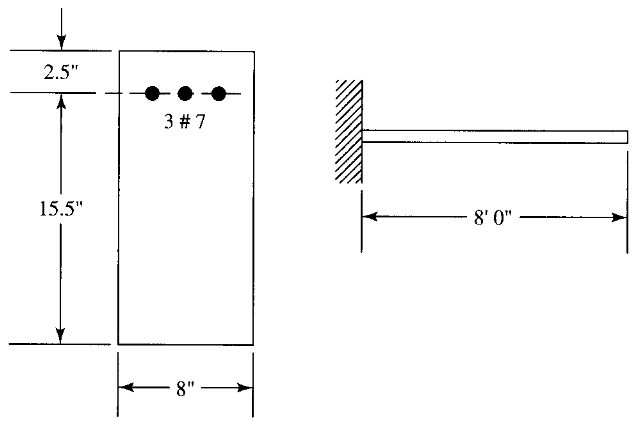


Figure 3.17 Example 3.4.

Solution

1. Calculate the external factored moment:

$$W_u = 1.2D + 1.6L = 1.2(1.5) + 1.6(0.9) = 3.24 \text{ K/ft}$$

$$M_u = W_u\frac{L^2}{2} = 3.24\frac{8^2}{2} = 103.68 \text{ K·ft} = 1244 \text{ K·in.}$$

2. Check ε_t:

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{1.8 \times 60}{0.85 \times 4 \times 8} = 3.97 \text{ in.}$$

$$c = \frac{a}{0.85} = 4.67 \text{ in.} \quad d_t = d = 15.5 \text{ in.} \quad \frac{c}{d_t} = 0.3 < 0.375$$

Also,

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right)0.003 = \left(\frac{15.5 - 4.67}{4.67}\right)0.003 = 0.007 > 0.005, \quad \phi = 0.9$$

or check

$$\rho = \frac{A_s}{bd} = \frac{1.8}{8 \times 15.5} = 0.0145 < \rho_{max} = 0.01806$$

(from Table 3.2). Therefore, it is a tension-controlled section and $\phi = 0.9$.

3. Calculate ϕM_n:

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right)$$

$$= 0.9(1.8)(60)\left(15.5 - \frac{3.97}{2}\right) = 1312 \text{ K·in.} > M_u$$

Then section is adequate.

Example 3.5

A simply supported beam has a span of 20 ft. If the cross section of the beam is as shown in Fig. 3.18, $f'_c = 3$ ksi, and $f_y = 60$ ksi, determine the allowable uniformly distributed service live load on the beam assuming the dead load is that due to beam weight. Given: $b = 12$ in., $d = 17$ in., total depth $h = 20$ in., and reinforced with three no. 8 bars ($A_s = 2.37$ in.^{2}).

Solution

1. Determine the design moment strength:

$$\rho = \frac{A_s}{bd} = \frac{3 \times 0.79}{12 \times 17} = 0.0116$$

$$\rho_{max} = 0.01356 \text{ (Table 3.2)}$$

$$\rho < \rho_{max}$$

Therefore it is a tension-controlled section and $\phi = 0.9$

Also, $\rho > \rho_{min} = \dfrac{200}{f_y} = 0.00333$.

2.
$$\phi M_n = \phi A_s f_y \left(d - \frac{A_s f_y}{1.7 f'_c b}\right)$$

$$= 0.9 \times 2.37 \times 60 \left(17 - \frac{2.37 \times 60}{1.7 \times 3 \times 12}\right) = 1878 \text{ K·in.} = 156.5 \text{ K·ft}$$

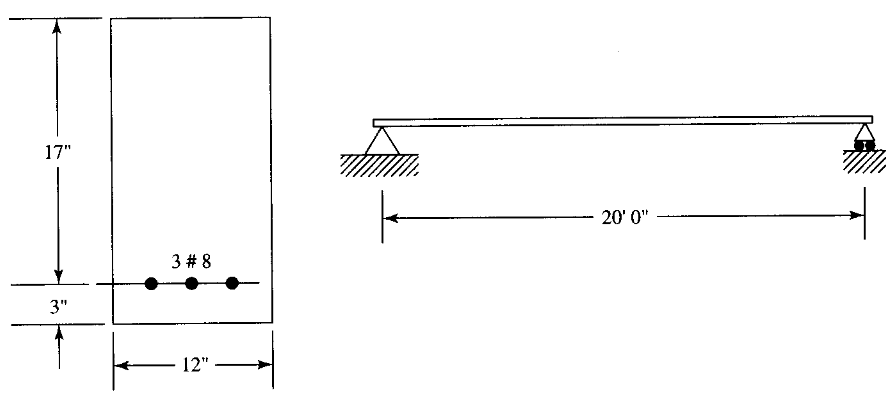


Figure 3.18 Example 3.5.

3. The dead load acting on the beam is self-weight (assumed):

$$w_D = \frac{12 \times 20}{144} \times 150 = 250 \text{ lb/ft} = 0.25 \text{ K/ft}$$

where 150 is the weight of reinforced concrete in pcf.

4. The external factored moment is

$$M_u = 1.2M_D + 1.6M_L$$

$$= 1.2\left(\frac{0.25}{8} \times 20^2\right) + 1.6\left(\frac{w_L}{8} \times 20^2\right) = 15.0 + 80w_L$$

where w_L = uniform service live load on the beam in K/ft.

5. Internal design moment equals the external factored moment:

$$156.5 = 15.0 + 80w_L \quad \text{and} \quad w_L = 1.77 \text{ K/ft}$$

The allowable uniform service live load on the beam is 1.77 K/ft.

Example 3.6: Minimum Steel Reinforcement

Check the design adequacy of the section shown in Fig. 3.19 to resist a factored moment $M_u = 30$ K·ft, using $f'_c = 3$ ksi and $f_y = 40$ ksi.

Solution

1. Check ρ provided in the section:

$$\rho = \frac{A_s}{bd} = \frac{3 \times 0.2}{10 \times 18} = 0.00333$$

2. Check $\rho_{\min}$ required according to the ACI Code:

$$\rho_{\min} = \frac{200}{f_y} = 0.005 > \rho = 0.00333$$

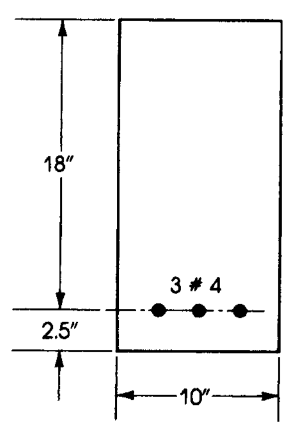


Figure 3.19 Example 3.6.

Therefore, use $\rho = \rho_{\min} = 0.005$.

$$A_{s\,\min} = \rho_{\min} bd = 0.005 \times 10 \times 18 = 0.90 \text{ in.}^2$$

Use three no. 5 bars ($A_s = 0.91\,\text{in.}^2$), because three no. 4 bars are less than the minimum specified by the code.

3. Check moment strength: $\phi M_n = \phi A_s f_y (d - a/2)$.

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{0.91 \times 40}{0.85 \times 3 \times 10} = 1.43 \text{ in.}$$

$$\phi M_n = 0.9 \times 0.91 \times 40 \left(18 - \frac{1.43}{2}\right) = 566 \text{ K·in.} = 47.2 \text{ K·ft}$$

4. An alternative solution, according to the ACI Code, Section 10.5, for three no. 4 bars, $A_s = 0.6\,\text{in.}^2$ is

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{0.6 \times 40}{0.85 \times 3 \times 10} = 0.94 \text{ in.}$$

$$\phi M_n = \frac{0.9}{12} \times 0.6 \times 40 \left(18 - \frac{0.94}{2}\right) = 31.55 \text{ K·ft}$$

$$A_s \text{ required for 30 K·ft} = \frac{30}{31.55} \times 0.6 = 0.57 \text{ in.}^2$$

The minimum A_s required according to the ACI Code, Section 10.5, is at least one-third greater than $0.57\,\text{in.}^2$:

$$\text{Minimum } A_s \text{ required} = 1.33 \times 0.57 = 0.76 \text{ in.}^2$$

which exceeds the $0.6\,\text{in.}^2$ provided by the no. 4 bars. Use three no. 5 bars, because $A_s = 0.91\,\text{in.}^2$ is greater than the $0.76\,\text{in.}^2$ required.

3.12 BUNDLED BARS

When the design of a section requires the use of a large amount of steel—for example, when $\rho_{\max}$ is used—it may be difficult to fit all bars within the cross-section. The ACI Code, 7.6, allows the use of parallel bars placed in a bundled form of two, three, or four bars, as shown in Fig. 3.20. Up to four bars (no. 11 or smaller) can be bundled when they are enclosed by stirrups.

The same bundled bars can be used in columns, provided that they are enclosed by ties. All bundled bars may be treated as a single bar for checking the spacing and concrete cover requirements. The single bar diameter shall be derived from the equivalent total area of the bundled bars.

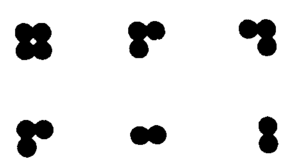


Figure 3.20 Bundled bar arrangement.

Summary: Singly Reinforced Rectangular Section

The procedure for determining the design moment of a singly reinforced rectangular section according to the ACI Code limitations can be summarized as follows:

1. Calculate the steel ratio in the section, $\rho = A_s/bd$.
2. Calculate the balanced and maximum steel ratios, Eqs. 3.18 and 3.31 or Table 3.2, for tension-controlled section. Also, calculate $\rho_{\min} = 200/f_y$ when $f'_c < 4500$ psi (f'_c and f_y are in psi units) and $\rho_{\min} = 3\sqrt{f'_c}/f_y$ when $f'_c \geq 4500$ psi.
3. If $\rho_{\min} \leq \rho \leq \rho_{\max}$, then the section meets the ACI Code limitations for tension-controlled section. If $\rho \leq \rho_{\min}$, the section is not acceptable (unless a steel ratio $\rho \geq \rho_{\min}$ is used). If $\rho \geq \rho_{\max}$, $\phi = 0.9$.
4. Calculate $a = \dfrac{A_s f_y}{0.85 f'_c b}$, c, ε_t, and ϕ.
5. Calculate $\phi M_n = \phi A_s f_y \left(d - \dfrac{a}{2}\right)$.

Flow charts representing this section and other sections are given on www.wiley.com/college/hassoun.

3.13 SECTIONS IN THE TRANSITION REGION (ϕ < 0.9)

In the case when the NTS, ε_t in the extreme tension steel lies between the compression-controlled strain limit (0.002 for $f_y = 60$ ksi) and the tension-controlled strain limit of 0.005, the strength reduction factor, ϕ, will be less than 0.9. Consequently, the design moment strength of the section ϕM_n will be smaller than ϕM_n with $\phi = 0.9$ (refer to Fig. 3.6). In the transition region, ε_t should not be less than 0.004 for flexural members (ACI Code, Section 10.3). (See Example 3.8.)

Example 3.7

Determine the design moment strength of a rectangular concrete section reinforced with four no. 9 bars in one row (Fig. 3.21).

Given: $b = 12$ in., $d = 16.5$ in., $h = 19$ in., $f'_c = 4$ ksi, and $f_y = 60$ ksi.

Solution

1. By the ACI Code provisions, for $f'_c = 4$ ksi, $f_y = 60$ ksi, and tension-controlled conditions ($\rho_b = 0.0285$ and $\rho_{\max} = 0.01806$), check $\rho = A_s/bd = 4/(12 \times 6) = 0.02083 > \rho_{\max}$. This indicates that the section is in the transition region and $\phi < 0.9$.

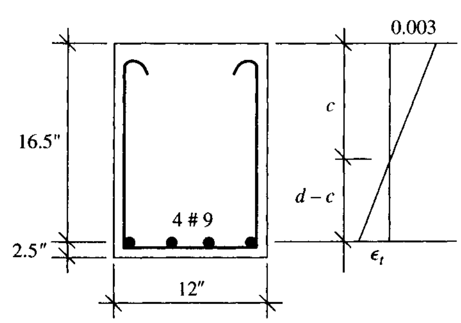


Figure 3.21 Example 3.7 ($d = d_t$).

2. Calculate a, c, and ε_t:

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{4 \times 60}{0.85 \times 4 \times 12} = 5.882 \text{ in.}$$

$$c = \frac{a}{0.85} = 6.92 \text{ in.} \quad d_t = d = 16.5 \text{ in.} \quad \frac{c}{d_t} = 0.42 > 0.375$$

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) 0.003 = \left(\frac{16.5 - 6.92}{6.92}\right) 0.003 = 0.004153 > 0.004$$

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) = 0.829$$

3. Calculate:

$$\phi M_n = \phi A_s f_y (d - a/2)$$

$$= 0.829(4)(60)\frac{\left(16.5 - \frac{5.882}{2}\right)}{12} = 224.9 \text{ K·ft}$$

Discussion

A slightly conservative approach can be used assuming tension-controlled section, $\rho = \rho_{\max} = 0.01806$ and $\phi = 0.9$. $A_{s\max} = 0.01806(12 \times 16.5) = 3.576 \text{ in.}^2$, $a = 5.259$ in., and $\phi M_n = 223.2$ K·ft (almost equal to the above ϕM_n).

Example 3.8: Two Rows of Bars

Determine the design moment strength of a rectangular concrete section reinforced with six no. 9 bars in two rows (Fig. 3.22).

Given: $b = 12$ in., $d = 23.5$ in., $h = 27$ in., $d_t = 24.5$ in., $f'_c = 4$ ksi, and $f_y = 60$ ksi.

Solution

1. For tension-controlled condition, $\varepsilon_t = 0.005$, $\rho_{\max} = 0.01806$ (Table 3.2) and $\rho_b = 0.0285$. Check

$$\rho = \frac{A_s}{bd} = \frac{6}{12 \times 23.5} = 0.02128 > \rho_{\max}$$

Section is in the transition region.

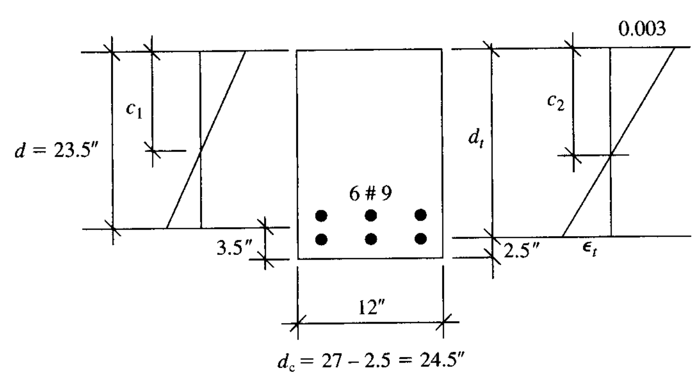


Figure 3.22 Example 3.8.

2. Calculate a, c, and ε_t:

$$a = \frac{A_s f_y}{0.85 f_c' b} = \frac{6 \times 60}{0.85 \times 4 \times 12} = 8.824 \text{ in.}$$

$$c = \frac{a}{0.85} = 10.38 \text{ in.} \quad d_t = h - 2.5 = 27 - 2.5 = 24.5$$

$$\frac{c}{d_t} = 0.424 > 0.375$$

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) 0.003 = \left(\frac{24.5 - 10.38}{10.38}\right) 0.003 = 0.00408 > 0.004$$

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) = 0.823$$

3. Calculate

$$\phi M_n = \phi A_s f_y (d - a/2)$$

$$= 0.823(6)(60)\frac{\left(23.5 - \frac{8.824}{2}\right)}{12} = 471 \text{ K·ft}$$

Discussion

For a tension-controlled section limitation, $\rho_{max} = 0.01806$ and $R_u = 820$ psi,

$$\phi M_n = R_u b d^2 = 0.82(12)\frac{(23.5)^2}{12} = 452.8 \text{ K·ft}$$

This is a conservative value: It is advisable to choose adequate reinforcement to produce tension-controlled condition with $\phi = 0.9$.

3.14 RECTANGULAR SECTIONS WITH COMPRESSION REINFORCEMENT

In concrete sections proportioned to resist the bending moments resulting from external loading on a structural member, the internal moment is equal to or greater than the external moment, but a concrete section of a given width and effective depth has a minimum capacity when ρ_{max} is used. If the external factored moment is greater than the design moment strength, more compressive and tensile reinforcement must be added.

Compression reinforcement is used when a section is limited to specific dimensions due to architectural reasons, such as a need for limited headroom in multistory buildings. Another advantage of compression reinforcement is that long-time deflection is reduced, as is explained in Chapter 6. A third use of bars in the compression zone is to hold stirrups, which are used to resist shear forces.

Two cases of doubly reinforced concrete sections will be considered, depending on whether compression steel yields or does not yield.

3.14.1 When Compression Steel Yields

Internal moment can be divided into two moments, as shown in Fig. 3.23. M_{u_1} is the moment produced by the concrete compressive force and an equivalent tension force in steel, A_{s_1}, acting as

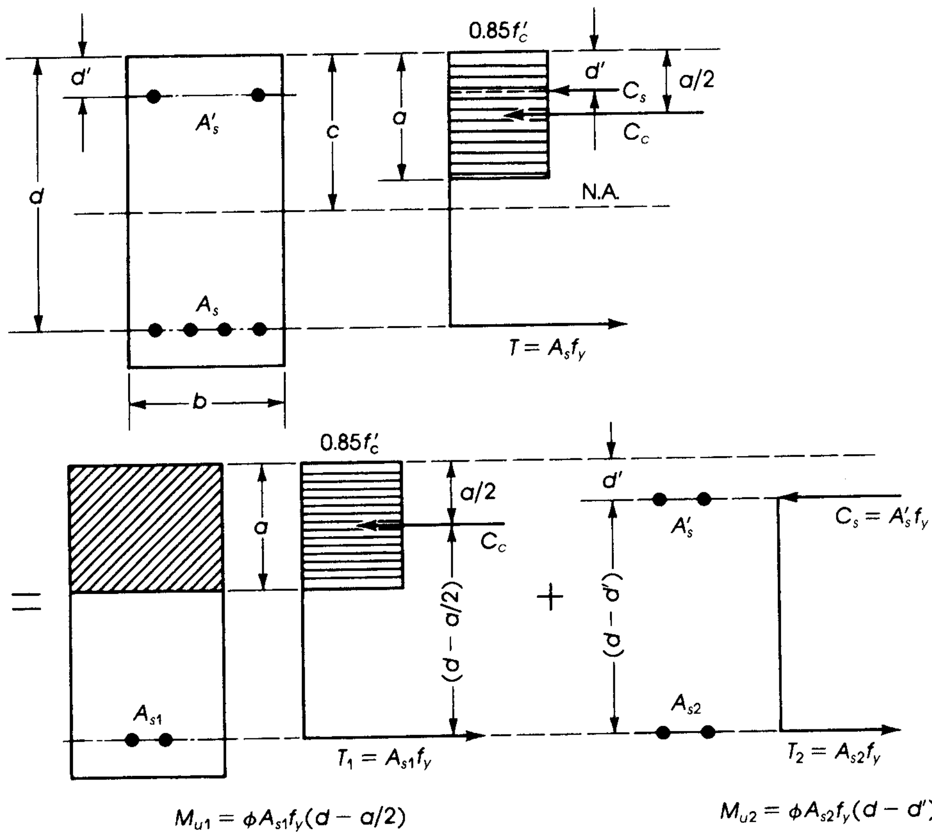


Figure 3.23 Rectangular section with compression reinforcement.

a basic section. M_{u_2} is the additional moment produced by the compressive force in compression steel A'_s and the tension force in the additional tensile steel, A_{s_2}, acting as a steel section.

The moment M_{u_1} is that of a singly reinforced concrete basic section,

$$T_1 = C_c \tag{3.36}$$

$$A_{s_1} f_y = C_c = 0.85 f'_c ab \tag{3.37}$$

$$a = \frac{A_{s_1} f_y}{0.85 f'_c b} \tag{3.38}$$

$$M_{u_1} = \phi A_{s_1} f_y \left(d - \frac{a}{2}\right) \tag{3.39}$$

The restriction for M_{u_1} is that $\rho_1 < A_{s_1}/bd$ shall be equal to or less than ρ_{max} for singly reinforced tension-controlled sections, as given in Eq. 3.31a.

Considering the moment M_{u_2} and assuming that the compression steel designated as A'_s yields,

$$M_{u_2} = \phi A_{s_2} f_y (d - d') \tag{3.40a}$$

$$M_{u_2} = \phi A'_s f_y (d - d') \tag{3.40b}$$

In this case $A_{s_2} = A'_s$, producing equal and opposite forces, as shown in Fig. 3.23. The total resisting moment, M_u, is then the sum of the two moments M_{u_1} and M_{u_2}:

$$\phi M_n = M_{u_1} + M_{u_2} = \phi\left[A_{s_1} f_y\left(d - \frac{a}{2}\right) + A'_s f_y (d - d')\right] \tag{3.41}$$

The total steel reinforcement used in tension is the sum of the two steel amounts A_{s_1} and A_{s_2}. Therefore,

$$A_s = A_{s_1} + A_{s_2} = A_{s_1} + A'_s \tag{3.42}$$

and

$$A_{s_1} = A_s - A'_s$$

Then, substituting $(A_s - A'_s)$ for A_{s_1} in Eqs. 3.38, 3.39, and 3.41,

$$a = \frac{(A_s - A'_s) f_y}{0.85 f'_c b} \tag{3.43}$$

$$\phi M_n = \phi\left[(A_s - A'_s) f_y\left(d - \frac{a}{2}\right) + A'_s f_y (d - d')\right] \tag{3.44}$$

and

$$(\rho - \rho') \le \rho_{\max} = \rho_b\left(\frac{0.003 + f_y/E_s}{0.008}\right) \tag{3.45}$$

For $f_y = 60$ ksi, $(\rho - \rho') \le 0.63375\ \rho_b$, $\phi = 0.9$, and $\varepsilon_t = 0.005$. Equation 3.45 must be fulfilled in doubly reinforced concrete sections, which indicates that the difference between total tension steel and the compression steel should not exceed the maximum steel for singly reinforced concrete tension-controlled sections. Failure due to yielding of the total tensile steel will then be expected, and sudden failure of concrete is avoided.

If $\rho_1 = (\rho - \rho') > \rho_{\max}$, the section will be in the transition region with a limit of $(\rho - \rho') \le \rho_{\max t}$ (Eq. 3.34*a*). In this case, $\phi < 0.9$ for M_{u_1} and $\phi = 0.9$ for M_{u_2}. Equation 3.44 becomes

$$\phi M_n = \phi\left[(A_s - A'_s) f_y\left(d - \frac{a}{2}\right)\right] + 0.9 A'_s f_y (d - d') \tag{3.44a}$$

Note that $(A_s - A'_s) \le \rho_{\max t}(bd)$.

In the compression zone, the force in the compression steel is $C_s = A'_s(f_y - 0.85 f'_c)$, taking into account the area of concrete displaced by A'_s. In this case,

$$T = A_s f_y = C_c + C_s = 0.85 f'_c ab + A'_s(f_y - 0.85 f'_c)$$

$$A_s f_y - A'_s f_y + 0.85 f'_c A'_s = 0.85 f'_c ab = C_c = A_{s_1} f_y \quad \text{(for the basic section)}$$

Dividing by bdf_y,

$$\rho - \rho'\left(1 - 0.85\frac{f'_c}{f_y}\right) = \rho_1, \qquad \text{where } \rho_1 = \frac{A_{s_1}}{bd} \le \rho_{\max}$$

Therefore,

$$\rho - \rho'\left(1 - 0.85\frac{f'_c}{f_y}\right) \le \rho_{\max} = \rho_b\left(\frac{0.003 + f_y/E_s}{0.008}\right) \tag{3.46}$$

Although Eq. 3.46 is more accurate than Eq. 3.45, it is quite practical to use both equations to check the condition for maximum steel ratio in rectangular sections when compression steel yields.

For example, if $f_c' = 3$ ksi and $f_y = 60$ ksi, Eq. 3.46 becomes $(\rho - 0.9575\rho') \leq 0.016$; if $f_c' = 4$ ksi and $f_y = 60$ ksi, then $(\rho - 0.9575\rho') \leq 0.02138$.

The maximum total tensile steel ratio, ρ, that can be used in a rectangular section when compression steel yields is as follows:

$$\text{Max } \rho = (\rho_{\max} + \rho') \tag{3.47}$$

where $\rho_{\max}$ is maximum tensile steel ratio for the basic singly reinforced tension-controlled concrete section. This means that maximum total tensile steel area that can be used in a rectangular section when compression steel yield is as follows:

$$\text{Max } A_s = bd(\rho_{\max} + \rho') \tag{3.47a}$$

In the preceding equations, it is assumed that compression steel yields. To investigate this condition, refer to the strain diagram in Fig. 3.24. If compression steel yields, then

$$\varepsilon_s' \geq \varepsilon_y = \frac{f_y}{E_s}$$

From the two triangles above the neutral axis, substitute $E_s = 29{,}000$ ksi and let f_y be in ksi. Then

$$\frac{c}{d'} = \frac{0.003}{0.003 + \frac{f_y}{E_s}} = \frac{87}{87 - f_y}$$

$$c = \left(\frac{87}{87 - f_y}\right) d' \tag{3.48}$$

From Eq. 3.37,

$$A_{s_1} f_y = 0.85 f_c' ab \tag{3.47}$$

but

$$A_{s_1} = A_s - A_s' \quad \text{and} \quad \rho_1 = (\rho - \rho')$$

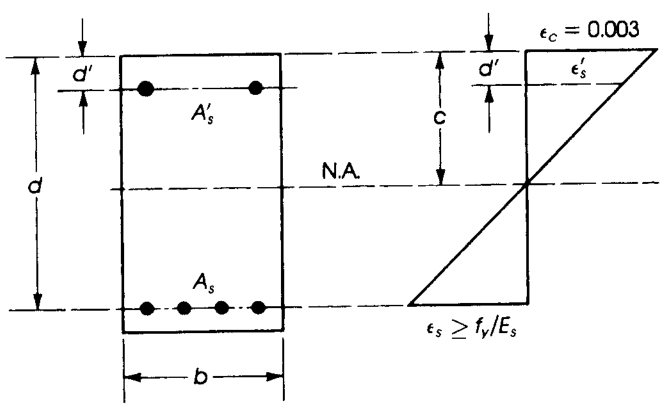


Figure 3.24 Strain diagram in doubly reinforced section.

Therefore, Eq. 3.37 becomes $(A_s - A'_s)f_y = 0.85f'_c ab$:

$$(\rho - \rho')bdf_y = 0.85f'_c ab$$

$$(\rho - \rho') = 0.85\left(\frac{f'_c}{f_y}\right)\left(\frac{a}{d}\right)$$

Also,

$$a = \beta_1 c = \beta_1\left(\frac{87}{87 - f_y}\right)d'$$

Therefore,

$$(\rho - \rho') = 0.85\beta_1\left(\frac{f'_c}{f_y}\right)\left(\frac{d'}{d}\right)\left(\frac{87}{87 - f_y}\right) = K \qquad (3.49)$$

The quantity $(\rho - \rho')$ is the steel ratio, or $(A_s - A'_s)/bd = A_{s_1}/bd = \rho_1$ for the singly reinforced basic section.

If $(\rho - \rho')$ is greater than the value of the right-hand side in Eq. 3.49, then compression steel will also yield. In Fig. 3.25 we can see that if A_{s_1} is increased, T_1 and, consequently, C_1

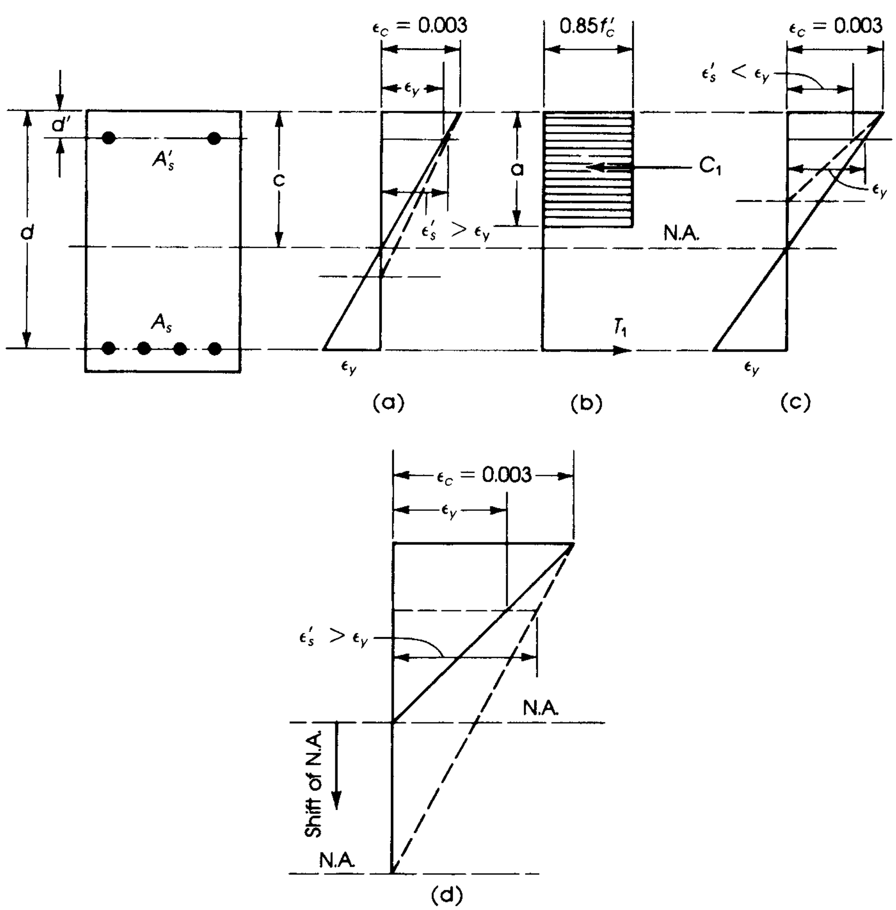


Figure 3.25 Yielding and nonyielding cases of compression reinforcement. Diagram (*d*), a close-up of (*a*), shows how the neutral axis responds to an increase in A_{s1}.

Table 3.4 Values of K for Different f'_c and f_y

f'_c (ksi)	f_y (ksi)	K	K (for $d' = 2.5$ in.)
3	40	$0.1003d'/d$	$0.251/d$
3	60	$0.1164d'/d$	$0.291/d$
4	60	$0.1552d'/d$	$0.388/d$
5	60	$0.1826d'/d$	$0.456/d$

will be greater and the neutral axis will shift downward, increasing the strain in the compression steel and ensuring its yield condition. If the tension steel used (A_{s_1}) is less than the right-hand side of Eq. 3.49, then T_1 and C_1 will consequently be smaller, and the strain in compression steel, ε'_s, will be less than or equal to ε_y, because the neutral axis will shift upward, as shown in Fig. 3.25*c*, and compression steel will not yield.

Therefore, Eq. 3.49 can be written

$$(\rho - \rho') \geq 0.85\,\beta_1 \frac{f'_c}{f_y} \times \frac{d'}{d} \times \frac{87}{87 - f_y} = K \qquad (3.49a)$$

where f_y is in ksi, and this is the condition for compression steel to yield.

For example, the values of K for different values of f'_c and f_y are as shown in Table 3.4.

Example 3.9

A rectangular beam has a width of 12 in. and an effective depth of $d = 22.5$ in. to the centroid of tension steel bars. Tension reinforcement consists of six no. 9 bars in two rows; compression reinforcement consists of two no. 7 bars placed as shown in Fig. 3.26. Calculate the design moment strength of the beam if $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. Check if compression steel yields:

$$A_s = 6.0 \text{ in.}^2 \quad \rho = \frac{A_s}{bd} = \frac{6.0}{12 \times 22.5} = 0.02222$$

$$A'_s = 1.2 \text{ in.}^2 \quad \rho' = \frac{A'_s}{bd} = \frac{1.2}{12 \times 22.5} = 0.00444$$

$$A_s - A'_s = 4.8 \text{ in.}^2 \quad \rho - \rho' = 0.01778$$

For compression steel to yield,

$$(\rho - \rho') \geq 0.85\,\beta_1 \frac{f'_c}{f_y} \times \frac{d'}{d} \times \frac{87}{87 - f_y} = K$$

β_1 is 0.85 because $f'_c = 4000$ psi; therefore,

$$K = (0.85)^2 \left(\frac{4}{60}\right)\left(\frac{2.5}{22.5}\right)\left(\frac{87}{87-60}\right) = 0.0175$$

$$(\rho - \rho') = 0.01778 > 0.0175$$

Therefore, compression steel yields.

2. Check that $(\rho - \rho') \leq \rho_{\max}$ (Eq. 3.45): For $f'_c = 4$ ksi and $f_y = 60$ ksi, $\rho_b = 0.0285$ and $\rho_{\max} = 0.01806$ (Table 3.2). $(\rho - \rho') \leq 0.01778 < \rho_{\max}$, and $\phi = 0.9$ (a tension-controlled condition).

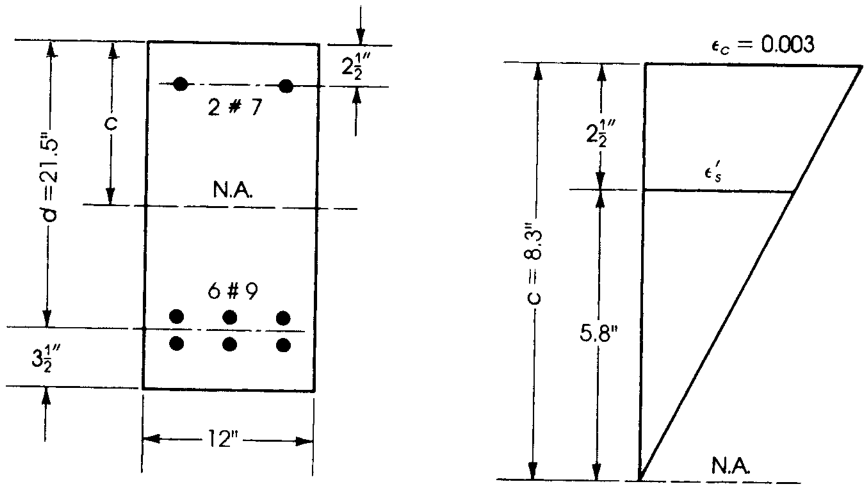


Figure 3.26 Example 3.9.

3. ϕM_n can be calculated by Eq. 3.44:

$$\phi M_n = \phi \left[(A_s - A'_s) f_y \left(d - \frac{a}{2} \right) + A'_s f_y (d - d') \right]$$

$$a = \frac{(A_s - A'_s) f_y}{0.85 f'_c b} = \frac{4.8 \times 60}{0.85 \times 4 \times 12} = 7.06 \text{ in.}$$

$$\phi M_n = (0.9) \left[4.8 \times 60 \left(22.5 - \frac{7.06}{2} \right) + 1.2 \times 60 (22.5 - 2.5) \right]$$

$$= 6213 \text{ K·in.} = 517.8 \text{ K·ft}$$

4. An alternative approach for checking if compression steel yields can be made as follows:

$$c = \frac{a}{0.85} = \frac{7.06}{0.85} = 8.3 \text{ in.}$$

$$\varepsilon'_s = \frac{5.8}{8.3} \times 0.003 = 0.0021 \qquad \varepsilon_y = \frac{f_y}{E_s} = \frac{60}{29{,}000} = 0.00207$$

Because ε'_s, exceeds ε_y, compression steel yields.

5. Check ε_t:$c = 8.3$ in., $d_t = 26 - 2.5 = 23.5$ in.

$$\varepsilon_t = \left(\frac{23.5 - 8.3}{8.3} \right) 0.003 = 0.0055 > 0.005$$

$$\text{or } \frac{c}{d} = 0.353 < 0.375 \qquad \text{(o.k.)}$$

6. The maximum total tension steel for this section, max A_s, is equal to

$$\text{Max } A_s = bd(\rho_{\max} + \rho') = 12 \times 22.5(0.01806 + 0.00444)$$

$$= 6.08 \text{ in.}^2 > A_s = 6.0 \text{ in.}^2 \qquad \text{(used in the section)}$$

3.14.2 When Compression Steel Does Not Yield

As was explained earlier, if

$$(\rho - \rho') < \left(0.85\beta_1 \times \frac{f'_c}{f_y} \times \frac{d'}{d} \times \frac{87}{87 - f_y}\right) = K \tag{3.50}$$

then compression steel does not yield. This indicates that if $(\rho - \rho') < K$, the tension steel will yield before concrete can reach its maximum strain of 0.003, and the strain in compression steel, ε'_s, will not reach ε_y at failure (Fig. 3.25). Yielding of compression steel will also depend on its position relative to the extreme compressive fibers d'. A higher ratio of d'/c will decrease the strain in the compressive steel, ε'_s, as it places compression steel A'_s nearer to the neutral axis.

If compression steel does not yield, a general solution can be performed by analysis based on statics. Also, a solution can be made as follows: Referring to Figs. 3.23 and 3.24,

$$\varepsilon'_s = 0.003\left(\frac{c - d'}{c}\right) \quad f'_s = E_s\varepsilon'_s = 29{,}000(0.003)\left(\frac{c - d'}{c}\right) = 87\left(\frac{c - d'}{c}\right)$$

Let $C_c = 0.85 f'_c \beta_1 cb$:

$$C_s = A'_s(f'_s - 0.85f'_c) = A'_s\left[87\frac{(c - d')}{c}0.85f'_c\right]$$

Because $T = A_s f_y = C_c + C_s$, then

$$A_s f_y = (0.85f'_c\beta_1 cb) + A'_s\left[87\left(\frac{c - d'}{c}\right) - 0.85f'_c\right]$$

Rearranging terms yields

$$(0.85f'_c\beta_1 b)c^2 + [(87A'_s) - (0.85f'_cA'_s) - A_s f_y]c - 87A'_s d' = 0$$

This is similar to $A_1c^2 + A_2c + A_3 = 0$, where

$$A_1 = 0.85f'_c\beta_1 b$$

$$A_2 = A'_s(87 - 0.85f'_c) - A_s f_y$$

$$A_3 = -87A'_s d'$$

Solve for c:

$$c = \frac{1}{2A_1}\left[-A_2 \pm \sqrt{A_2^2 - 4A_1A_3}\right] \tag{3.51}$$

Once c is determined, then calculate f'_s, a, C_c and C_s.
$f'_s = 87[(c - d')/c]$; $a = \beta_1 c$; $C_c = 0.85f'_c ab$; and $C_s = A'_s(f'_s - 0.85f'_c)$.

$$\phi M_n = \phi\left[C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right] \tag{3.52}$$

When compression steel does not yield, $f'_s < f_y$, and the maximum total tensile steel reinforcement needed for a rectangular section is

$$\text{Max } A_s = \rho_{\max}bd + A'_s\frac{f'_s}{f_y} = bd\left(\rho_{\max} + \frac{\rho' f'_s}{f_y}\right) \tag{3.53}$$

Using steel ratios and dividing by bd:

$$\text{Max } \rho = \frac{\max A_s}{bd} \leq \rho_{\max} + \rho' \frac{f'_s}{f_y} \tag{3.54}$$

or

$$\left(\rho - \rho' \frac{f'_s}{f_y}\right) \leq \rho_{\max} \tag{3.55}$$

where $\rho_{\max}$ is the maximum steel ratio for the tension-controlled singly reinforced rectangular section (Eq. 3.31).

In this case,

$$a = \frac{A_s f_y - A'_s f'_s}{0.85 f'_c b} \tag{3.56}$$

$$\phi M_n = \phi \left[(A_s f_y - A'_s f'_s)\left(d - \frac{a}{2}\right) + A'_s f'_s (d - d')\right] \tag{3.57}$$

In summary, the procedure for analyzing sections with compression steel is as follows:

1. Calculate ρ, ρ', and $(\rho - \rho')$. Also calculate $\rho_{\max}$ and $\rho_{\min}$.
2. Calculate

$$K = 0.85\,\beta_1 \left(\frac{f'_c}{f_y}\right)\left(\frac{87}{87 - f_y}\right)\left(\frac{d'}{d}\right)$$

   Use ksi units.
3. If $(\rho - \rho') \geq K$, then compression steel yields, and $f'_s = f_y$; if $(\rho - \rho') < K$, then compression steel does not yield, and $f'_s < f_y$.
4. If compression steel yields, then
   a. Check that $\rho_{\max} \geq (\rho - \rho') \geq \rho_{\min}$ (to use $\phi = 0.9$) or check $\varepsilon_t \geq 0.005$.
   b. Calculate

$$a = \frac{(A_s - A'_s) f_y}{0.85 f'_c b}.$$

   c. Calculate

$$\phi M_n = \phi \left[(A_s - A'_s) f_y \left(d - \frac{a}{2}\right) + A'_s f_y (d - d')\right].$$

   d. The maximum A_s that can be used in the section is Max $A_s = bd(\rho_{\max} + \rho') \geq A_s$ (used).
5. If compression steel does *not* yield, then
   a. Calculate the distance to the neutral axis c by using analysis (Example 3.10) or by using the quadratic equation (3.51).
   b. Calculate

$$f'_s = 87\left(\frac{c - d'}{c}\right) \text{ (ksi)}.$$

c. Check that $(\rho - \rho' f_s'/f_y) \leq \rho_{\max}$ or max A_s that can be used in the section is greater than or equal to the A_s used.

$$\text{Max } A_s = bd\left(\rho_{\max} + \frac{\rho' f_s'}{f_y}\right) \geq A_s \quad \text{(used)}$$

d. Calculate

$$a = \frac{(A_s f_y - A_s' f_s')}{0.85 f_c' b} \quad \text{or } a = \beta_1 c.$$

e. Calculate

$$\phi M_n = \phi\left[(A_s f_y - A_s' f_s')\left(d - \frac{a}{2}\right) + A_s' f_s'(d - d')\right].$$

For flow charts, refer to Chapter 21.

Example 3.10

Determine the design moment strength of the section shown in Fig. 3.27 using $f_c' = 5$ ksi, $f_y = 60$ ksi, $A_s' = 2.37$ in.^{2} (three no. 8 bars), and $A_s = 7.62$ in.^{2} (six no. 10 bars).

Solution

1. Calculate ρ and ρ':

$$\rho = \frac{A_s}{bd} = \frac{7.62}{14 \times 22.5} = 0.0242 \quad \rho' = \frac{A_s'}{bd} = \frac{2.37}{14 \times 22.5} = 0.00753$$

$$(\rho - \rho') = 0.01667$$

2. Apply Eq. 3.50, assuming $\beta_1 = 0.8$ for $f_c' = 5000$ psi.

$$K = 0.85\beta_1 \times \frac{f_c'}{f_y} \times \frac{d'}{d} \times \frac{87}{87 - f_y} = 0.85 \times 0.8\left(\frac{5}{60}\right)\left(\frac{2.5}{22.5}\right)\left(\frac{87}{87 - 60}\right) = 0.0203$$

(or from Table 3.3, $K = 0.456/d = 0.0203$).

$$(\rho - \rho') = 0.01667 < 0.0203$$

Therefore, compression steel does not yield, and $f_s' < 60$ ksi.

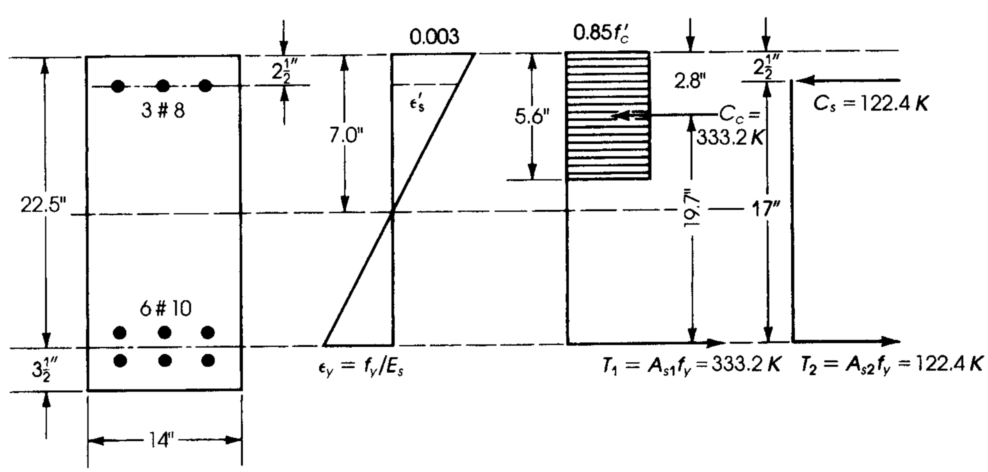


Figure 3.27 Example 3.10 analysis solution.

For $f_c' = 5$ ksi and $f_y = 60$ ksi, $\rho_b = 0.0335$ and $\rho_{\max} = 0.02123$ [Table 3.2 $(\rho - \rho') < \rho_{\max}$, for the basic section]. $\phi = 0.9$, so this is a tension-controlled condition.

3. Calculate ϕM_n by analysis. Internal forces:

$$C_c = 0.85 f_c' ab \qquad a = \beta_1 c = 0.8c$$

$$C_c = 0.85 \times 5(0.8c) \times 14 = 47.7c$$

$$C_s = \text{the force in compression steel}$$

$$= A_s' f_s' - \text{the force in displaced concrete}$$

$$= A_s'(f_s' - 0.85 f_c')$$

From strain triangles,

$$\varepsilon_s' = 0.003 \frac{c - d'}{c}$$

$$f_s' = E_s \varepsilon_s' \text{ (since steel is in the elastic range)}$$

$$= 29{,}000 \left[\frac{0.003(c - d')}{c}\right] = \frac{87(c - d')}{c} \text{ (ksi)}$$

Therefore,

$$C_s = 2.37[87 \frac{(c - d')}{c} - (0.85 \times 5)](\text{kips}) = \left[\frac{206.2(c - 2.5)}{c}\right] - 10.07$$

$$T = T_1 + T_2 = (A_{s1} + A_{s2}) f_y = S_s f_y = 7.62(60) = 457.2 \text{ kips}$$

4. Equate internal forces to determine the position of the neutral axis (the distance c):

$$T = C = C_c + C_s$$

$$457.2 = 47.6c + \frac{206.2(c - 2.5)}{c} - 10.07$$

$$c^2 - 5.48c - 10.83 = 0$$

$$c = 7.0 \text{ in.} \qquad a = 0.8c = 5.6 \text{ in.}$$

Equation 3.51 can also be used to calculate c and a.

5. Calculate f_s', C_c, and C_s:

$$f_s' = \frac{87(c - 2.5)}{c} = \frac{87(7.0 - 2.5)}{7.0} = 55.9 \text{ ksi}$$

which confirms that compression steel does not yield.

$$C_c = 47.6c = 47.6(7.0) = 333.2 \text{ kips}$$

$$C_s = (A_s' f_s' - 10.07) = 2.37(55.90) - 10.07 = 122.40 \text{ kips}$$

6. To calculate ϕM_n, take moments about the tension steel A_s:

$$\phi M_n = \phi \left[C_c \left(d - \frac{a}{2}\right) + C_s(d - d')\right] = 0.9[333.2(22.5 - 2.8) + 122.40(22.5 - 2.5)]$$

$$= 8110.8 \text{ K·in.} = 675.9 \text{ K·ft}$$

7. Check that $(\rho - \rho' f'_s / f_y) \le \rho_{\max}$ (Eq. 3.55):

$$0.0242 - 0.00754\left(\frac{55.9}{60}\right) = 0.0171 < \rho_{\max} = 0.02123$$

The maximum total tension steel that can be used in this section is calculated by Eq. 3.50

$$\max A_s = bd\left(\rho_{\max} + \frac{\rho' f'_s}{f_y}\right)$$

$$= 14(22.5)\left(0.02123 + \frac{0.00753 \times 55.9}{60}\right) = 8.9 \text{ in.}^2 > 7.62 \text{ in.}^2 \qquad \text{(o.k.)}$$

8. ε_t can be checked as follows: $c = 7.0$ in., $d_t = 23.5$ in.

$$\frac{c}{d_t} = 0.3 < 0.375 \quad \text{or}$$

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right)0.003 = \left(\frac{23.5 - 7}{7}\right)0.003 = 0.0071 > 0.005$$

Tension-controlled section.

3.15 ANALYSIS OF T- AND I-SECTIONS

3.15.1 Description

It is normal to cast concrete slabs and beams together, producing a monolithic structure. Slabs have smaller thicknesses than beams. Under bending stresses, those parts of the slab on either side of the beam will be subjected to compressive stresses, depending on the position of these parts relative to the top fibers and relative to their distances from the beam. The part of the slab acting with the beam is called the flange, and it is indicated in Fig. 3.28*a* by area bt. The rest of the section confining the area $(h - t)b_w$ is called the *stem*, or *web*.

In an I-section there are two flanges, a compression flange, which is actually effective, and a tension flange, which is ineffective, because it lies below the neutral axis and is thus neglected completely. Therefore, the analysis and design of an I-beam is similar to that of a T-beam.

3.15.2 Effective Width

In a T-section, if the flange is very wide, the compressive stresses are at a maximum value at points adjacent to the beam and decrease approximately in a parabolic form to almost 0 at a distance x from the face of the beam. Stresses also vary vertically from a maximum at the top fibers of the flange to a minimum at the lower fibers of the flange. This variation depends on the position of the neutral axis and the change from elastic to inelastic deformation of the flange along its vertical axis.

An equivalent stress area can be assumed to represent the stress distribution on the width b of the flange, producing an equivalent flange width, b_e, of uniform stress (Fig. 3.28*c*).

Analysis of equivalent flange widths for actual T-beams indicate that b_e is a function of span length of the beam [7]. Other variables that affect the effective width b_e are (Fig. 3.29).

- Spacing of beams
- Width of stem (web) of beam b_w
- Relative thickness of slab with respect to the total beam depth

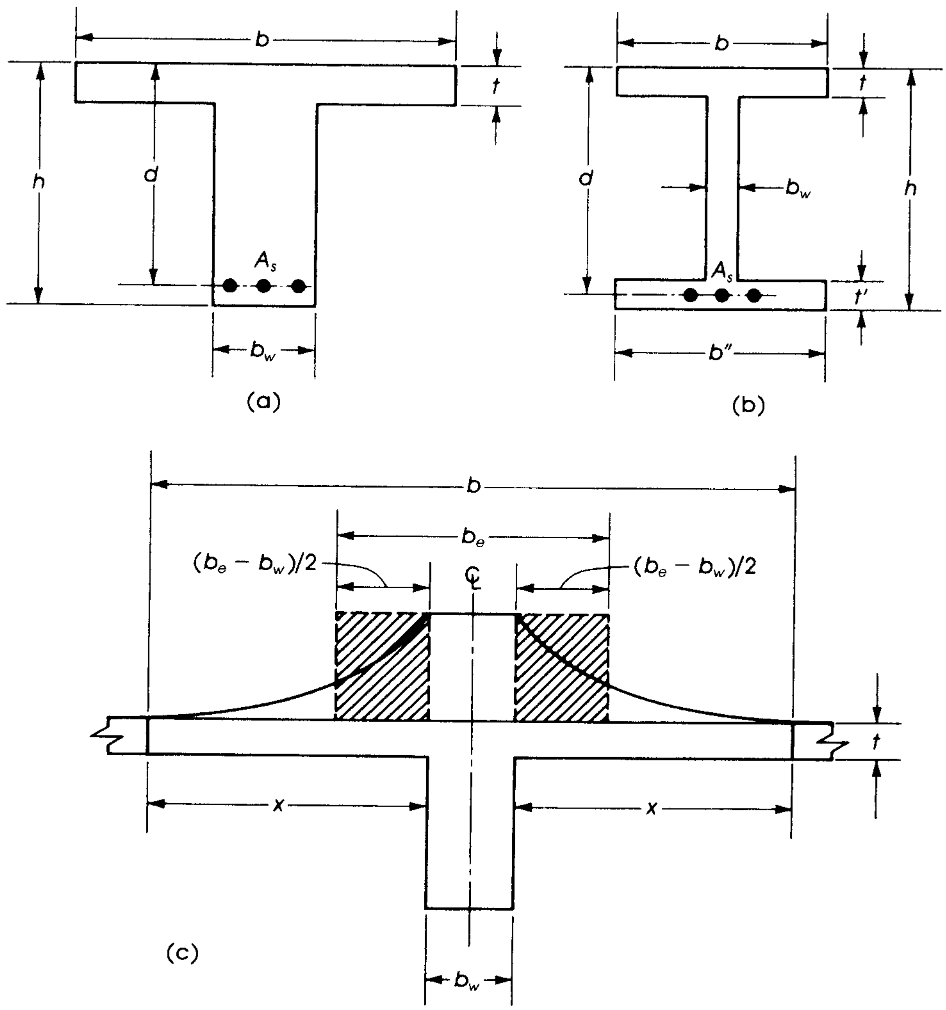


Figure 3.28 (*a*) T-section and (*b*) I-section, with (*c*) illustration of effective flange width b_e.

- End conditions of the beam (simply supported or continuous)
- The way in which the load is applied (distributed load or point load)
- The ratio of the length of the beam between points of zero moment to the width of the web and the distance between webs

The ACI Code, Section 8.10.2, prescribes the following limitations on the effective flange width b_e, considering that the span of the beam is equal to L:

1. $b_e = L/4$
2. $b_e = 16t + b_w$
3. $b_e = b$, where b is the distance between centerlines of adjacent slabs

The *smallest* of the aforementioned three values must be used.

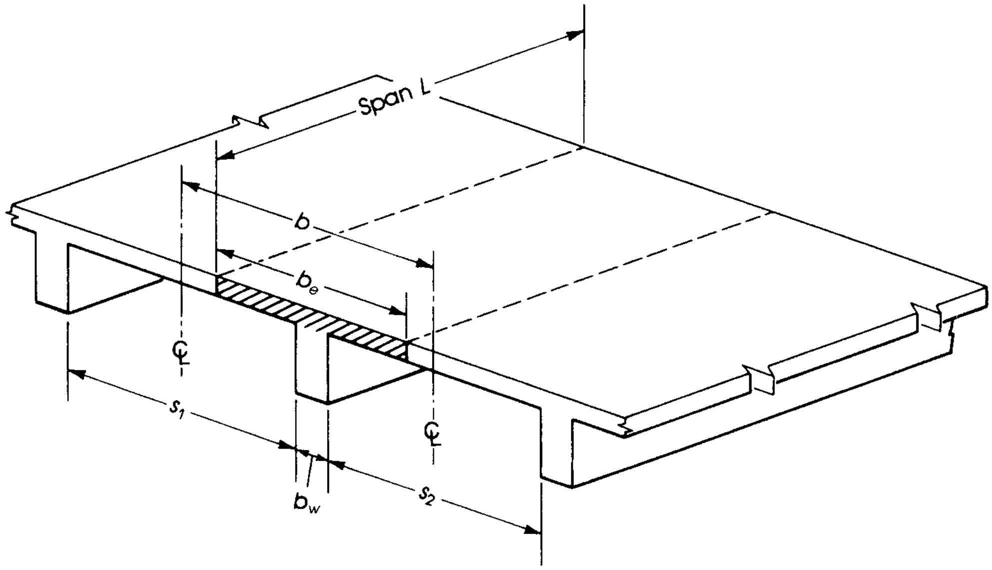


Figure 3.29 Effective flange width of T-beams.

These values are conservative for some cases of loading and are adequate for other cases. A similar effective width of flange can be adopted for I-beam sections. Investigations indicate that the effective compression flange increases as load is increased toward the ultimate value [6]. Under working loads, stress in the flange is within the elastic range.

A T-shaped or I-shaped section may behave as a rectangular section or a T-section. The two cases are investigated as follows.

3.15.3 T-Sections Behaving as Rectangular Sections

In this case, the depth of the equivalent stress block a lies within the flange, with extreme position at the level of the bottom fibers of the compression flange ($a \leq t$). When the neutral axis lies within the flange (Fig. 3.30a), the depth of the equivalent compressive distribution stress lies within the flange, producing a compressed area equal to $b_e a$. The concrete below the neutral axis is assumed ineffective, and the section is considered singly reinforced, as explained earlier, with b_e replaced by b. Therefore,

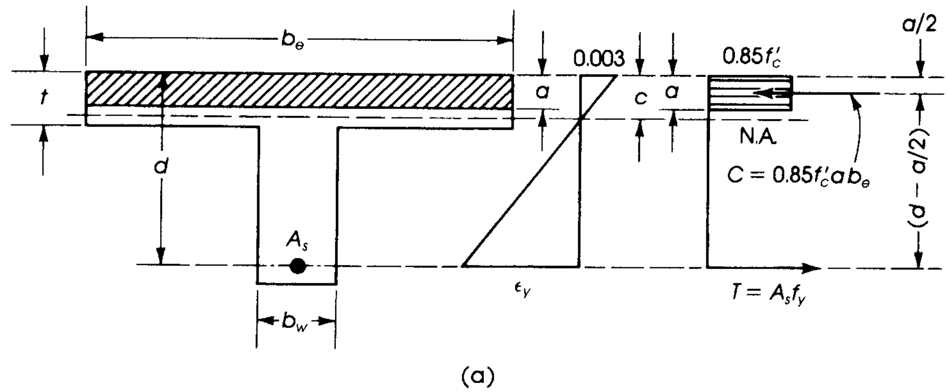
$$a = \frac{A_s f_y}{0.85 f_c' b_e} \tag{3.58}$$

and

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right) \tag{3.59}$$

If the depth a is increased such that $a = t$, then the factored moment capacity is that of a singly reinforced concrete section:

$$\phi M_n = \phi A_s f_y \left(d - \frac{t}{2}\right) \tag{3.60}$$



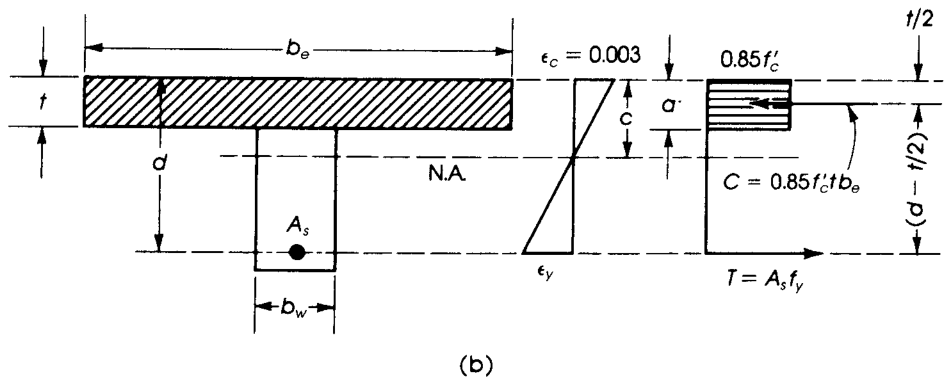


Figure 3.30 Rectangular section behavior (*a*) when the neutral axis lies within the flange and (*b*) when the stress distribution depth equals the slab thickness.

In this case

$$t = \frac{A_s f_y}{0.85 f'_c b_e} \text{ or } A_s = \frac{0.85 f'_c b_e t}{f_y} \tag{3.61}$$

In this analysis, the limit of the steel area in the section should apply: $A_s \le A_{s\,\max}$, and $\varepsilon_t \ge 0.005$.

3.15.4 Analysis of a T-Section

In this case the depth of the equivalent compressive distribution stress lies below the flange; consequently, the neutral axis also lies in the web. This is due to an amount of tension steel A_s more than that calculated by Eq. 3.61. Part of the concrete in the web will now be effective in resisting the external moment. In Fig. 3.31, the compressive force C is equal to the compression area of the flange and web multiplied by the uniform stress of 0.85 f'_c:

$$C = 0.85 f'_c [b_e t + b_w (a - t)]$$

The position of C is at the centroid of the T-shaped compressive area at a distance z from top fibers.

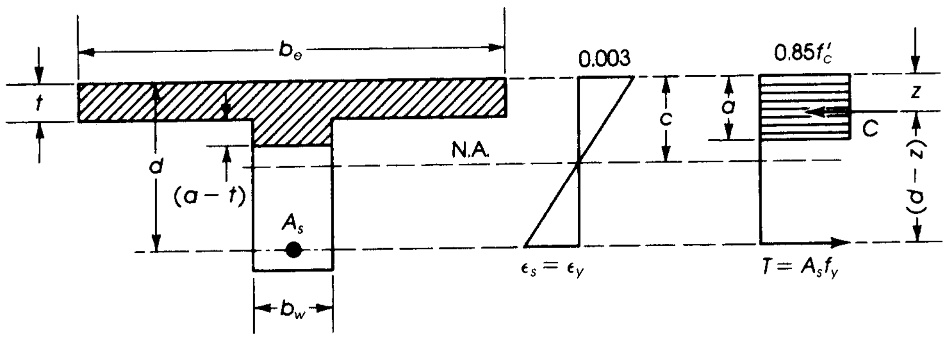


Figure 3.31 T-section behavior.

The analysis of a T-section is similar to that of a doubly reinforced concrete section, considering an area of concrete $(b_e - b_w)t$ as equivalent to the compression steel area A'_s. The analysis is divided into two parts, as shown in Fig. 3.32:

1. A singly reinforced rectangular basic section, $b_w d$, and steel reinforcement A_{s1}. The compressive force, C_1, is equal to $(0.85\ f'_c a b_w)$, the tensile force, T_1, is equal to $A_{s1} f_y$, and the moment arm is equal to $(d - a/2)$.
2. A section that consists of the concrete overhanging flange sides $2 \times [(b_e - b_w)t]/2$ developing the additional compressive force (when multiplied by $0.85\ f'_c$) and a moment arm equal to $(d - t/2)$. If A_{sf} is the area of tension steel that will develop a force equal to the

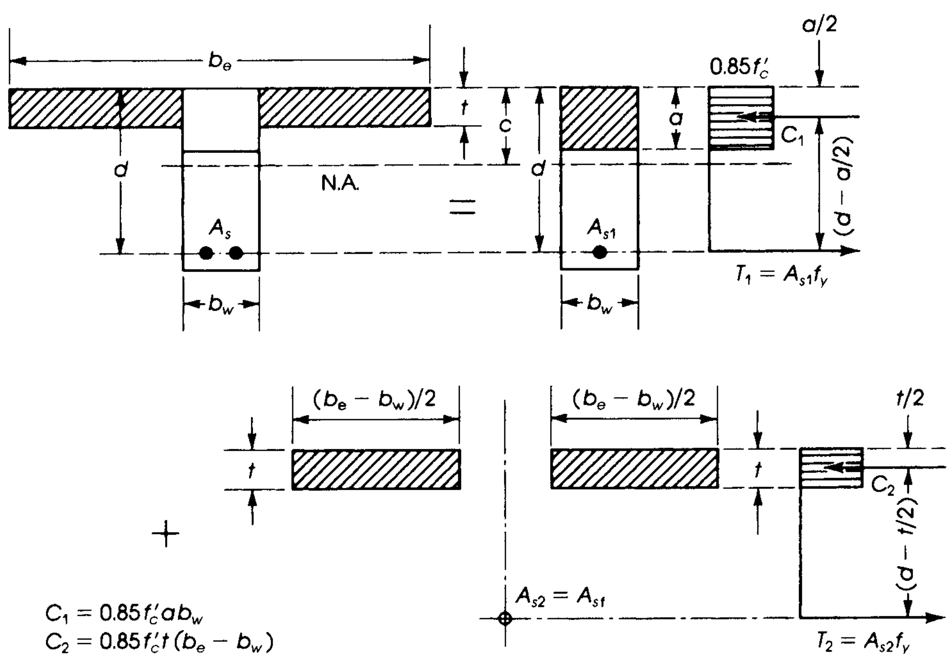


Figure 3.32 T-section analysis.

compressive strength of the overhanging flanges, then

$$A_{sf} f_y = 0.85 f_c' (b_e - b_w) t$$

$$A_{sf} = \frac{0.85 f_c' t (b_e - b_w)}{f_y} \tag{3.62}$$

The total steel used in the T-section A_s is equal to $A_{s1} + A_{sf}$, or

$$A_{s1} = A_s - A_{sf} \tag{3.63}$$

The T-section is in equilibrium, so $C_1 = T_1$, $C_2 = T_2$, and $C_1 = C_1 + C_2 = T_1 + T_2 = T$. Considering equation $C_1 = T_1$ for the basic section, then $A_{s1} f_y = 0.85\ f_c' a b_w$ or $(A_s - A_{sf}) f_y = 0.85\ f_c' a b_w$; therefore,

$$a = \frac{(A_s - A_{sf}) f_y}{0.85 f_c' b_w} \tag{3.64}$$

Note that b_w is used to calculate a. The factored moment capacity of the section is the sum of the two moments M_{u1} and M_{u2}.

$$\phi M_n = M_{u1} + M_{u2}$$

$$M_{u1} = \phi A_{s1} f_y \left(d - \frac{a}{2}\right) = \phi (A_s - A_{sf}) f_y \left(d - \frac{a}{2}\right)$$

where

$$A_{s1} = (A_s - A_{sf}) \quad \text{and} \quad a = \frac{(A_s - A_{sf}) f_y}{0.85\ f_c' b_w}$$

$$M_{u2} = \phi A_{sf} f_y \left(d - \frac{t}{2}\right) \tag{3.65}$$

$$\phi M_n = \phi \left[(A_s - A_{sf}) f_y \left(d - \frac{a}{2}\right) + A_{sf} f_y \left(d - \frac{t}{2}\right) \right]$$

Considering the web section $b_w d$, the net tensile strain (NTS), ε_t, can be calculated from a, c, and d_t as follows:
If $c = a/\beta_1$ (from Eq. 3.64) and $d_t = h - 2.5$ in., then $\varepsilon_t = 0.003\ (c - d_t)/c$. For tension-controlled section in the web, $\varepsilon_t \geq 0.005$.

The design moment strength of a T-section or I-section can be calculated from the preceding equation(3.65). It is necessary to check the following:

1. The total tension steel ratio relative to the web effective area is equal to or greater than ρ_{min}.

$$\rho_w = A_s / b_w d \geq \rho_{min}$$

$$\rho_{min} = (3\sqrt{f_c'})/f_y \geq 200/f_y \tag{3.66}$$

2. Also, check that the NTS is equal to or greater than 0.005 for tension-controlled sections.
3. The maximum tension steel (Max A_s), in a T-section must be equal to or greater than the steel ratio used, A_s, for tension-controlled sections, with $\phi = 0.9$.

$$\text{Max } A_s = A_{sf}\ (\text{flange}) + \rho_{max}(b_w d)\ (\text{web}) \tag{3.67}$$

$$\text{Max } A_s = (1/f_y)[0.85 f_c' t (b - b_w)] + \rho_{max}(b_w d) \tag{3.68}$$

In steel ratios, relative to the web only, divide Eq. 3.67 by $b_w d$:

$$\rho_w = A_s/b_w d \leq (\rho_{max} + A_{sf}/b_w d) \quad (3.69)$$

Or

$$(\rho_w - \rho_f) \leq \rho_{max} \text{ (web)} \quad (3.70)$$

where ρ_{max} is the maximum steel ratio for the basic singly reinforced web section (Table 3.2), and $\rho_f = A_{sf}/b_w d$.

A general equation for calculating (Max A_s) in a T-section when $a > t$ can be developed as follows:

$$C = 0.85 f_c'[(b_e - b_w)t + ab_w]$$

For $\varepsilon_c = 0.003$ and $\varepsilon_t = 0.005$, then $c/d = 0.003/(0.003 + 0.005) = 0.375$ (for the web). Hence, $a = \beta_1 c = 0.375\beta_1 d$.

The maximum steel area is equal to C/f_y and therefore

$$\text{Max } A_s = (0.85 f_c'/f_y)[(b_e - b_w)t + 0.375\beta_1 b_w d] \quad (3.71)$$

where Max A_s is the maximum tension steel area that can be used in a T-section when $a > t$. For example for $f_c' = 3$ ksi and $f_y = 60$ ksi, the preceding equation is reduced to:

$$\text{Max } A_s = 0.0425[(b_e - b_w)t + 0.319 b_w d] \quad (3.72)$$

For $f_c' = 4$ ksi and $f_y = 60$ ksi,

$$\text{Max } A_s = 0.0567[(b_e - b_w)t + 0.319 b_w d] \quad (3.73)$$

In summary, the procedure to analyze a T-section or inverted L-section is as follows:

1. Determine the effective width of the flange b_e (refer to Section 3.15.3). Calculate ρ_{max} and ρ_{min} (or take from tables).

2. Check if $a \leq t$ as follows: $a' = A_s f_y/(0.85 f_c' b_e)$

3. If $a' \leq t$, it is a rectangular section analysis.

  a. Calculate $\phi M_n = \phi A_s f_y(d - a/2)$, $a = a'$
  Note that $c = a/\beta_1$ and $\varepsilon_t = 0.003\ (d_t - c)/c \geq 0.005$ for tension-controlled section and $\phi = 0.9$.

  b. Check that $\rho_w = A_s/b_w$d $\geq \rho_{min}$.

  c. Max A_s can be calculated from Eq. 3.68 and should be $\geq A_s$ used. When $a < t$, normally this condition is met.

4. If $a' > t$, it is a T-section analysis:

  a. Calculate $A_{sf} = 0.85 f_c' t(b_e - b_w)/f_y$

  b. Check that $(\rho_w - \rho_f) \leq \rho_{max}$ (relative to the web area), where

$$\rho_w = A_s/b_w d \quad \text{and} \quad \rho_f = A_{sf}/b_w d$$

  Or check that Max $A_s \geq A_s$ used in the section, for $\phi = 0.9$, (Eq. 3.71).

  c. Check that $\rho_w = A_s/b_w d \geq \rho_{min}$. This condition is normally met when $a > t$.

  d. Calculate $a = (A_s - A_{sf})/0.85 f_c' b_w$ (for the web section).

  e. Calculate ϕM_n from Eq. 3.65.

Example 3.11

A series of reinforced concrete beams spaced at 7 ft, 10 in. on centers have a simply supported span of 15 ft. The beams support a reinforced concrete floor slab 4 in. thick. The dimensions and reinforcement of the beams are shown in Fig. 3.33. Using $f_c' = 3$ ksi and $f_y = 60$ ksi, determine the design moment strength of a typical interior beam.

Solution

1. Determine the effective flange width b_e. The effective flange width is the smallest of

$$16t + b_w = (16 \times 4) + 10 = 74 \text{ in.}$$

$$\text{Span}/4 = 15 \times 12/4 = 45 \text{ in.}$$

$$\text{Center to center of beams} = (7 \times 12) + 10 = 94 \text{ in.}$$

Therefore, $b_e = 45$ in. controls.

2. Check the depth of the stress block. If the section behaves as a rectangular one, then the stress block lies within the flange (Fig. 3.30). In this case, the width of beam used is equal to 45 in.

$$a' = A_s f_y/(0.85 f_c' b) = 2.37 \times 60/(0.85 \times 3 \times 45) = 1.24 \text{ in.} < t$$

Therefore, it is a rectangular section with $a = a' = 1.24$ in.

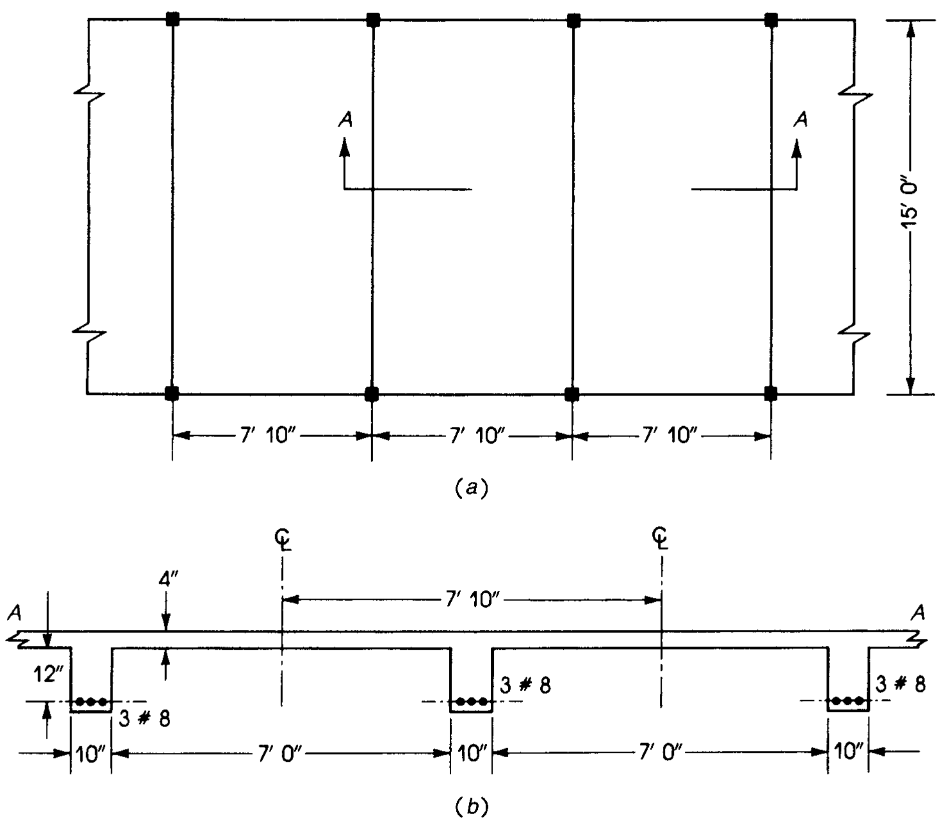


Figure 3.33 Example 3.11: (*a*) plan of slab-beam roof and (*b*) section *A*-*A*.

3. Check that

$$\rho_w = \frac{A_s}{b_w d} \geq \rho_{min} = 0.00333$$

$$\rho_w = \frac{2.37}{(10 \times 16)} = 0.0148 > 0.00333$$

4. Check ε_t : $a = 1.24$ in., $c = 1.24/0.85 = 1.46$ in., $d_t = d = 16$ in.

$$\varepsilon_t = 0.003(d_t - c)/c = 0.003(16 - 1.46)/1.46 = 0.0299 > 0.005, \phi = 0.9$$

5. Calculate $\phi M_n = \phi A_s f_y (d - a/2) = 0.9(2.37)(60)(16 - 1.24/2)$

$$= 1968 \text{ K·in.} = 164 \text{ K·ft.}$$

6. You may check that A_s used is less than or equal to A_s (Eq. 3.72), which is not needed when $a < t$:

$$\text{Max } A_s = 0.0425[(45 - 10) + 0.31 \times 10 \times 16] = 8.11 \text{ in.}^2 > 2.37 \text{ in.}^2$$

Example 3.12

Calculate the design moment strength of the T-section shown in Fig. 3.34 using $f'_c = 3.5$ ksi and $f_y = 60$ ksi.

Solution

1. Given $b = b_e = 36$ in., $b_w = 10$ in., $d = 17$ in., and $A_s = 6.0$ in.^{2}, check if $a \leq t$:

$$a' = A_s f_y/(0.85 f'_c b) = 6 \times 60/(0.85 \times 3.5 \times 36) = 3.36 \text{ in.}$$

Since $a' > t$, it is a T-section analysis.

2. $A_{sf} = 0.85 f'_c t(b - b_w)/f_y = 0.85 \times 3.5 \times 3(36 - 10)/60 = 3.87$ in.^{2}.$(A_s - A_{sf}) = A_{s1}$ (web) $= 6 - 3.87 = 2.13$ in.^{2}
3. Check ε_t:a (web) $= A_{s1} f_y/(0.85 f'_c b_w) = 2.13 \times 60/(0.85 \times 3.5 \times 10) = 4.3$ in. c $= 4.3/0.85 = 5.06$ in., $d_t = 20.5 - 2.5 = 18$ in., and $c/d_t = 0.281 < 0.375$ Or, $\varepsilon_t = 0.003(d_t - c)/c = 0.0077 > 0.005$, then $\phi = 0.9$
4. Check that $A_s > A_{s\,min}$, $\rho_{min} = 0.00333$

$$A_{s\,min} = 0.00333 \times 10 \times 17 = 0.57 \text{ in.}^2$$

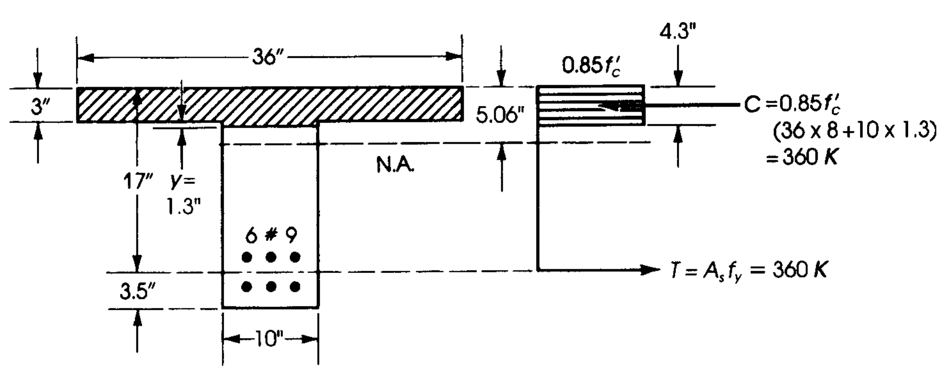


Figure 3.34 Example 3.12.

5. Calculate ϕM_n using Eq. 3.65:

$$\phi M_n = \phi\left[(A_s - A_{sf})f_y\left(d - \frac{a}{2}\right) + A_{sf}f_y\left(d - \frac{t}{2}\right)\right]$$

$$= 0.9\left[2.13 \times 60\left(17 - \frac{4.3}{2}\right) + 3.87 \times 60\left(17 - \frac{3}{2}\right)\right]$$

$$= 4947 \text{ K·in.} = 412.3 \text{ K·ft}$$

Another approach to check whether $a \le t$ is to calculate the tension force, $T = A_s f_y$, and compare it to the compressive force in the total flange (Fig. 3.34).

$$T = A_s f_y = 60 \times 60 = 360 \text{ K}$$

$$C = 0.85 f_c' t b_e = 0.85 \times 3.5 \times 3 \times 36 = 321.3 \text{ K}$$

Since T exceeds C, then $a \le t$, and the section acts as a T-section.

An additional area of concrete should be used to provide the difference of (360 − 321.3) = 38.7 K. This area has a width of 10 in. and a depth of y. Therefore,

$$b_w y(0.85 f_c') = 38.7 \text{ K or } 10(y)(0.85 \times 3.5) = 38.7 \text{ K}$$

$y = 1.3$ in., and $a = y + t = 1.3 + 3 = 4.3$ in., as calculated earlier.

3.16 DIMENSIONS OF ISOLATED T-SHAPED SECTIONS

In some cases, isolated beams with the shape of a T-section are used in which additional compression area is provided to increase the compression force capacity of sections. These sections are commonly used as prefabricated units.

The ACI Code, Section 8.10.4, specifies the size of isolated T-shaped sections as follows:

1. Flange thickness, t, shall be equal to or greater than one-half of the width of the web, b_w.
2. Total flange width b shall be equal to or less than four times the width of the web, b_w (Fig. 3.35).

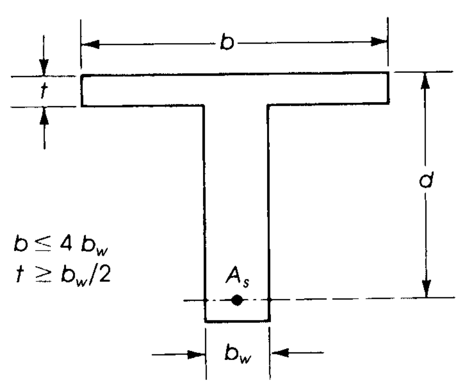


Figure 3.35 Isolated T-shaped sections.

3.17 INVERTED L-SHAPED SECTIONS

In slab-beam girder floors, the end beam is called a *spandrel beam*. This type of floor has part of the slab on one side of the beam and is cast monolithically with the beam. The section is unsymmetrical under vertical loading (Fig. 3.36*a*). The loads on slab S_1 cause torsional moment uniformly distributed on the spandrel beam B_1. Design for torsion is explained later. The over-hanging flange width $(b - b_w)$ of a beam with the flange on one side only is limited by the ACI Code, Section 8.10.2, to the smallest of the following:

1. One-twelfth of the span of the beam
2. Less than or equal to six times the thickness of the slab
3. Less than or equal to one-half the clear distance to the next beam.

If this is applied to the spandrel beam in Fig. 3.36*b*, then

1. $(b - 12) \leq (20 \times 12)/12 = 20\,\text{in.}$ (controls)
2. $(b - 12) \leq 6 \times 6 = 36\,\text{in.}$
3. $(b - 12) \leq 3.5 \times 12 = 42\,\text{in.}$

Therefore, the effective flange width is $b = 20 + 12 = 32\,\text{in.}$, and the effective dimensions of the spandrel beam are as shown in Fig. 3.36*d*.

3.18 SECTIONS OF OTHER SHAPES

Sometimes a section different from the previously defined sections is needed for special requirements of structural members. For instance, sections such as those shown in Fig. 3.37 may be used in the precast concrete industry. The analysis of such sections is similar to that of a rectangular section, taking into consideration the area of the removed or added concrete. The next example explains the analysis of such sections.

Example 3.13

The section shown in Fig. 3.38 represents a beam in a structure containing prefabricated elements. The total width and total depth are limited to 14 and 21 in., respectively. Tension reinforcement used is four no. 9 bars. Using $f'_c = 4$ ksi and $f_y = 60$ ksi., determine the design moment strength of the section.

Solution

1. Determine the position of the neutral axis based on $T = 4 \times 60 = 240$ K.

$$240 = 0.85 f'_c [2(4 \times 5) + 14(a - 4)]$$

where $a =$ depth of the equivalent compressive block needed to produce a total compressive force of 240 K.

If $240 = (0.85 \times 4)(40 + 14a - 56)$, then $a = 6.18\,\text{in.}$ and $c = a/0.85 = 7.28\,\text{in.}$

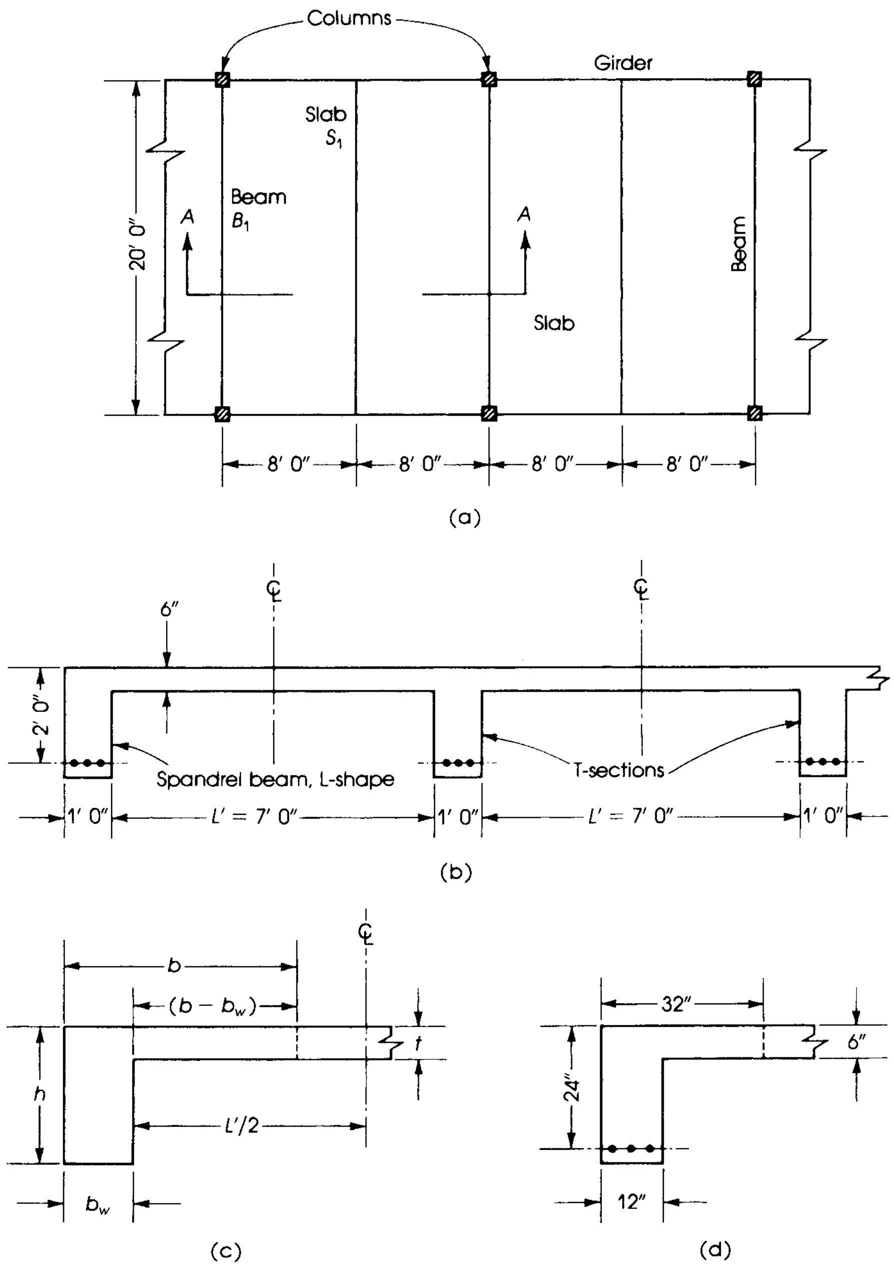


Figure 3.36 Slab-beam-girder floor, showing (*a*) plan, (*b*) section including spandrel beam, (*c*) dimensions of the spandrel beam, and (*d*) its effective flange width.

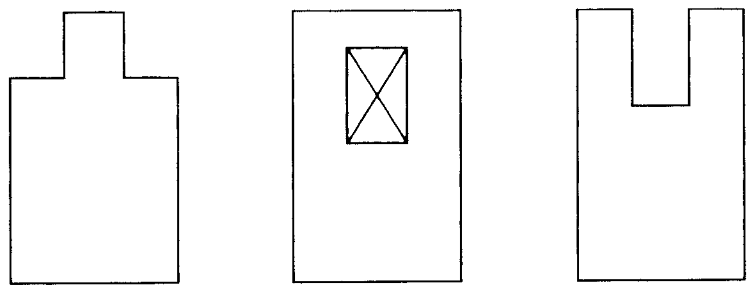


Figure 3.37 Sections of other shapes.

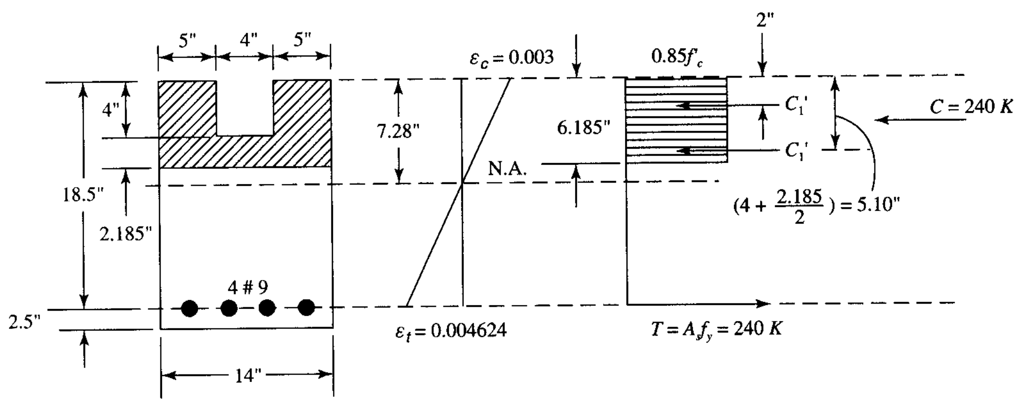


Figure 3.38 Example 3.13: (*a*) balanced and (*b*) under-reinforced sections.

2. Calculate M_n by taking moments of the two parts of the compressive forces (each by its arm), about the tension steel.

$$C_1' = \text{compressive force on the two small areas, } 4 \times 5 \text{ in.}$$
$$= 0.85 \times 4\,(2 \times 4 \times 5) = 136 \text{ K.}$$
$$C_1'' = \text{compressive force on area, } 14 \times 2.185$$
$$= 0.85 \times 4 \times 14 \times 2.185 = 104 \text{ K.}$$
$$M_n = C_1'(d - 2) + C_1''(d - 5.10)$$
$$= 136 \times 16.5 + 104 \times 13.4 = 3637.6 \text{ K·in.} = 303.1 \text{ K·ft}$$

3. Calculate $\varepsilon_t = 0.003(d_t - c)/c$, where $d_t = 18.5$ in.

$$\varepsilon_t = 0.003(18.5 - 7.28)/7.28 = 0.004624 < 0.005 \text{ but} > 0.004$$

Section is in the transition region and $\phi < 0.9$.

$$\phi = 0.48 + 83\varepsilon_t = 0.864$$
$$\phi M_n = 0.864(303.1) = 261.9 \text{ K·ft}$$

3.19 ANALYSIS OF SECTIONS USING TABLES

Reinforced concrete sections can be analyzed and designed using tables shown in Appendix A (for U.S. customary units) and Appendix B (for SI units). The tables give the value of R_u as related to the steel ratio, ρ, in addition to the maximum and minimum values for ρ and R_u. When the section dimensions are known, R_u is calculated; then ρ and A_s are determined from tables. The values in the tables are calculated based on tension-controlled sections with $\phi = 0.9$. If ϕ is less than 0.9 (transition region), the values of R_u should be multiplied by the ratio $\phi/0.9$.

$$\phi M_n = R_u bd^2 \qquad R_u = M_u/bd^2 = \phi\rho f_y[1 - \rho f_y/1.7f_c']$$

$$A_s = \rho bd \quad \text{and} \quad \rho = A_s/bd$$

For any given value of ρ, R_u can be determined from tables. Then ϕM_n can be calculated. The values of ρ and R_u range between a minimum value of R_u (min) when ρ minimum is used, to a maximum value as limited by the ACI Code, when ρ is equal to ρ (max), for tension controlled sections with $\phi = 0.9$.

The use of tables will reduce the manual calculation time. The next example explains the use of tables.

Example 3.14

Calculate the design moment strength of the section shown in Example 3.2, Fig. 3.14 using tables. Use $b = 12$ in., $d = 21$ in., $f_c' = 3$ ksi, $f_y = 60$ ksi and three no. 9 bars.

Solution

1. Using three no. 9 bars, $A_s = 3.0$ in.^{2}, $\rho = A_s/bd = 3.0/(12 \times 21) = 0.0119$. From Table 3.2, $\rho_{max} = 0.01356 > \rho$ used. Therefore, $\phi = 0.9$, and it is a tension-controlled section.
   From Table A1, for $\rho = 0.0119$, $f'_c = 3$ ksi and $f_y = 60$ ksi, get $R_u = 553$ psi (by interpolation).
2. Calculate $\phi M_n = R_u bd^2 = 0.553\ (12)(21)^2 = 2926$ K·in. $= 243.8$ K·ft

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3.20 ADDITIONAL EXAMPLES

The following examples are introduced to enhance the understanding of the analysis and design applications.

Example 3.15

Calculate the design moment strength of the precast concrete section shown in Fig. 3.39 using $f_c' = 4$ ksi and $f_y = 60$ ksi.

Solution

1. The section behaves as a rectangular section with $b = 14$ in., and $d = 21.5$ in. Note that the width b is that of the section on the compression side.
2. Check that $\rho = A_s/bd = 5/(14 \times 21.5) = 0.01661$, which is less than the maximum steel ratio of 0.018 for tension-controlled sections. Therefore, $\phi = 0.9$. Also $\rho > \rho_{min} = 0.00333$. Therefore, ρ is within the limits of a tension-controlled section.
3. Calculate a : $a = A_s f_y/(0.85 f_c' b) = 5 \times 60/(0.85 \times 4 \times 14) = 6.3$ in.

$$\phi M_n = \phi A_s f_y(d - a/2) = 0.9 \times 5 \times 60(21.5 - 6.3/2) = 4954.5 \text{ K·in} = 412.9 \text{ K·ft.}$$

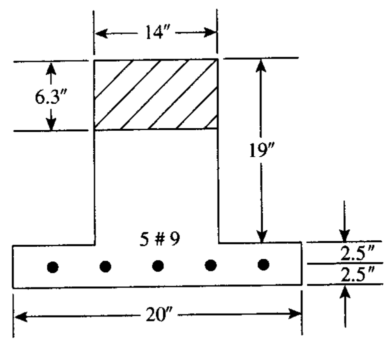

Figure 3.39 Example 3.15.

Example 3.16

A reinforced concrete beam was tested to failure and had a rectangular section, $b = 14$ in., and $d = 18.5$ in. At ultimate moment (failure), the strain in the tension steel was recorded and was equal to 0.004106. The strain in the concrete at failure may be assumed to be 0.003. If $f_c' = 3$ ksi and $f_y = 60$ ksi, it is required to:

1. Check if the tension steel has yielded.
2. Calculate the steel area provided in the section to develop the above strains. Then calculate the applied moment.
3. Calculate the design moment strength based on the ACI Code provisions. (Refer to Fig. 3.40.)

Solution

1. Check the strain in the tension steel relative to the yield strain. The yield strain $\varepsilon_y = f_y/E_s = 60/29{,}000 = 0.00207$. The measured strain in the tension steel is equal to 0.004106, which is much greater than 0.00207, indicating that the steel bars have yielded and in the elastoplastic range. The concrete strain was 0.003 indicating that the concrete has failed and started to crush. Therefore, the tension steel has yielded.
2. Calculate the depth of the neutral axis c from the strain diagram. (Fig. 3.40). From the triangles of the strain diagram,

$$c/d = 0.003/(0.003 + 0.004106) \quad \text{and} \quad c = 18.5\left(\frac{3}{7.106}\right) = 7.81 \text{ in.}$$

$$a = \beta_1 c = 0.85 \times 7.81 = 6.64 \text{ in.}$$

The compression force in the concrete, $C_c = 0.85\, f_c' ab = 0.85 \times 3 \times 6.64 \times 14 = 237$ K. The tension steel $A_s = C_c/f_y = 237/60 = 3.95 \text{ in.}^2$ (section has five no. 8 bars).

$$M_n = A_s f_y\left(d - \frac{a}{2}\right) = 3.95 \times 60(18.5 - 6.64/2) = 3597.6 \text{ K·in} = 299.8 \text{ K·ft}$$

3. Check $\varepsilon_t = 0.003(d_t - c)/c$.

$$c = 7.81 \text{ in.}, d_t = h - 2.5 \text{ in.} = 22 - 2.5 = 19.5 \text{ in.}$$

$\varepsilon_t = 0.003(19.5 - 7.81)/7.81 = 0.0049$, which is less than 0.005 for tension-controlled sections, but greater than 0.004. Section is in the transition region, and $\phi < 0.9$.

$$\phi = 0.48 + 83\varepsilon_t = 0.853$$

The allowable design moment $= \phi M_n = 0.863 \times 299.8 = 255.6$ K·ft.

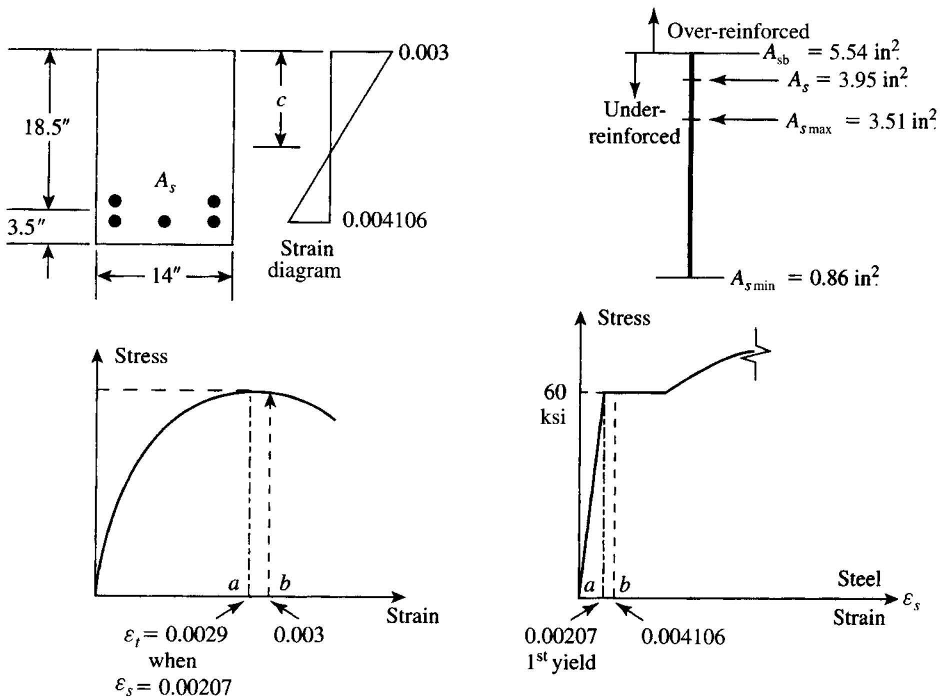

Figure 3.40 Example 3.16.

Discussion

From Table 3.2, $\rho_b = 0.0214$ and $\rho_{\max} = 0.01356$. For comparison, A_s (max) $= 0.01356(14 \times 18.5) = 3.51\,\text{in.}^2$ for $\phi = 0.9$, and A_s (balanced) $= 5.54\,\text{in.}^2$. The ratio of $A_s/A_{s\,\max} = 3.95/3.51 = 1.125$ and $A_s/A_{\text{sb}} = 0.713$. If $A_s = A_{\max} = 3.51\,\text{in.}^2$ is used with $\phi = 0.9$, then

$$a = 3.51 \times 60/(0.85 \times 3 \times 14) = 5.9\text{ in.}$$

and

$$\phi M_n = 0.9 \times 3.51 \times 60(18.5 - 5.9/2) = 2947.2\text{ K·in.} = 245.6\text{ K·ft.}$$

which is equal to 96% of the moment calculated above. Figure 3.40 shows the behavior of the tested beam.

3.21 EXAMPLES USING SI UNITS

The following equations are some of those mentioned in this chapter but converted to SI units. The other equations, which are not listed here, can be used for both U.S. Customary and SI units. Note that f'_c and f_y are in MPa (N/mm^2).

$$\rho_b = 0.85\beta_1(f'_c/f_y)[600/(600 + f_y)] \tag{3.18}$$

For tension-controlled condition,

$$\rho_{max} = (0.003 + f_y/E_s)\rho_b/0.008 \tag{3.31}$$

$$(\rho - \rho') \geq 0.85\beta_1(f_c'/f_y)(d'/d)[600/(600 - f_y)] = K \tag{3.49}$$

Example 3.17

Determine the design moment strength and the position of the neutral axis of a rectangular section that has $b = 300$ mm, $d = 500$ mm, and is reinforced with five 20-mm-diameter bars. Given $f_c' = 20$ MPa and $f_y = 400$ MPa.

Solution

1. Area of five 20-mm bars is 1570 mm^2.

$$\rho = A_s/bd = 1570/(300 \times 500) = 0.01047 \quad \rho_{min} = 1.4/f_y = 0.0035$$

For $f_c' = 20$ MPa and $f_y = 400$ MPa, $\rho_b = 0.0217$ and $\rho_{max} = 0.01356$. Note that $E_s = 200{,}000$ MPa and $f_y/E_s = 0.002$. Because $\rho < \rho_{max}$, it is a tension-controlled section with $\phi = 0.9$. Also $\rho > \rho_{min}$.

2. Calculate the design moment strength:

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right)$$

$$a = A_s f_y/(0.85 f_c' b) = 1570 \times 400/(0.85 \times 20 \times 300) = 123 \text{ mm}$$

$$\phi M_n = 0.9 \times 1570 \times 400 \left(500 - \frac{123}{2}\right) \times 10^{-6} = 247.8 \text{ KN·m}$$

Note that the moment was multiplied by 10^{-6} to get the answer in KN·m. The distance to the neutral axis from the compression fibers $(c) = a/\beta_1$, where $\beta_1 = 0.85$ for $f_c' < 30$ MPa. Therefore, $c = 123/0.85 = 145$ mm.

Example 3.18

A 2.4-m-span cantilever beam has a rectangular section with $b = 300$ mm, $d = 490$ mm, and is reinforced with three bars, 25 mm in diameter. The beam carries a uniform dead load (including its own weight) of 25.5 KN/m and a uniform live load of 32 KN/m. Check the adequacy of the section if $f_c' = 30$ MPa and $f_y = 400$ MPa.

Solution

1. $U = 1.2D + 1.6L = 1.2 \times 25.5 + 1.6 \times 32 = 81.8$ KN/m. External factored moment $= M_u = UL^2/2 = 81.8(2.4^2)/2 = 235.6$ KN·m.
2. Calculate the design moment strength:

$$A_s = 1470 \text{ mm}^2 \quad \rho = A_s/bd = 1470/(300 \times 490) = 0.01$$

$$\rho_b = 0.85\beta_1(f_c'/f_y)[600/(600 + f_y)] = 0.0325$$

$$\rho_{max} = (0.005/0.008)\rho_b = \left(\frac{5}{8}\right)(0.0325) = 0.0203, \rho_{min} = \frac{1.4}{400} = 0.0035$$

Since $\rho < \rho_{max}$ but $> \rho_{min}$, it is a tension-controlled section and $\phi = 0.9$. $a = A_s f_y/(0.85 f_c' b) = 1470 \times 400/(0.85 \times 30 \times 300) = 77$ mm, $c = 90$ mm. $\phi M_n = \phi A_s f_y(d - a/2) = 0.9 \times 1470 \times 400(490 - 77/2) \times 10^{-6} = 238.9$ KN·m. $\varepsilon_t = 0.003(d_t - c)/c = 0.003\ (490 - 90)/90 = 0.01333 > 0.005$, $\phi = 0.9$ as assumed.

3. The internal design moment strength is greater than the external factored moment. Therefore, the section is adequate.

Example 3.19

Calculate the design moment strength of a rectangular section with the following details: $b = 250$ mm, $d = 440$ mm, $d' = 60$ mm, tension steel is six bars 25 mm in diameter (in two rows), compression steel is three bars 20 mm in diameter, $f_c' = 20$ MPa and $f_y = 350$ MPa.

Solution

1. Check if compression steel yields:

$$A_s = 490 \times 6 = 2940 \text{ mm}^2, \quad A_s' = 314 \times 3 = 942 \text{ mm}^2 \quad A_s - A_s' = 1998 \text{ mm}^2$$

$$\rho = 2940/(250 \times 440) = 0.0267 \quad \rho' = 942/(250 \times 440) = 0.00856$$

$$\rho - \rho' = 0.01814.$$

For compression steel to yield:

$$(\rho - \rho') \geq 0.85 \times 0.85 \times (20/350)(60/440)(600/600 - 350) = 0.01351$$

$$(\rho - \rho') = 0.01814 > 0.01351. \text{ Therefore, compression steel yields.}$$

2. Calculate M_n:

$$a = (A_s - As')/0.85 f_c' b = 1998/(0.85 \times 20 \times 250) = 164 \text{ mm}$$

$$M_n = [1998 \times 350 \left(440 - \frac{164}{2}\right) + 942 \times 350(440 - 60)] \times 10^{-6} = 417.3 \text{ KN·m}$$

3. Check ϕ based on $\varepsilon_t \geq 0.005$.

$$\varepsilon_t = 0.003(d_t - c)/c \quad a = 164 \text{ mm} \quad c = 164/0.85 = 193 \text{ mm}$$

$$d_t = h - 65 \text{ mm} = d + 25 \text{ mm for two rows of tension bars.}$$

$$d_t = 440 + 25 = 465 \text{ mm}$$

$\varepsilon_t = 0.003(465 - 193)/193 = 0.04228$, which is less than 0.005, but greater than the 0.004 limit. $\phi = 0.48 + 83 \times \varepsilon_t, = 0.831$, and $\phi M_n = 0.831\ (417.3) = 346.8$ KN·m.

SUMMARY

Flow charts for the analysis of sections are given on www.wiley.com/college/hassoun.

Section 3.1–3.8

1. The type of failure in a reinforced concrete flexural member is based on the amount of tension steel used, A_s.
2. Load factors for dead and live loads are $U = 1.2D + 1.6L$. Other values are given in the text.
3. The reduction strength factor for beams (ϕ) = 0.9 for tension controlled sections with $\varepsilon_t \geq 0.005$.
4. An equivalent rectangular stress block can be assumed to calculate the design moment strength of the beam section, ϕM_n.
5. Design provisions are based on four conditions, Section 3.5

Section 3.9–3.13: Analysis of a Singly Reinforced Rectangular Section

Given: f'_c, f_y, b, d, and A_s. Required: the design moment strength, ϕM_n.

To determine the design moment strength of a singly reinforced concrete rectangular section,

1. Calculate the compressive force, $C = 0.85 f'_c ab$ and the tensile force, $T = A_s f_y$. Calculate $a = A_s f_y/(0.85 f'_c b)$. Calculate $\phi M_n = \phi C(d - a/2) = \phi T(d - a/2) = \phi A_s f_y(d - a/2)$. Check $\varepsilon_t = 0.003(d_t - c)/c \geq 0.005$ for $\phi = 0.9$ (tension-controlled section). (See Section 3.6.)
2. Calculate the balanced, maximum, and minimum steel ratios:

$$\rho_b = 0.85\beta_1(f'_c/f_y)[87/(87 + f_y) \quad \rho_{\max} = (0.003 + f_y/E_s)\rho_b/0.008$$

$$\rho_{\min} = 0.2/f_y \text{ for } f'_c \leq 4.5 \text{ ksi}$$

(where f'_c and f_y are in ksi. (See Section 3.9.2.) The steel ratio in the section is $\rho = A_s/bd$. Check that $\rho_{\min} \leq \rho \leq \rho_{\max}$.

3. Another form of the design moment strength is

$$M_n = \rho f_y(bd^2)(1 - \rho f_y/1.7 f'_c) = R_n bd^2$$

$$R_n = \rho f_y[1 - (\rho f_y/1.7 f'_c)] \quad \text{and} \quad R_u = \phi R_n$$

4. For $f_y = 60$ ksi and $f'_c = 3$ ksi (Table 3.2), $\rho_{\max} = 0.01356$, $\rho_{\min} = 0.00333$, $R_n = 686$ psi, and $R_u = 615$ psi.

   For $f_y = 60$ ksi and $f_c{}' = 4$ ksi, $\rho_{\max} = 0.01806$, $\rho_{\min} = 0.00333$, $R_n = 911$ psi, and $R_u = 820$ psi.

Section 3.14: Analysis of Rectangular Section with Compression Steel

Given: b, d, d', A_s, A'_s, f'_c, and f_y. Required: the deign moment strength, ϕM_n.

1. Calculate $\rho = A_s/bd$, $\rho' = A_s/bd$, and $(\rho - \rho')$.
2. Calculate ρ_b, $\rho_{\max}$, and $\rho_{\min}$ as given above (or see Section 3.10)
3. Calculate $K = 0.85\beta_1(f'_c/f_y)(d'/d)[87/(87 - f_y)]$. ($f'_c$ and f_y are in ksi.)
4. When compression steel yields,
   a. Check that $\rho \geq \rho_{\min}$.
   b. Check that $(\rho - \rho') \geq K$ for compression steel to yield. If not, then compression steel does not yield.
   c. If compression steel yields, then $f'_s = f_y$.
   d. Check that $\rho \leq (\rho_{\max} + \rho')$ or $(\rho - \rho') \leq \rho_{\max}$.
   e. Calculate $a = (A_s - A'_s)f_y/(0.85 f'_c b)$.
   f. Calculate $\phi M_n = \phi(A_s - A'_s)f_y(d - a/2) + \phi A'_s f_y(d - d')$.
   g. If $(\rho - \rho') > \rho_{\max}$ but $< \rho_{\max t}$ (for the transition region), then $\phi < 0.9$ for M_{u1} and $\phi = 0.9$ for M_{u2} (Eq. 3.44 *a*).
5. When compression steel does not yield,
   a. Compression steel does not yield when $(\rho - \rho') < K$. The value of f'_s is not known.
   b. Calculate c = the distance to the neutral axis from the compression fibers as follows:

$$A_1 c^2 + A_2 c + A_3 = 0,$$

where

$$A_1 = 0.85 f_c' \beta_1 b$$
$$A_2 = A_s'(87 - 0.85 f_c) - A_s f_y$$
$$A_3 = -87 A_s' d'.$$

Solve for c.

An alternative solution to calculate c is as follows:

$$C + C' = T$$
$$C = 0.85 f_c'(\beta_1 cb - A_s') \quad C' = A_s'[87(c - d')/c] - 0.85 f_c' A_s'$$

and

$$T = A_s f_y$$

Solve for c.

c. Calculate $f_s' = 87(c - d')/c \leq f_y$(in ksi).

d. Check that $\rho \leq [\rho_{\max} + \rho'(f_s'/f_y)]$ or $A_s \leq [\rho_{\max}(bd) + A_s'(f_s'/f_y)]$.

e. Calculate a:

$$a = (A_s f_y - A_s' f_s')/(0.85 f_c' b) \quad \text{or} \quad a = \beta_1 c$$

f. Calculate ϕM_n:

$$\phi M_n = \phi[(A_s f_y - A_s' f_s')(d - a/2) + A_s' f_s'(d - d')]$$

Note that $(A_s f_y - A_s' f_y) = A_{s1} = A_s - A_{s2} = A_s - (A_s' f_s'/f_y)$ and $A_{s2} f_y = A_s' f_s'$. Also, $a = A_{s1} f_y/(0.85 f_c' b)$

Sections 3.15–3.17: Analysis of T-Sections

Given: f_c', f_y, A_s, and section dimensions. Required: design moment strength, ϕM_n. Two possible cases may develop. (Determine the effective flange width, b_e, first.)

Case 1

1. If $a \leq t$ (the slab thickness), then it is a T-section shape but acts as a singly reinforced rectangular section using $b = b_e$ (the flange effective width) to calculate ϕM_n.

$$a' = A_s f_y/(0.85 f_c' b_e) \leq t$$

Or, check that A_c (the area of concrete in compression) $= A_s f_y/(0.85 f_c') \leq bt$. If $A_c \geq bt$, then it is a T-section analysis.

2. If $a' \leq t$ or $A_c \leq bt$, then $a' = a$ and $\phi M_n = \phi A_s f_y(d - a/2)$.

3. Check that ρ_w (steel ratio in web) $= A_s/b_w d \geq \rho_{\min}$.

4. Check that $A_s \leq A_{s\max}$ from Eq. 3.71. (Normally, this is o.k. for this case.)

$$A_{s\max} = 0.6375(f_c'/f_y)[t(b - b_w) + (0.375) b_w \beta_1 d]$$

5. Check that $\varepsilon_t \geq 0.005$ for $\phi = 0.9$. (Normally this is o.k. for this case.)

6. The effective flange width $b = b_e$ is the smallest of

a. Span/4

b. Center to center of adjacent slabs
c. $(b_w + 16t)$, where t = slab thickness

Case 2

1. When $a > t$ or $A_c > bt$, it is a T-section analysis.
2. For the flange, $C_f = 0.85f'_c t(b - b_w) = A_{sf} f_y$, calculate $A_{sf} = C_f/f_y$.
3. For the web,

$$A_{sw} = \text{tension steel in the web} = A_s - A_{sf}$$

$$a = (A_s - A_{sf}) f_y/(0.85f'_c b_w)$$

$$C_w(\text{web}) = 0.85f'_c a b_w = A_{sw} f_y$$

4. $$\begin{aligned}\phi M_n &= \phi[M_w(\text{web}) + M_f(\text{flange})] = \phi[C_w(d - a/2) + C_f(d - t/2)] \\ &= \phi[0.85f'_c a b_w(d - a/2) + 0.85f'_c t(b - b_w)(d - t/2)] \\ &= \phi[(A_s - A_{sf}) f_y(d - a/2) + A_{sf} f_y(d - t/2)]\end{aligned}$$

5. Check that $\varepsilon_t \geq 0.005$ for tension-controlled section and $\phi = 0.9$. (See Example 3.12).
6. Check that $A_{s\,\min} \leq A_s \leq A_{s\,\max}$. (See case 1.)

Sections 3.18–3.21

1. Analysis of nonuniform sections is explained in Example 3.13.
2. Tables in Appendix A may be used for the analysis of rectangular sections.
3. Examples in SI units are introduced.

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PROBLEMS

3.1 Singly reinforced rectangular sections. Determine the design moment strength of the sections given in the following table, knowing that $f_c' = 4$ ksi and $f_y = 60$ ksi. (Answers are given in the right column.)

No.	*b* (in.)	*d* (in.)	A_s (in.^{2})	ϕM_n (K·ft)
a	14	22.5	5.08 (4 no. 10)	441.2
b	18	28.5	7.62 (6 no. 10)	849.1
c	12	23.5	4.00 (4 no. 9)	370.1
d	12	18.5	3.16 (4 no. 8)	230.0
e	16	24.5	6.35 (5 no. 10)	600
f	14	26.5	5.00 (5 no. 9)	525.3
g	10	17.5	3.00 (3 no. 9)	200.5
h	20	31.5	4.00 (4 no. 9)	535.2

For problems in SI units, 1 in. = 25.4 mm, 1 in.^{2} = 645 mm^2, 1 ksi = 6.9 MPa (N/mm^2), and 1 M_u (K·ft) = 1.356 kN·m.

3.2 Rectangular section with compression steel. Determine the design moment strength of the sections given in the following table, knowing that $f_c'4$ ksi, $f_y = 60$ ksi, and $d' = 2.5$ in. (Answers are given in the right column. In the first four problems, $f_s' = f_y$)

No.	*b* (in.)	*d* (in.)	A_s (in.^{2})	A_s'(in.^{2})	ϕM_n (K·ft)
a	15	22.5	8.0 (8 no. 9)	2.0 (2 no. 9)	692.2
b	17	24.5	10.08 (8 no. 10)	2.54 (2 no. 10)	950
c	13	22	7.00 (7 no. 9)	1.8 (3 no. 7)	590.2
d	10	21.5	5.08 (4 no. 10)	1.2 (2 no. 7)	464.7
e	14	20.5	7.62 (6 no. 10)	2.54 (2 no. 10)	597.9
f	16	20.5	9.0 (9 no. 9)	4.0 (4 no. 9)	716.3
g	20	18.0	12.0 (12 no. 9)	6.0 (6 no. 9)	820.3
h	18	20.5	10.16 (8 no. 10)	5.08 (4 no. 10)	813.7

For problems in SI units: 1 in. = 25.4 mm, 1 in.^{2} = 645 mm^2, 1 ksi = 6.9 MPa (N/mm^2), and 1 M_u(K·ft) = 1.356 kN·m.

3.3 T-sections. Determine the design moment strength of the T-sections given in the following table, knowing that $f'_c = 3$ ksi and $f_y = 60$ ksi. (Answers are given in the right column. In the first three problems, $a < t$.)

No.	b (in.)	b_w (in.)	t (in.)	d (in.)	A_s (in.^{2})	ϕM_n (K·ft)
a	54	14	3	17.5	5.08 (4 no. 10)	374.8
b	48	14	4	16.5	4.0 (4 no. 9)	279.4
c	72	16	4	18.5	10.16 (8 no. 10)	769.9
*d	32	16	3	15.5	6.0 (6 no. 9)	N.G.
e	44	12	4	20.5	8.0 (8 no. 9)	660.1
f	50	14	3	16.5	7.0 (7 no. 9)	466.8
g	40	16	3	16.5	6.35 (5 no. 10)	415.0
h	42	12	3	17.5	6.0 (6 no. 9)	425.8

For problems in SI units: 1 in. = 25.4 mm, 1 in.^{2} = 645 mm^2, 1 ksi = 6.9 MPa (N/mm^2), and 1 M_u (K·ft) = 1.356 kN·m.
*Answer = 325.5 K·ft if ρ_{max} is used.

3.4 Calculate ρ_b, ρ_{max}, R_u(max), R_u, a/d, and max (a/d) for a rectangular section that has a width of b = 12 in. (300 mm) and an effective depth of d = 20 in. (500 mm) for the following cases:

a. $f'_c = 3$ ksi, $f_y = 40$ ksi, A_s = four no. 8 bars
b. $f'_c = 4$ ksi, $f_y = 60$ ksi, A_s = four no. 7 bars
c. $f'_c = 4$ ksi, $f_y = 75$ ksi, A_s = four no. 9 bars
d. $f'_c = 5$ ksi, $f_y = 60$ ksi, A_s = four no. 9 bars
e. $f'_c = 30$ MPa, $f_y = 400$ MPa, $A_s = 3 \times 30$ mm
f. $f'_c = 20$ MPa, $f_y = 300$ MPa, $A_s = 3 \times 25$ mm
g. $f'_c = 30$ MPa, $f_y = 500$ MPa, $A_s = 4 \times 25$ mm
h. $f'_c = 25$ MPa, $f_y = 300$ MPa, $A_s = 4 \times 20$ mm

3.5 Using the ACI Code requirements, calculate the design moment strength of a rectangular section that has a width of b = 250 mm (10 in.) and an effective depth of d = 550 mm (22 in.) when $f'_c = 20$ MPa (3 ksi), $f_y = 420$ MPa (60 ksi), and the steel used is as follows:

a. 4 × 20 mm b. 3 × 25 mm c. 4 × 30 mm
d. 2 no. 9 bars e. 6 no. 9 bars

3.6 A reinforced concrete simple beam has a rectangular section with a width of b = 8 in. (200 mm) and effective depth of d = 18 in. (450 mm). At design moment (failure), the strain in the steel was recorded and was equal to 0.0015. (The strain in concrete at failure may be assumed to be 0.003.) Use $f'_c = 3$ ksi (20 MPa) and $f_y = 50$ ksi (350 MPa) for all parts.

a. Check if the section is balanced, under-reinforced, or over-reinforced.
b. Determine the steel area that will make the section balanced.
c. Calculate the steel area provided in the section to produce the aforementioned strains, and then calculate its moment. Compare this value with the design moment strength allowed by the ACI Code using ρ_{max}.
d. Calculate the design moment strength of the section if the steel percentage used is $\rho = 1.4\%$.

3.7 A 10-ft.- (3-m-)span cantilever beam has an effective cross-section (bd) of 12 in. by 24 in. (300 by 600 mm) and is reinforced with five no. 8 (5 × 25 mm) bars. If the uniform load due to its own weight and the dead load are equal to 685 lb/ft (10 kN/m), determine the allowable uniform live load on the beam using the ACI load factors. Given: $f_c' = 3$ ksi (20 MPa) and $f_y = 60$ ksi (400 MPa).

3.8 The cross-section of a 17-ft- (5-m-) span simply supported beam is 10 by 28 in. (250 by 700 mm), and it is reinforced symmetrically with eight no. 6 bars (8 × 20 mm) in two rows. Determine the allowable concentrated live load at midspan considering the total acting dead load (including self-weight) is equal to 2.55 K/ft (37 kN/m). Given: $f_c' = 3$ ksi (20 MPa) and $f_y = 40$ ksi (300 MPa).

3.9 Determine the design moment strength of the sections shown in Fig. 3.41. Neglect the lack of symmetry in (b). Given: $f_c' = 4$ ksi (30 MPa) and $f_y = 60$ ksi (400 MPa).

3.10 A rectangular concrete section has a width of $b = 12$ in. (300 mm), an effective depth of $d = 18$ in. (450 mm), and d' = 2.5 in. (60 mm). If compression steel consisting of two no. 7 bars (2 × 20 mm) is used, calculate the allowable moment strength that can be applied on the section if the tensile steel, A_s, is as follows:

a. Four no. 7 (4 × 20 mm) bars **b.** Eight no. 7 (8 × 20 mm) bars
Given: $f_c' = 3$ ksi (20 MPa) and $f_y = 40$ ksi (300 MPa).

3.11 A 16-ft- (4.8-m-) span simply supported beam has a width of $b = 12$ in. (300 mm), $d = 22$ in. (500 mm), $d' = 2.5$ in. (60 mm), and $A_s' =$ three no. 6 bars (3 × 20 mm). The beam carries a uniform dead load of 2 K/ft (30 kN/m), including its own weight. Calculate the allowable uniform live load that can be safely applied on the beam. Given: $f_c' = 4$ ksi (20 MPa) and $f_y = 60$ ksi (400 MPa). (Hint: Use ρ_{max} for the basic section to calculate M_u.)

3.12 Check the adequacy of a 10-ft- (3-m-)span cantilever beam, assuming a concrete strength of $f_c' = 4$ ksi (30 MPa) and a steel yield strength of $f_y = 60$ ksi (400 MPa) are used. The dimensions of the beam section are $b = 10$ in. (250 mm), $d = 20$ in. (500 mm), $d' = 2.5$ in. (60 mm), $A_s =$ six no. 7 bars (6 × 20 mm), $A_s' =$ two no. 5 bars (2 × 15 mm). The dead load on the beam, excluding its own weight, is equal to 2 K/ft (30 kN/m), and the live load equals 1.25 K/ft (20 kN/m). (Compare the internal M_u with the external factored moment.)

3.13 A series of reinforced concrete beams spaced at 9 ft (2.7 m) on centers are acting on a simply supported span of 18 ft (5.4 m). The beam supports a reinforced concrete floor slab 4 in. (100 mm) thick. If the width of the web is $b_w = 10$ in. (250 mm), $d = 18$ in. (450 mm), and the beam is reinforced with three

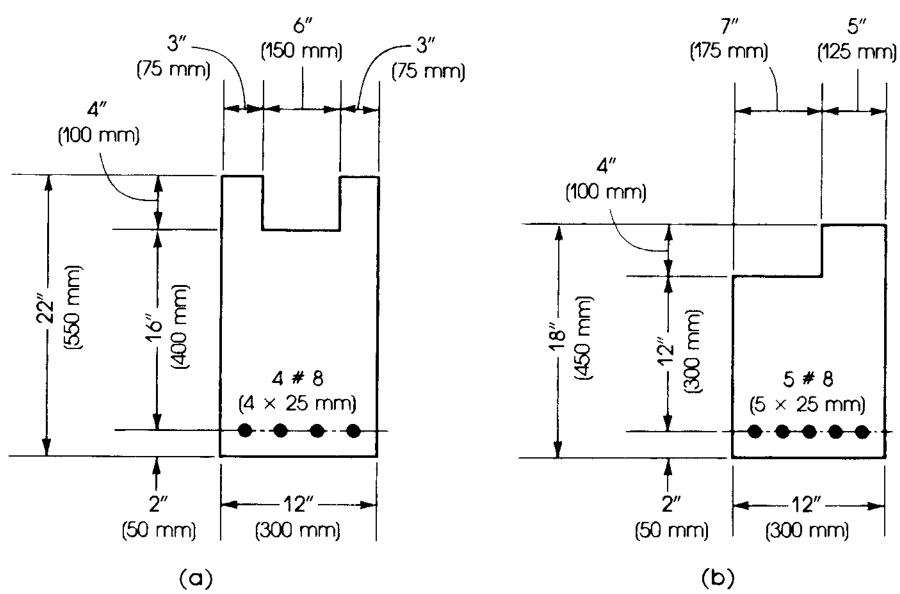


Figure 3.41 Problem 3.9.

no. 9 bars (3 × 30 mm), determine the moment strength of a typical interior beam. Given: $f_c' = 4$ ksi (30 MPa) and $f_y = 60$ ksi (400 MPa).

3.14 Calculate the design moment strength of a T-section that has the following dimensions:

- Flange width = 30 in. (750 mm)
- Flange thickness = 3 in. (75 mm)
- Web width = 10 in. (250 mm)
- Effective depth (d) = 18 in. (450 mm)
- Tension reinforcement: six no. 8 bars (6 × 25 mm)
- $f_c' = 3$ ksi (20 MPa)
- $f_y = 60$ ksi (400 MPa)

3.15 Repeat Problem 3.14 if $d = 24$ in. (600 mm).

3.16 Repeat Problem 3.14 if the flange is an inverted L shape with the same flange width projecting from one side only. (Neglect lack of symmetry.)

CHAPTER 4

FLEXURAL DESIGN OF REINFORCED CONCRETE BEAMS

Reinforced conrete office building, Amman, Jordan.

4.1 INTRODUCTION

In the previous chapter, the analysis of different reinforced concrete sections was explained: Details of the section were given, and we had to determine the design moment of the section. In this chapter, the process is reversed: The external moment is given, and we must find safe, economic, and practical dimensions of the concrete section and the area of reinforcing steel that provide adequate internal moment strength.

4.2 RECTANGULAR SECTIONS WITH REINFORCEMENT ONLY

From the analysis of rectangular singly reinforced sections (Section 3.9), the following equations were derived for tension-controlled sections, where f'_c and f_y are in ksi:

$$\rho_b = 0.85\ \beta_1\ \frac{f'_c}{f_y}\left(\frac{87}{87+f_y}\right) \tag{3.18}$$

$$\rho_{\max} = \rho_b \left(\frac{0.003 + \dfrac{f_y}{E_s}}{0.008}\right) \tag{3.31}$$

For $f_y = 60$ ksi,

$$\rho_{\max} = 0.63375\rho_b\ (\text{or } 0.634\rho_b) \tag{3.32}$$

Also,

$$\text{for} f_y = 40 \text{ ksi}, \rho_{\max} = 0.5474\ \rho_b$$
$$\text{for} f_y = 50 \text{ ksi}, \rho_{\max} = 0.5905\ \rho_b$$
$$\text{for} f_y = 75 \text{ ksi}, \rho_{\max} = 0.6983\ \rho_b$$

The value of β_1 in ρ_b is 0.85 when $f'_c \leq 4000$ psi (30 N/mm^2) and decreases by 0.05 for every increase of 1000 psi (7 N/mm^2) in concrete strength. The steel percentage of balanced section, ρ_b, and the maximum allowable steel percentage, $\rho_{\max}$, can be calculated for different values of f'_c and f_y, as shown in Table 4.1 or Table A.4 in Appendix A.

It should be clarified that the designer has a wide range of choice between a large concrete section and relatively small percentage of steel, ρ, producing high ductility and a small section with a high percentage of steel with low ductility. A high value of the net tensile strain, ε_t, indicates a high ductility and a relatively low percentage of steel. The limit of the net tensile strain for tension-controlled sections is 0.005, with $\phi = 0.9$. The strain limit of 0.004 can be used with a reduction in ϕ. If the ductility index is represented by the ratio of the net tensile strain, ε_t, to the yield strain, $\varepsilon_y = f_y/E_s$, the relationship between ε_t, ρ/ρ_b, ϕ, and $\varepsilon_t/\varepsilon_y$ is shown in Table 4.2 for $f_y = 60$ ksi. Also, the ACI Code, Section 8.4, indicates that ε_t should be ≥ 0.0075 for the redistribution of moments in continuous flexural members producing a ductility index of 3.75. It can be seen that adopting $\varepsilon_t \geq 0.005$ is preferable to the use of a higher steel ratio, ρ/ρ_b, with $\varepsilon_t = 0.004$, because the increase in M_n is offset by a lower ϕ. The value of $\varepsilon_t = 0.004$ represents the use of minimum steel percentage of 0.00333 for $f'_c = 4$ ksi and $f_y = 60$ ksi. This case should be avoided. The value of ϕ between $\varepsilon_t = 0.005$ and 0.004 can be calculated from Eq. 3.8: $\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right)$.

Table 4.1 Suggested Design Steel Ratios, ρ_s

f'_c (ksi)	f_y (ksi)	% ρ_b	% $\rho_{\max}$	% ρ_s	Ratio ρ_s/ρ_b	Ratio $\rho_s/\rho_{\max}$	R_{us} (psi)	$R_{u\max}$ (psi)
3	40	3.71	2.031	1.4	0.38	0.69	450	614
	60	2.14	1.356	1.2	0.56	0.89	556	615
4	60	2.85	1.806	1.4	0.49	0.78	662	820
	75	2.07	1.445	1.2	0.58	0.83	702	820
5	60	3.35	2.123	1.6	0.48	0.75	766	975
	75	2.43	1.700	1.4	0.58	0.82	830	975

Table 4.2 Relation Between ε_t, ρ/ρ_b, ϕ, and $\varepsilon_t/\varepsilon_y$ ($f_y = 60$ ksi)

ε_t	0.004	0.005	0.006	0.007	0.0075	0.008	0.009	0.010	0.040
$\frac{\rho}{\rho_b}$	0.714	0.625	0.555	0.500	0.476	0.454	0.417	0.385	0.117
$\frac{\varepsilon_t}{\varepsilon_y}$	2.0	2.5	3.0	3.5	3.75	4.0	4.5	5.0	20
ϕ	0.82	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9

The design moment equations were derived in the previous chapter in the following forms:

$$\phi M_n = M_u = R_u bd^2 \tag{3.21}$$

where

$$R_u = \phi \rho f_y \left(1 - \frac{\rho f_y}{1.7 f'_c}\right) = \phi R_n \tag{3.22}$$

where $\phi = 0.9$, for tension-controlled sections and less than 0.9 for sections in the transition region.

$$\phi M_n = M_u = \phi A_s f_y \left(d - \frac{A_s f_y}{1.7 f'_c b}\right) \tag{3.19a}$$

Also,

$$\phi M_n = M_u = \phi \rho f_y bd^2 \left(1 - \frac{\rho f_y}{1.7 f'_c}\right) \tag{3.20}$$

We can see that for a given factored moment and known f'_c and f_y, there are three unknowns in these equations: the width, b, the effective depth of the section, d, and the steel ratio, ρ. A unique solution is not possible unless values of two of these three unknowns are assumed. Usually ρ is assumed (using $\rho_{\max}$, for instance), and b can also be assumed.

Based on the preceding discussion, the following cases may develop for a given M_u, f'_c, and f_y:

1. If ρ is assumed, then R_u can be calculated from Eq. 3.22, giving $bd^2 = M_u/R_u$. The ratio of d/b usually varies between 1 and 3, with a practical ratio of 2. Consequently, b and d can be determined, and $A_s = \rho bd$. The ratio ρ for a singly reinforced rectangular section must be equal to or less than $\rho_{\max}$, as given in Eq. 3.31. It is a common practice to assume a value of ρ that ranges between $\frac{1}{2}\rho_{\max}$ and $\frac{1}{2}\rho_b$. Table 4.1 gives suggested values of the steel ratio ρ to be used in singly reinforced sections when ρ is not assigned. For example, if $f_y = 60$ ksi, the value $\rho_s = 1.4\%$ is suggested for $f'_c = 4$ ksi 1.6% for $f'_c = 5$ ksi and 1.2% for $f'_c = 3$ ksi. The designer may use ρ up to $\rho_{\max}$, which produces the minimum size of the singly reinforced concrete section. Using ρ_{min} will produce the maximum concrete section. If b is assumed in addition to ρ, then d can be determined as follows:

$$d = \sqrt{\frac{M_u}{R_u b}} \tag{4.1}$$

If $d/b = 2$, then $d = \sqrt[3]{2M_u/R_u}$ and $b = d/2$, rounded to the nearest higher inch.

2. If b and d are given, then the required reinforcement ratio ρ can be determined by rearranging Eq. 3.20 to obtain

$$\rho = \frac{0.85 f'_c}{f_y}\left[1 - \sqrt{1 - \frac{4M_u}{1.7\phi f'_c bd^2}}\right] \tag{4.2}$$

$$= \frac{0.85f'_c}{f_y}\left[1 - \sqrt{1 - \frac{2R_n}{0.85f'_c}}\right] \quad (4.2a)$$

or

$$\rho = \frac{f'_c}{f_y}[0.85 - \sqrt{(0.85)^2 - Q}]$$

where

$$Q = \left(\frac{1.7}{\phi f'_c}\right)\frac{M_u}{bd^2} = \left(\frac{1.7}{\phi f'_c}\right)R_u \quad (4.3)$$

$$A_s = \rho bd = \left(\frac{f'_c}{f_y}\right)bd[0.85 - \sqrt{(0.85)^2 - Q}] \quad (4.4)$$

where all units are in kips (or pounds) and inches, and Q is dimensionless. For example, if $M_u = 2440$ K·in., $b = 12$ in., $d = 18$ in., $f'_c = 3$ ksi, and $f_y = 60$ ksi, then $\rho = 0.01389$ (from Eq. 4.2) and $A_s = \rho bd = 0.01389(12)(18) = 3.0\,\text{in.}^2$, or directly from Eq. 4.4, $Q = 0.395$ and $A_s = 3.0\ \text{in}^2$. When b and d are given, it is better to check if compression steel is or is not required because of a small d. This can be achieved as follows:

a. Calculate $\rho_{\max}$ and $R_u(\max) = \phi\rho_{\max} f_y[1 - (\rho_{\max} f_y/1.7f'_c)]$.

b. Calculate $\phi M_n(max) = R_u bd^2$ = the design moment strength of a singly reinforced concrete section.

c. If $M_u < \phi M_{n\,\max}$, then no compression reinforcement is needed. Calculate ρ and A_s from the preceding equations.

d. $M_u > \phi M_{n\ \max}$, then compression steel is needed. In this case, the design procedure is explained in Section 4.4.

3. If ρ and b are given, calculate R_u:

$$R_u = \phi\rho f_y\left(1 - \frac{\rho f_y}{1.7f'_c}\right)$$

Then calculate d from Eq. 4.1:

$$d = \sqrt{\frac{M_u}{R_u b}} \quad \text{and} \quad A_s = \rho bd$$

4.3 SPACING OF REINFORCEMENT AND CONCRETE COVER

4.3.1 Specifications

Figure 4.1 shows two reinforced concrete sections. The bars are placed such that the clear spacings shall be at least equal to nominal bar diameter D but not less than 1 in. (25 mm). Vertical clear spacings between bars in more than one layer shall not be less than 1 in. (25 mm), according to the ACI Code, Section 7.6.

The width of the section depends on the number, n, and diameter of bars used. Stirrups are placed at intervals; their diameters and spacings depend on shear requirements, to be explained

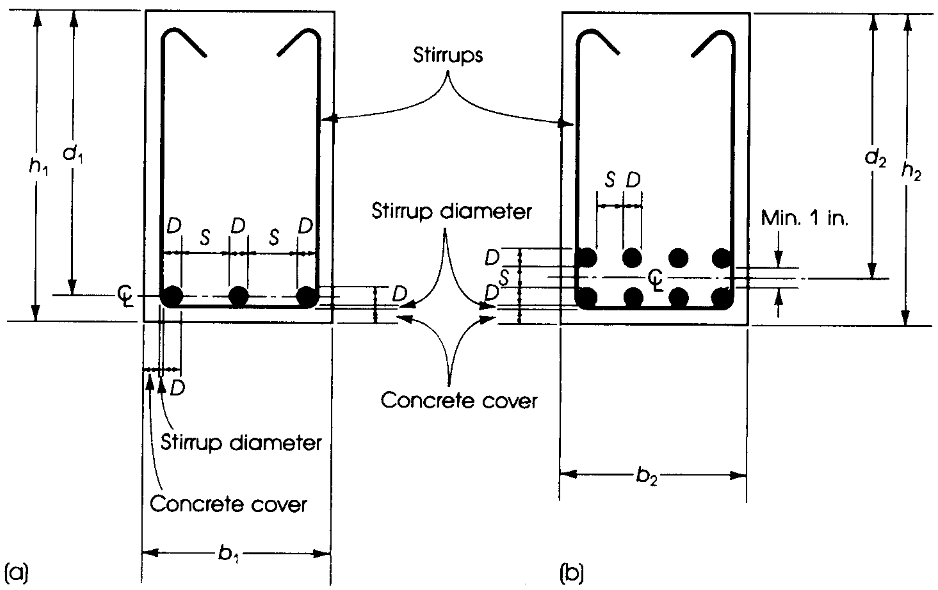


Figure 4.1 Spacing of steel bars (a) in one row or (b) two rows.

later. At this stage, stirrups of $\frac{3}{8}$ in. (10 mm) diameter can be assumed to calculate the width of the section. There is no need to adjust the width, b, if different diameters of stirrups are used. The specified concrete cover for cast-in-place and precast concrete is given in the ACI Code, Section 7.7. Concrete cover for beams and girders is equal to $\frac{3}{2}$ in. (38 mm), and that for slabs is equal to $\frac{3}{4}$ in. (20 mm), when concrete is not exposed to weather or in contact with ground.

4.3.2 Minimum Width of Concrete Sections

The general equation for the minimum width of a concrete section can be written in the following form:

$$b_{\min} = nD + (n-1)s + 2(\text{stirrup's diameter}) + 2(\text{concrete cover}) \tag{4.5a}$$

where

n = number of bars
D = diameter of the largest bar used
s = spacing between bars (equal to D or 1 in., whichever is larger)

If the stirrup's diameter is taken equal to $\frac{3}{8}$ in. (10 mm) and concrete cover equals $\frac{3}{2}$ in. (38 mm), then

$$b_{\min} = nD + (n-1)s + 3.75 \text{ in. } (95 \text{ mm}) \tag{4.5b}$$

This equation, if applied to the concrete sections in Fig. 4.1, becomes

$$b_1 = 3D + 2S + 3.75 \text{ in. } (95 \text{ mm})$$

$$b_2 = 4D + 3S + 3.75 \text{ in. } (95 \text{ mm})$$

To clarify the use of Eq. 4.5, let the bars used in sections of Fig. 4.1 be no. 10 (32-mm) bars. Then

$$b_1 = 5 \times 1.27 + 3.75 = 10.10 \text{ in. } (s = D) \quad \text{say, 11 in.}$$
$$b_1 = 5 \times 32 + 95 = 225 \text{ mm} \quad \text{say, 250 mm}$$
$$b_2 = 7 \times 1.27 + 3.75 = 12.64 \text{ in.} \quad \text{say, 13 in.}$$
$$b_1 = 7 \times 232 + 95 = 319 \text{ mm} \quad \text{say, 320 mm}$$

If the bars used are no. 6 (20 mm), the minimum widths become

$$b_1 = 3 \times 0.75 + 2 \times 1 + 3.75 = 8.0 \text{ in.} \quad s = 1 \text{ in.}$$
$$b_1 = 3 \times 20 + 2 \times 25 + 95 = 205 \text{ mm} \quad \text{say, 210 mm}$$
$$b_2 = 4 \times 0.75 + 3 \times 1 + 3.75 = 9.75 \text{ in.} \quad \text{say, 10 in.}$$
$$b_2 = 4 \times 20 + 3 \times 25 + 95 = 250 \text{ mm}$$

The width of the concrete section shall be increased to the nearest inch. Table A.7 in Appendix A gives the minimum beam width for different numbers of bars in the section.

4.3.3 Minimum Overall Depth of Concrete Sections

The effective depth, d, is the distance between the extreme compressive fibers of the concrete section and the centroid of the tension reinforcement. The minimum total depth is equal to d plus the distance from the centroid of the tension reinforcement to the extreme tension concrete fibers, which depends on the number of layers of the steel bars. In application to the sections shown in Fig. 4.1,

$$h_1 = d_1 + \frac{D}{2} + \frac{3}{8} \text{ in.} + \text{concrete cover} \tag{4.6a}$$
$$= d_1 + \frac{D}{2} + 1.857 \text{ in. (50 mm)}$$

for one row of steel bars and

$$h_2 = d_2 = 0.5 + D + \frac{3}{8} \text{ in.} + \text{concrete cover} \tag{4.6b}$$
$$= d_2 + D + 2.375 \text{ in. (60 mm)}$$

for two layers of steel bars. The overall depth, h, shall be increased to the nearest half inch (10 mm) or, better, to the nearest inch (20 mm in SI). For example, if $D = 1$ in. (25 mm), $d_1 =$ 18.9 in. (475 mm), and $d_2 = 20.1$ in. (502 mm),

$$\text{Minimum } h_1 = 18.9 + 0.5 + 1.875 = 21.275 \text{ in.}$$

say, 21.5 in. or 22 in.,

$$h_1 = 475 + 13 + 50 = 538 \text{ mm}$$

say, 540 mm or 550 mm, and

$$\text{Minimum } h_2 = 20.1 + 1.0 + 2.375 = 23.475 \text{ in.}$$

say, 23.5 in. or 24 in.,

$$h_2 = 502 + 25 + 60 = 587 \text{ mm}$$

say, 590 mm or 600 mm.

If no. 9 or smaller bars are used, a practical estimate of the total depth, h, can be made as follows:

$$h = d + 2.5 \text{ in. (65 mm)}, \quad \text{for one layer of steel bars}$$

$$h = d + 3.5 \text{ in. (90 mm)}, \quad \text{for two layers of steel bars}$$

For more than two layers of steel bars, a similar approach may be used.

It should be mentioned that the minimum spacing between bars depends on the maximum size of the coarse aggregate used in concrete. The nominal maximum size of the coarse aggregate shall not be larger than one-fifth of the narrowest dimension between sides of forms, nor one-third of the depth of slabs, nor three-fourths of the minimum clear spacing between individual reinforcing bars or bundles of bars (ACI Code, Section 3.3).

Example 4.1

Design a simply reinforced rectangular section to resist a factored moment of 361 K·ft using the maximum steel percentage ρ_{max} for tension-controlled sections. Given: $f'_c = 3$ ksi and $f_y = 60$ ksi.

Solution

For $f'_c = 3$ ksi, $f_y = 60$ ksi, and $\beta_1 = 0.85$, ρ_{max} for a tension-controlled section is calculated as follows ($\phi = 0.9$):

$$\rho_b = (0.85)\beta_1 \left(\frac{f'_c}{f_y}\right)\left[\frac{87}{(87 + f_y)}\right],$$

$$\rho_b = (0.85)^2 \left(\frac{3}{60}\right)\left(\frac{87}{147}\right) = 0.0214$$

$$\rho_{max} = \rho_b \left(\frac{0.003 + \frac{f_y}{E_s}}{0.008}\right) = 0.63375\rho_b = 0.01356 \qquad \text{(Table 4.1)}$$

$$R_{u\,max} = \phi\rho_{max} f_y \left(1 - \frac{\rho_{max} f_y}{1.7 f'_c}\right)$$

$$= 0.9 \times 0.01356 \times 60 \times \left(1 - \frac{0.01356 \times 60}{1.7 \times 3}\right) = 0.615 \text{ ksi}$$

(Or, use the tables in Appendix A or Table 4.1.)

Since $M_u = R_u bd^2$,

$$bd^2 = \frac{M_u}{R_u} = \left(\frac{361 \times 12}{0.615}\right) = \frac{4332}{0.615} = 7043 \text{ in.}^3$$

Thus, for the following assumed b, calculate d and $A_s = \rho bd$:

$b = 10$ in.	$d = 26.5$ in.	$A_s = 4.24$ in.^{2}	
$b = 12$ in.	$d = 24.2$ in.	$A_s = 4.65$ in.^{2}	6 no. 8 bars ($A_s = 4.71$ in.^{2})
$b = 14$ in.	$d = 22.4$ in.	$A_s = 5.01$ in.^{2}	5 no. 9 bars ($A_s = 5.0$ in.^{2})
$b = 16$ in.	$d = 21.0$ in.	$A_s = 5.37$ in.^{2}	

The choice of the effective depth d depends on three factors:

1. The width b required. A small width will result in a deep beam that decreases the headroom available. Furthermore, a deep narrow beam may lower the design moment strength of the structural member due to possible lateral deformation.
2. The amount and distribution of reinforcing steel. A narrow beam may need more than one row of steel bars, thus increasing the total depth of the section.
3. The wall thickness. If cement block walls are used, the width b is chosen to be equal to the wall thickness. Exterior walls in buildings in most cases are thicker than interior walls. The architectural plan of the structure will show the different thicknesses.

A reasonable choice of d/b varies between 1 and 3, with practical value about 2. It can be seen from the previous calculations that the deeper the section, the more economical it is, as far as the quantity of concrete used, expressed by the area bd of a 1-ft length of the beam. Alternatively, calculate $bd^2 = M_u/R_u$ and then choose adequate b and d.

The area of the steel reinforcement, A_s, is equal to ρbd. The area of steel needed for the different choices of b and d for this example was shown earlier. Because the steel percentage required is constant ($\rho_{max} = 0.01356$), A_s is proportional to bd. For a choice of a 12×24.2-in. section, the required A_s is 4.65 in.^{2} Choose six no. 8 bars in two rows (actual $A_s = 4.71$ in.^{2}). The minimum b required for three no. 8 bars in one row is 8.9 in. $<$ 12 in., and total $h = 24.2 + 3.5 = 27.7$ in., say, 28 in. (actual $d = 24.6$ in.). Another choice is a section with a 14×22.4-in. section with a total depth (h) of 25 in. and five no. 9 bars in one row. The choice of bars depends on

1. Adequate placement of bars in the section, normally in one or two rows, fulfilling the restrictions of the ACI Code for minimum spacing between bars
2. The area of steel bars chosen closest to the required calculated steel area

The final section is shown in Fig. 4.2.

Example 4.2

Solve Example 4.1 using a steel percentage ρ of about 1% and $b = 14$ in.

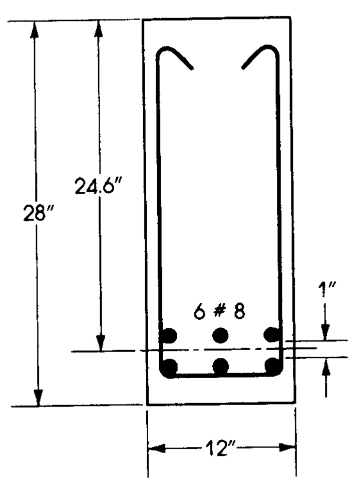


Figure 4.2 Example 4.1.

Solution

1. For $f'_c = 3$ ksi and $f_y = 60$ ksi, $\rho_{max} = 0.01356$ for a tension-controlled section:

$$R_u = \phi \rho f_y \left(1 - \frac{\rho f_y}{1.7 f'_c}\right)$$

$$= 0.9 \times 0.01 \times 60 \left(1 - \frac{0.01 \times 60}{1.7 \times 3}\right) = 0.476 \text{ ksi}$$

(From the tables in Appendix A, for $\rho = 0.01$, $R_u = 476$ psi.)

2. $bd^2 = M_u/R_u = 4332/0.476 = 9100$ in.^{3}. Choosing $b = 14$ in. and $d = 25.5$ in.,

$$A_s = \rho bd = 0.01 \times 14 \times 25.5 = 3.57 \text{ in.}^2$$

Choose four no. 9 bars in one layer; $A_s = 4.00$ in.^{2}

$$b_{min} = nD + (n-1)s + 3.75$$

$$= 7 \times 1.128 + 3.75 = 11.7 \text{ in.} < 14 \text{ in.}$$

$$h_{min} = d + \frac{D}{2} + 1.875$$

$$= 25.5 + \frac{1.138}{2} + 1.875 = 27.94 \text{ in.} \qquad \text{say, 28 in.} \qquad (d = 25.5 \text{ in.})$$

3. Because the actual A_s used is greater than the calculated A_s, a smaller depth can be adopted. Therefore, take $h = 26$ in. Then $d = 26 - 1.138/2 - 1.875 = 23.5$ in.

For small variation in depth, $A_s = 3.57(25.5/23.5) = 3.87$ in.^{2}, which is less than the 4.00 in.^{2} used (Fig. 4.3). A check of the design moment strength of the section can be made:

$$\text{actual } \rho = \frac{4}{14 \times 23.5} = 0.0121$$

Since $\rho < \rho_{max} = 0.01356$ for a tension-controlled section ($\phi = 0.9$),

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{4.0 \times 60}{0.85 \times 3 \times 14} = 6.72 \text{ in.}$$

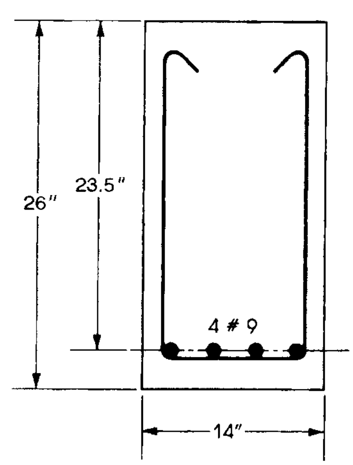


Figure 4.3 Example 4.2.

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right)$$

$$= 0.9 \times 4 \times 60 \left(23.5 - \frac{6.72}{2}\right) = 4350 \text{ K·in.} > 4332 \text{ K·in.}$$

which is acceptable.

4. Check the net tensile strain, ε_t. For $f_y = 60$ ksi,

$$\varepsilon_t = \left(\frac{0.005}{\frac{\rho}{\rho_b}}\right) - 0.003 \tag{3.25}$$

$$\rho_b = 0.0214 \qquad \text{(Table 4.1)}$$

$$\frac{\rho}{\rho_b} = \frac{0.0121}{0.0214} = 0.5654$$

$$\varepsilon_t = \frac{0.005}{0.5654} - 0.003 = 0.00584 > 0.005 \qquad \text{(tension-controlled section)}$$

Or, alternatively, $c = a/0.85 = 7.9$ in., $d_t = 26 - 2.5 = 23.5$ in., $c/d_t = 0.336 < 0.375$, which is o.k.

Example 4.3

Find the necessary reinforcement for a given section that has a width of 10 in. and a total depth of 20 in. (Fig. 4.4) if it is subjected to an external factored moment of 163 K·ft. Given: $f'_c = 4$ ksi and $f_y = 60$ ksi.

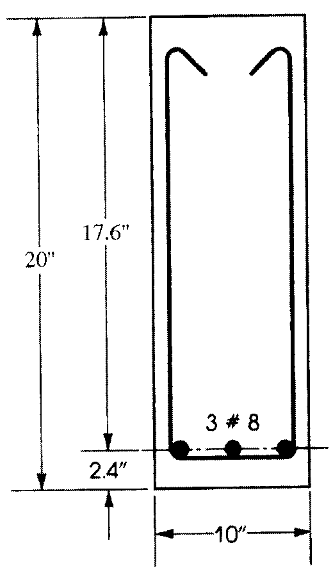


Figure 4.4 Example 4.3.

Solution

1. Assuming one layer of no. 8 steel bars (to be checked later), $d = 20 - 0.5 - 1.875 = 17.625$ in. (or $d = 20 - 2.5$ in. $= 17.5$ in.).
2. Check if the section is adequate without compression reinforcement. Compare the moment strength of the section (using $\rho_{\max}$ for tension-controlled condition). For $f'_c = 4$ ksi and $f_y = 60$ ksi, $\rho_{\max} = 0.01806$.

$$R_{u\max} = \phi\rho_{\max} f_y \left(1 - \frac{\rho_{\max} f_y}{1.7 f'_c}\right) = 820 \text{ psi} \qquad \text{(from Table 4.1)}$$

The moment strength of a singly reinforced basic section is

$$\begin{aligned}\phi M_{n\max} &= R_{u\max} bd^2 = 0.82(10)(17.5)^2 \\ &= 2511 \text{ K·in.} > 163 \times 12 = 1956 \text{ K·in.}\end{aligned}$$

Therefore, $\rho < \rho_{\max}$ and the section is singly reinforced, and tension controls ($\phi = 0.9$).

3. Calculate ρ from Eq 4.2 or 4.3:

$$Q = \left(\frac{1.7}{\phi f'_c}\right) \times \frac{M_u}{bd^2} = \left(\frac{1.7}{0.9 \times 4}\right) \times \left(\frac{1956}{10 \times 17.5^2}\right) = 0.302$$

$$\rho = \frac{f'_c}{f_y}\left(0.85 - \sqrt{(0.85)^2 - Q}\right) = 0.0134 < \rho_{\max} \qquad \text{(tension-controlled condition)}$$

$A_s = \rho bd = 0.0134(10)(17.5) = 2.345$ in.^{2} Use three no. 8 bars ($A_s = 2.35$ in.^{2}) in one row, $b_{\min} < 10$ in. The final section is shown in Fig. 4.4.

Example 4.4

Find the necessary reinforcement for a given section, $b = 15$ in., if it is subjected to a factored moment of 313 K·ft. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. For $f'_c = 4$ ksi and $f_y = 60$ ksi, and from Table 4.1: $\rho_b = 0.0285$, $\rho_{\max} = 0.01806$ (tension-controlled section), $R_{u\max} = 820$ psi.
2. Using $\rho_{\max} = 0.01806$ and $R_u = 820$ psi,

$$bd^2 = \frac{M_u}{R_u} = \frac{313(12)}{0.820} = 4581 \text{ in.}^3$$

For $b = 15$ in. and $d = 17.50$,

$$A_s = \rho bd = 0.01806(15)(17.5) = 4.74 \text{ in.}^2$$

Choose four no. 10 bars, $A_s = 5.08$ in.$^2 > 4.74$ in.^{2}. Bars can be placed in one row, $b_{\min} = 12.7$ in. in Table A.7. Total depth $(h) = 17.5 + 2.5 = 20$ in.

Discussion

1. Since a steel area of 5.08 in.^{2} used is greater than 4.74 in.^{2} required (the limit for a tension-controlled section with $\phi = 0.9$), the section is in the transition zone. Actually, the section is under-reinforced and the nominal moment $= M_n = A_s f_y(d - a/2) = 368.6$ K·ft. ($A_s = 5.08$ in.^{2} and $a = 5.976$ in.). If $\phi = 0.9$ is used then $\phi M_n = 331.7$ K·ft.

2. The ACI Code indicates that for sections in the transition zone, $\phi < 0.9$, and $\varepsilon_t \geq 0.004$.

Checking $\varepsilon_t = \left(\dfrac{0.005}{\dfrac{\rho}{\rho_b}}\right) - 0.003,$

$$\rho = \frac{5.08}{15 \times 17.5} = 0.01935 \qquad \frac{\rho}{\rho_b} = 0.679$$

$$\varepsilon_t = \left(\frac{0.00507}{0.679}\right) - 0.003 = 0.004467 > 0.004$$

Or, alternatively, calculate $a = 5.08 \times 60/(0.85 \times 4 \times 15) = 5.976$, $c = a/.85 = 7.03$, $d_t = d = 17.5$ in. Then $\varepsilon_t = 0.003(d_t - c)/c = 0.004467$. Calculate

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) = 0.856$$

$$\phi M_n = 0.856(368.6) = 315.4 \text{ K}\cdot\text{ft}$$

3. It can be noticed that despite an additional amount of steel, $5.08 - 4.67 = 0.41$ in.^{2} (or about 9%), the design moment strength remained the same. This is because the strength reduction factor, ϕ, was decreased. Therefore, the design of sections within the tension-controlled zone with $\phi = 0.9$ gives a more economical design based on the ACI Code limitations.

4.4 RECTANGULAR SECTIONS WITH COMPRESSION REINFORCEMENT

A singly reinforced section has its moment strength when ρ_{max} of steel is used. If the applied factored moment is greater than the internal moment strength, as in the case of a limited cross-section, a doubly reinforced section may be used, adding steel bars in both the compression and the tension zones. Compression steel will provide compressive force in addition to the compressive force in the concrete area.

4.4.1 Assuming One Row of Tension Bars

The procedure for designing a rectangular section with compression steel when M_u, f'_c, b, d, and d' are given can be summarized as follows:

1. Calculate the balanced and the maximum steel ratio, ρ_{max}, using Eqs. 3.18 and 3.31.

$$\rho_b = 0.85\beta_1 \frac{f'_c}{f_y}\left(\frac{87}{87 + f_y}\right)$$

Calculate $A_{s\,max} = A_{s1} = \rho_{max} bd$ (maximum steel area as singly reinforced).

2. Calculate $R_{u\,max}$ using ρ_{max} ($\phi = 0.9$):

$$R_{u\,max} = \phi\rho_{max} f_y \left(1 - \frac{\rho_{max} f_y}{1.7 f'_c}\right)$$

($R_{u\,max}$ can be obtained from the tables in Appendix A or Table 4.1.)

3. Calculate the moment strength of the section, M_{u1}, as singly reinforced, using ρ_{max} and $R_{u\,max}$.

$$M_{u1} = R_{u\,max} bd^2$$

If $M_{u1} < M_u$ (the applied moment), then compression steel is needed. Go to the next step. If $M_{u1} > M_u$, then compression steel is not needed. Use Eq. 4.2 to calculate ρ and $A_s = \rho bd$, as explained earlier.

4. Calculate $M_{u2} = M_u - M_{u1}$ = the moment to be resisted by compression steel.
5. Calculate A_{s2} from $M_{u2} = \phi A_{s2} f_y (d - d')$.
   Then calculate the total tension reinforcement, A_s:

$$A_s = A_{s1} + A_{s2}$$

6. Calculate the stress in the compression steel as follows:
   a. Calculate $f'_s = 87[(c - d')/c]$ ksi $\leq f_y$. (f'_s cannot exceed f_y.)
   b. Or, ε'_s can be calculated from the strain diagram, and $f'_s = (\varepsilon'_s \cdot E_s)$. If $\varepsilon'_s \geq \varepsilon_y$, then compression steel yields and $f'_s = f_y$.
   c. Calculate A'_s from $M_{u2} = \phi A'_s f'_s (d - d')$. If $f'_s = f_y$, then $A'_s = A_{s2}$. If $f'_s < f_y$, then $A'_s > A_{s2}$, and $A'_s = A_{s2}(f_y/f'_s)$.
7. Choose bars for A_s and A'_s to fit within the section width, b. In most cases, A_s bars will be placed in two rows, whereas A'_s bars are placed in one row.
8. Calculate $h = d + 2.5$ in. for one row of tension bars and $h = d + 3.5$ in. for two rows of tension steel. Round h to the next higher inch. Now check that $\lfloor \rho - \rho\prime(f'_s/f_y) \rfloor < \rho_{\max}$ using the new d, or check that $A_{s\,\max} = bd\lfloor \rho_{\max} + \rho'(f'_s/f_y) \rfloor \geq A_s$ (used).

$$\rho = \frac{A_s}{(bd)} \quad \text{and} \quad \rho\prime = \frac{A'_s}{(bd)}$$

   This check may not be needed if $\rho_{\max}$ is used in the basic section.
9. If desired, the design moment strength of the final section, ϕM_n, can be calculated and compared with the applied moment, M_u: $\phi M_n \geq M_u$. Note that a steel ratio ρ smaller than $\rho_{\max}$ can be assumed in step 1, say $\rho = 0.6\rho_b$ or $\rho = 0.9\rho_{\max}$, so that the final tension bars can be chosen to meet the given $\rho_{\max}$ limitation.
10. The strain at the bars level can be checked as follows:

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) 0.003 \geq 0.005$$

4.4.2 Assuming Two Rows of Tension Bars

In the case of two rows of bars, it can be assumed that $d = h - 3.5$ in. and $d_t = h - 2.5$ in. $= d + 1.0$ in.

Two approaches may be used to design the section.

1. One approach is to assume a strain at the level of the centroid of the tension steel equal to 0.005 or $\varepsilon_s = 0.005$ (at d level). In this case, the strain in the lower row of bars is greater than 0.005. $\varepsilon_t = (d_t - c/c)0.003 > 0.005$, which still meets the ACI Code limitation. For this case, follow the above steps 1 to 9. Example 4.6, solution 1 explains this approach.
2. A second approach is to assume a strain $\varepsilon_t = 0.005$ at the level of the lower row of bars, d_t. In this case, the stain at the level of the centroid of bars is less than 0.005: $\varepsilon_s = [(d_t - c)/c]0.003 < 0.005$, which is still acceptable. Example 4.6, solution 2 explains this approach. The solution can be summarized as follows:

a. Calculate $d_t = h - 2.5$ in., and then form the strain diagram and calculate c, the depth of the neutral axis

$$c = \left(\frac{0.003}{0.003 + \varepsilon_t}\right) d_t$$

For $\varepsilon_t = 0.005$,

$$c = \left(\frac{3}{8}\right) d_t \text{ and } a = \beta_1 c$$

b. Calculate the compression force in the concrete.

$$C_1 = 0.85 f'_c ab = T_1 = A_{s1} f_y$$

Determine A_{s1}. Calculate $M_{u1} = \phi A_{s1} f_y (d - a/2)$. $\rho_1 = A_{s1}/bd$, $\phi = 0.9$.

c. Calculate $M_{u2} = M_u - M_{u1}$; assume $d' = 2.5$ in.

d. Calculate A_{s2}: $M_{u2} = \phi A_{s2} f_y (d - d')$, $f'_c = f_y$, $\phi = 0.9$. Total $A_s = A_{s1} + A_{s2}$.

e. Check if compression steel yields similar to step 6 above in section 4.4.1.

Example 4.5

A beam section is limited to a width of $b = 10$ in. and a total depth of $h = 22$ in. and has to resist a factored moment of 226.5 K·ft. Calculate the required reinforcement. Given: $f'_c = 3$ ksi and $f_y = 50$ ksi.

Solution

1. Determine the design moment strength that is allowed for the section as singly reinforced based on tension-control conditions. This is done by starting with $\rho_{\max}$. For $f'_c = 3$ ksi and $f_y = 50$ ksi, and from Eqs. 3.18, 3.22, and 3.31,

$$\rho_b = 0.0275 \qquad \rho_{\max} = 0.01624 \qquad R_u = 614 \text{ psi}$$

$$M_u = R_u bd^2 \qquad b = 10 \text{ in.} \qquad d = 22 - 3.5 = 18.5 \text{ in.}$$

$$M_u = 226.5 \times 12 = 2718 \text{ K·in.}$$

(This calculation assumes two rows of steel, to be checked later.) $M_{u1} = 0.614 \times 10 \times (18.5)^2 = 2101$ K·in. = max ϕM_n, as singly reinforced. Design $M_u = 2718$ K·in. > 2101 K·in. Therefore, compression steel is needed to carry the difference.

2. Compute A_{s1}, M_{u1}, and M_{u2}:

$$A_{s1} = \rho_{\max} bd = 0.01624 \times 10 \times 18.5 = 3.0 \text{ in.}^2$$

$$M_{u1} = 2101 \text{ K·in.}$$

$$M_{u2} = M_u - M_{u1} = 2718 - 2102 = 617 \text{ K·in.}$$

3. Calculate A_{s2} and A'_s, the additional tension and compression steel due to M_{u2}. Assume $d' = 2.5$ in.; $M_{u2} = \phi\ A_{s2} f_y\ (d - d')$.

$$A_{s2} = \frac{M_{u2}}{\phi f_y (d - d')} = \frac{617}{0.9 \times 50(18.5 - 2.5)} = 0.86 \text{ in.}^2$$

Total tension steel is equal to A_s.

$$A_s = A_{s1} + A_{s2} = 3.0 + 0.86 = 3.86 \text{ in.}^2$$

The compression steel has $A'_s = 0.86$ in.^{2} (in A'_s yields).

4. Check if compression steel yields:

$$\varepsilon_y = \frac{f_y}{29{,}000} = \frac{50}{29{,}000} = 0.00172$$

Let $a = (A_{s1} f_y)/(0.85 f'_c b) = (3.0 \times 50)/(0.85 \times 3 \times 10) = 5.88$ in.

$$c(\text{distance to neutral axis}) = \frac{a}{\beta_1} = \frac{5.88}{0.85} = 6.92 \text{ in.}$$

$$\varepsilon'_s = \text{strain in compression steel (from strain triangles)}$$

$$= 0.003 \times \left(\frac{5.88 - 2.5}{5.88}\right) = 0.00173 > \varepsilon_y = 0.001724$$

5. Check ε_t:

$$\rho_1 = \frac{3}{10 \times 18.5} = 0.016216$$

$$\frac{\rho_1}{\rho_b} = 0.5897 \qquad f_y = 50$$

From Eq. 3.24, $\varepsilon_{t_s} = 0.005$ is assumed at the centroid of the tension steel for $\rho_{\max}$ and R_u used. Calculate ε_t (at the lower row of bars):

$$d_t = 22 - 2.5 = 19.5 \text{ in.}$$

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) 0.003$$

$$= \left(\frac{19.5 - 6.92}{6.92}\right) 0.003$$

$$= 0.00545 > 0.005$$

as expected.

6. Choose steel bars as follows: $A_s = 3.86 \text{ in.}^2$ Choose five no. 8 bars ($A_s = 3.95 \text{ in.}^2$) in two rows, as assumed. $A'_s = 0.86 \text{ in.}^2$ Choose two no. 6 bars ($A'_s = 0.88 \text{ in.}^2$).

7. Check actual d: Actual $d = 22 - (1.5 + 0.375 + 1.5) = 18.625$ in. It is equal approximately to the assumed depth. The final section is shown in Fig. 4.5.

Example 4.6

A beam section is limited to $b = 12$ in. and to a total depth of $h = 20$ in. and is subjected to a factored moment $M_u = 298.4$ K·ft. Determine the necessary reinforcement using $f'_c = 4$ ksi and $f_y = 60$ ksi. (Refer to Fig. 4.6.)

Solution 1: Two Solutions Are Presented

1. Determine the maximum moment capacity of the section as singly reinforced based on tension-controlled conditions. For $f'_c = 4$ ksi and $f_y = 60$ ksi, $\rho_{\max} = 0.01806$ and $R_u = 820$ psi (Table 4.1). Assuming two rows of bars, $d = 20 - 3.5 = 16.5$ in.

$$\text{Max } M_{u1} = R_{u\,\max} b d^2 = 0.82(12)(16.5)^2 = 2679 \text{ K·in.} = 223.25 \text{ K·ft.}$$

The design moment is $M_u = 298.4 \times 12 = 3581$ K·in. $> M_{u1}$; therefore, compression steel is needed.

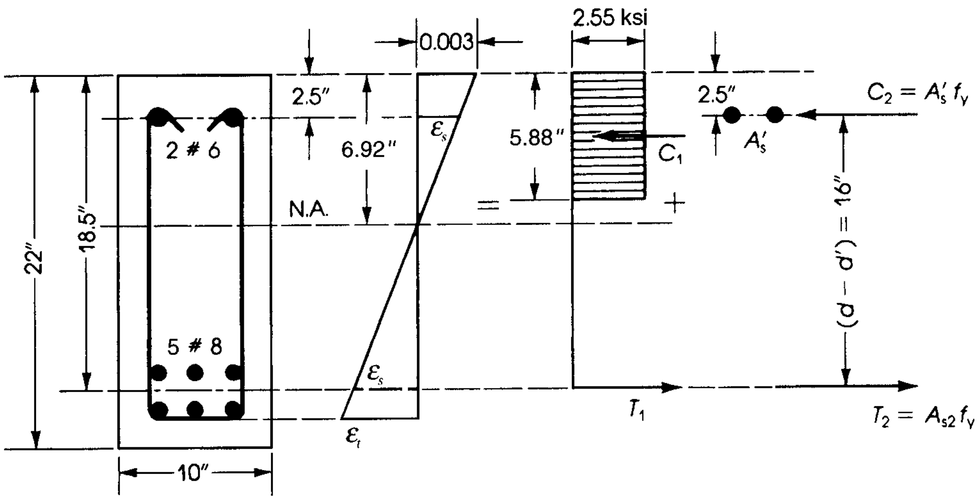


Figure 4.5 Example 4.5: doubly reinforced concrete section.

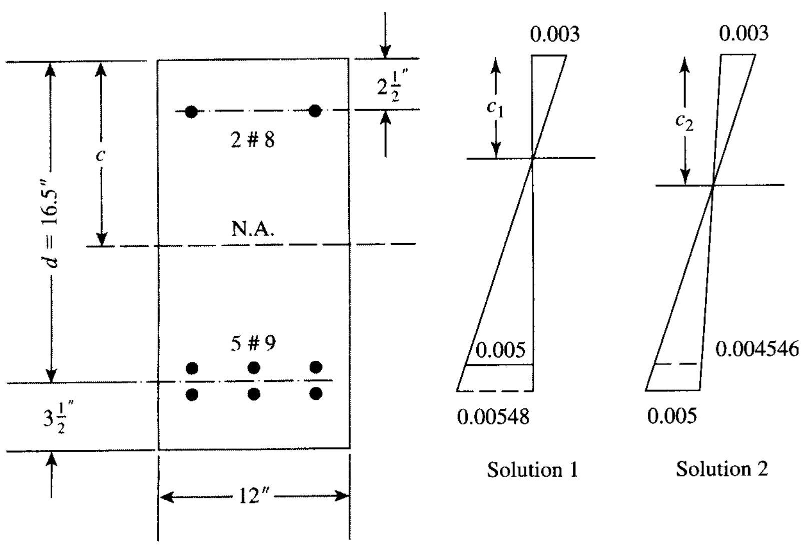


Figure 4.6 Example 4.6.

2. Calculate A_{s1}, M_{u2}, A_{s2}, and A_s.

$$A_{s1} = \rho_{\max} bd = 0.01806(12)(16.5) = 3.576 \text{ in.}^2$$

$$M_{u2} = M_u - M_{u1} = 3581 - 2679 = 902 \text{ K·in.}$$

$$M_{u2} = \phi A_{s2} f_y (d - d'), \text{ assume } d' = 2.5 \text{ in.}$$

$$902 = 0.9 A_{s2}(60)(16.5 - 2.5), \; A_{s2} = 1.19 \text{ in.}^2$$

$$\text{Total } A_s = A_{s1} + A_{s2} = 3.576 + 1.19 = 4.77 \text{ in.}^2 \qquad \text{(five no. 9 bars)}$$

3. Check if compression steel yields by Eq. 3.46. Compression steel yields if

$$\rho - \rho' \geq K = 0.85\beta_1 \frac{f'_c}{f_y}\left(\frac{d'}{d}\right)\left(\frac{87}{87 - f_y}\right)$$

$$K = (0.85)^2\left(\frac{4}{60}\right)\left(\frac{2.5}{16.5}\right)\left(\frac{87}{27}\right) = 0.0235$$

$$\rho - \rho' = \frac{A_{s1}}{bd} = \frac{3.576}{(12)(16.5)} = 0.01806 \leq K$$

Therefore, compression steel does not yield: $f'_s < f_y$

4. Calculate f'_s : $f'_s = 87[(c - d')/c] \leq f_y$. Determine c from A_{s1}: $A_{s1} = 3.576\,\text{in.}^2$,

$$a = \frac{A_{s1} f_y}{0.85 f'_c b} = \frac{3.576 \times 60}{0.85 \times 4 \times 12} = 5.26\ \text{in.}$$

$$c = \frac{a}{\beta_1} = \frac{5.26}{0.85} = 6.19\ \text{in.}$$

$$f'_s = 87 \times \left(\frac{6.19 - 2.5}{6.19}\right) = 51.8\ \text{ksi} < 60\ \text{ksi}$$

5. Calculate A'_s from $M_{u2} = \phi A'_s f'_s (d - d')$:

$$902 = 0.9A'_s(51.8)(16.5 - 2.5)$$

Thus, $A'_s = 1.38\ \text{in.}^2$, or calculate A'_s from $A'_s = A_{s2}(f_y/f'_s) = 1.38\ \text{in.}^2$ (two no. 8 bars). Note that the condition $[\rho - \rho'(f'_s/f_y)] = (\rho - \rho') \leq \rho_{\max}$ is already met.

$$\left(\rho - \rho'\frac{f'_s}{f_y}\right) = \frac{1}{bd}(A_s - A_{s2}) = \frac{3.576}{12 \times 16.5} = 0.01806$$

as assumed in the solution.

6. These calculations using $\rho_{\max}$ and R_u are based on a strain of 0.005 at the centroid of the tension steel.

$$\varepsilon_t(\text{at bottom row}) = \left(\frac{d_t - c}{c}\right)0.003$$

$$d_t = 20 - 2.5 = 17.5\ \text{in.} \qquad \varepsilon_t = \left(\frac{17.5 - 6.19}{6.19}\right)0.003 = 0.00548 > 0.005$$

as expected.

Solution 2

Assuming two rows of tension bars and a strain at the lower row, $\varepsilon_t = 0.005$, the solution will be as follows:

1. Calculate $d_t = 20 - 2.5 = 17.5\,\text{in.}$ From the strain diagram:

$$\frac{c}{d_t} = \frac{0.003}{0.003 + \varepsilon_t} = \frac{0.003}{0.008} = 0.375$$

$$c = 0.375(17.5) = 6.5625\ \text{in.} \qquad a = 0.85c = 5.578\ \text{in.}$$

2. The compression force in the concrete $= C_1 = 0.85\ f'_c ab$

$$C_1 = 0.85(4)(5.578)(12) = 227.6\text{ K} = T_1 \text{ (as singly reinforced)}$$

$$A_{s1} = \frac{C_1}{f_y} = \frac{T_1}{f_y} = \frac{227.6}{60} = 3.793\text{ in.}^2$$

$$d = 20 - 3.5 = 16.5\text{ in.}$$

$$M_{u1} = \phi A_{s1} f_y \left(d - \frac{a}{2}\right) = 0.9(3.793)(60)\left(16.5 - \frac{5.578}{2}\right) = 2808\text{ K}\cdot\text{in.}$$

$$= 234\text{ K}\cdot\text{ft}$$

$$R_{u1} = \frac{M_{u1}}{bd^2} = \frac{2808.3}{12(16.5)^2} = 0.86\text{ ksi} = 860\text{ psi}$$

$$\rho_1 = \frac{A_{s1}}{bd} = 0.01916$$

3. Since $M_u = 3581$ K·in. $> M_{u1}$, compression steel is needed.

$$M_{u2} = 3581 - 2808 = 773\text{ K}\cdot\text{in.}$$

$$M_{u2} = 0.9 A_{s2} f_y (d - d')$$

$$773 = 0.9 A_{s2}(60)(16.5 - 2.5) \qquad A_{s2} = 1.022\text{ in.}^2$$

$$\text{Total } A_s = A_{s1} + A_{s2} = 3.793 + 1.022 = 4.815\text{ in.}^2$$

Use five no. 9 bars.

4. Check if compression steel yields as in step 3 in the first solution.

$$K = 0.0235(\rho - \rho') = \rho_1 = 0.01916 < K$$

Compression steel does not yield.

$$f'_s = 87\left(\frac{c - d'}{c}\right) = \left(\frac{6.56 - 2.5}{6.56}\right)87 = 53.84\text{ ksi}$$

Calculate A_{s2}:

$$M_{u2} = \phi A'_s f'_s (d - d')$$

$$773 = 0.9 A'_s (53.84)(16.5 - 2.5) \qquad A'_s = 1.14\text{ in.}^2$$

Use two no. 7 bars ($A'_s = 1.2$ in.^{2}).

5. Check the design moment strength.

$$A_s = 5.0\text{ in.}^2 \qquad A'_s = 1.2\text{ in.}^2 \qquad A_{s1} = (A_s - A'_s) = 3.8\text{ in.}^2$$

$$\phi M_n = \phi\left[A_{s1} f_y \left(d - \frac{a}{2}\right) + A'_s f'_s (d - d')\right]$$

$$= 0.9[3.8(60)(16.5 - 5.578/2) + 1.2(53.84)(16.5 - 2.5)]$$

$$= 3627.6\text{ K}\cdot\text{in.} = 302.3\text{ K}\cdot\text{ft.}$$

which is adequate. Note that the strain ε_s at the centroid level of the tension steel is less than 0.005.

$$\varepsilon_s = \left(\frac{d - c}{c}\right)0.003 = \left(\frac{16.5 - 6.56}{6.56}\right)0.003 = 0.004546$$

Both solutions are adequate.

Discussion

1. In the first solution, the net tensile strain, $\varepsilon_t = 0.005$, was assumed at the centroid of the tension steel. In this case $\rho_{\max}$ and $R_{u\max}$ can be determined from Table 4.1 or tables in Appendix A. The strain in the lower row of bars will be always greater than 0.005, which meets the ACI Code requirement.
2. In the second solution, the strain limit, $\varepsilon_t = 0.005$, is assumed at the lower row. In this case, the strain at the centroid of the two rows of bars will be less than 0.005 and its value depends on the depth of the section. Moreover, ρ and R_u for this case are not known and their values depend on the effective depth d.
3. Comparing the two solutions, the neutral axis depth, c_1, in solution 1 is slightly smaller than c_2 for the second solution because of the strain limitations, producing a smaller A_{s1} and then higher A_{s2}. Total A_s will normally be very close. It is clear that solution 1 is easier to use because of the use of tables.
4. Note that solution 1 can have the same results as solution 2 by calculating A_{s1} as follows: $A_{s1} = \rho_{\max}\, bd_t = 0.01806\ (12 \times 17.5) = 3.793\ \text{in}^2$, which is the same A_{s1} calculated in solution 2, producing $\varepsilon_t = 0.005$ at the lower row of bars.

4.5 DESIGN OF T-SECTIONS

In slab-beam-girder construction, the slab dimensions as well as the spacing and position of beams are established first. The next step is to design the supporting beams, namely, the dimensions of the web and the steel reinforcement. Referring to the analysis of T-section in the previous chapter, we can see that a large area of the compression flange, forming a part of the slab, is effective in resisting a great part or all of the compression force due to bending. If the section is designed on this basis, the depth of the web will be small; consequently, the moment arm is small, resulting in a large amount of tension steel, which is not favorable. Shear requirements should be met, and this usually requires quite a deep section.

In many cases web dimensions can be known based on the flexural design of the section at the support in a continuous beam. The section at the support is subjected to a negative moment, the slab being under tension and considered not effective, and the beam width is that of the web.

In the design of a T-section for a given factored moment, M_u, the flange thickness, t, and width, b, would have been already established from the design of the slab and the ACI Code limitations for the effective flange width, b, as given in Section 3.15. The web thickness, b_w, can be assumed to vary between 8 in. and 20 in., with a practical width of 12 to 16 in. Two more unknowns still need to be determined, d and A_s. Knowing that M_u, f'_c, and f_y are always given, two cases may develop as follows:

1. When d is given and we must calculate A_s,
   - **a.** Check if the section acts as a rectangular or T-section by assuming $a = t$ and calculating the moment strength of the whole flange:

$$\phi M_{\text{nf}}(\text{flange}) = \phi(0.85\ f'_c)bt\left(d - \frac{t}{2}\right) \tag{4.7}$$

     If $M_u > \phi M_{\text{nf}}$, then $a > t$. If $M_u < \phi M_{\text{nf}}$, then $a < t$, and the section behaves as a rectangular section.
   - **b.** If $a < t$, then calculate ρ using Eq. 4.2, and $A_s = \rho bd$. Check that $\rho_w \geq \rho_{\min}$.

c. If $a > t$, determine A_{sf} for the overhanging portions of the flange, as explained in Section 3.15.4.

$$A_{sf} = 0.85f_c'(b - b_w)t/f_y \tag{4.8}$$

$$M_{u2} = \phi A_{sf} f_y \left(d - \frac{t}{2}\right) \tag{4.9}$$

The moment resisted by the web is

$$M_{u1} = M_u - M_{u2}$$

Calculate ρ_1 using M_{u1}, b_w, and d in Eq. 4.2 and determine $A_{s1} = \rho_1 b_w d$.

$$\text{Total } A_s = A_{s1} + A_{sf}$$

Then check that $A_s \leq A_{s\,\max}$, as explained in Section 3.15. Also check that $\rho_w = A_s/(b_w d) \geq \rho_{\min}$.

d. If $a = t$, then $A_s = \phi(0.85f_c')bt/f_y$.

2. When d and A_s are not known, the design may proceed as follows:

a. Assume $a = t$ and calculate the amount of total steel, A_{sft}, needed to resist the compression force in the whole flange, bt.

$$A_{sft} = \frac{(0.85\ f_c')bt}{f_y} \tag{4.10}$$

b. Calculate d based on A_{sft} and $a = t$ from the following equation:

$$M_u = \phi A_{sft} f_y \left(d - \frac{t}{2}\right) \tag{4.11}$$

If the depth, d, is acceptable, then $A_s = A_{sft}$ and $h = d + 2.5$ in. for one row of bars or $h = d + 3.5$ in. for two rows of bars.

c. If a new d_1 is adopted greater than the calculated d, then the section behaves as a rectangular section, and ρ can be calculated using Eq. 4.2; $A_s = \rho \mathrm{bd} < A_{sft}$.

d. If a new d_2 is adopted that is smaller than the calculated d, then the section will act as a T-section, and the final A_s will be greater then A_{sft}. In this case, proceed as in step 1(c) to calculate A_s.

Example 4.7

The T-beam section shown in Fig. 4.7 has a web width, b_w, of 10 in., a flange width, b, of 40 in., a flange thickness of 4 in., and an effective depth, d, of 14.5 in. Determine the necessary reinforcement if the applied factored moment is 3350 K·in. Given: $f_c' = 3$ ksi and $f_y = 60$ ksi.

Solution

1. Check the position of the neutral axis; the section may be rectangular. Assume the depth of compression block a is 4 in.; that is, $a = t = 4$ in. Then

$$\phi M_n = \phi(0.85f_c')bt\left(d - \frac{t}{2}\right) = 4590 \text{ K·in.} > M_u = 3350 \text{ K·in.}$$

The design moment that the concrete flange can resist is greater than the factored applied moment. Therefore, the section behaves as a rectangular section.

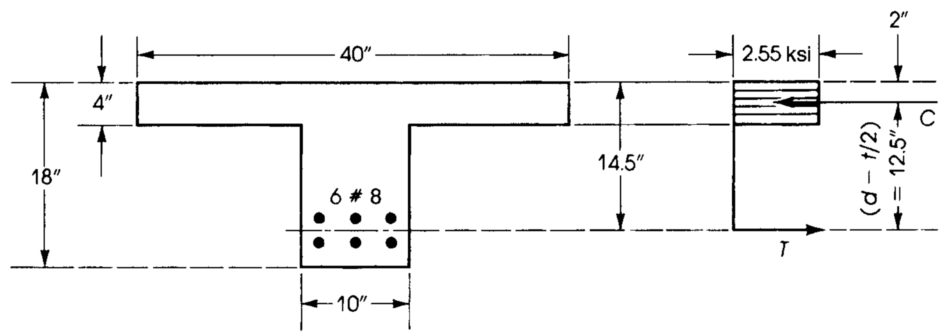


Figure 4.7 Example 4.7: T-section.

2. Determine the area of tension steel, considering a rectangular section, $b = 40$ in.

$$R_u = \phi M_n/(bd^2) = \frac{3{,}350{,}000}{40 \times 14.5^2} = 398 \text{ psi}$$

From Eq. 4.2 or from tables in Appendix A, for $R_u = 398$ psi, and $\rho = 0.00817$,

$$A_s = \rho bd = 0.00817 \times 40 \times 14.5 = 4.74 \text{ in.}^2$$

Use six no. 8 bars, $A_s = 4.74$ in.^{2} (in two rows).

3. Check that $\rho_w = A_s/b_w d \geq \rho_{\min}$; $\rho_w = 4.74/(10 \times 14.5) = 0.0327 > \rho_{\min} = 0.00333$. Note that A_s used is less than $A_{s\max}$ of 7.06 in.^{2} Calculated by Eq. 3.72.

Also, $a = 2.788$ in., $c = 3.28$ in., $d_t = 14.5$ in., and $\varepsilon_t = 0.003(d_t - c)/c = 0.01 > 0.005$, which is o.k.

Example 4.8

The floor system shown in Fig. 4.8 consists of 3-in. slabs supported by 14-ft-span beams spaced at 10 ft on centers. The beams have a web width, b_w, of 14 in. and an effective depth, d, of 18.5 in. Calculate the necessary reinforcement for a typical interior beam if the factored applied moment is 5080 K·in. Use $f'_c = 3$ ksi and $f_y = 60$ ksi.

Solution

1. Find the beam flange width: Flange width is the smallest of

$$b = 16t + b_w = 3 \times 16 + 12 = 60 \text{ in.}$$

$$b = \frac{\text{span}}{4} = \frac{14 \times 12}{4} = 42 \text{ in.}$$

Center-to-center of adjacent slabs is $10 \times 12 = 120$ in. Use $b = 42$ in.

2. Check the position of the neutral axis, assuming $a = t$.

$$\phi M_n \text{ (based on flange)} = \phi \times 0.85 f'_c bt \left(d - \frac{t}{2}\right)$$

$$= 0.9 \times 0.85 \times 3 \times 42 \times 3(18.5 - 1.5) = 4916 \text{ K·in.}$$

The applied moment is $M_u = 5080$ K·in. > 4916 K·in.; the beam acts as a T-section, so $a > t$.

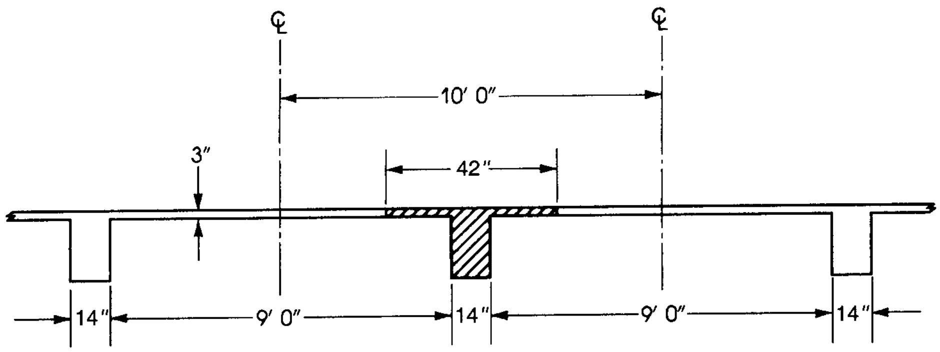


Figure 4.8 Example 4.8: effective flange width.

3. Find the portion of the design moment taken by the overhanging portions of the flange (Fig. 4.9). First calculate the area of steel required to develop a tension force balancing the compressive force in the projecting portions of the flange:

$$A_{sf} = \frac{0.85 f_c'(b - b_w)t}{f_y} = \frac{0.85 \times 3 \times (42 - 14) \times 3}{60} = 3.57 \text{ in.}$$

$\phi M_n = M_{u1} + M_{u2}$, that is, the sum of the design moment of the web and the design moment of the flanges.

$$M_{u2} = \phi A_{sf} f_y \left(d - \frac{t}{2}\right)$$

$$= 0.9 \times 3.57 \times 60 \left(18.5 - \frac{3}{2}\right) = 3272 \text{ K·in.}$$

4. Calculate the design moment of the web (as a singly reinforced rectangular section):

$$M_{u1} = M_u - M_{u2} = 5080 - 3272 = 1808 \text{ K·in.}$$

$$R_u = \frac{M_{u1}}{(b_w d^2)} = \frac{1,808,000}{14 \times (18.5)^2} = 377 \text{ psi}$$

From Eq. 4.2 or the tables in Appendix A, for $R_u = 377$ psi, $\rho_1 = 0.0077$.

$$A_{s1} = \rho_1 b_w d = 0.0077(14)(18.5) = 1.99 \text{ in.}^2$$

Total $A_s = A_{sf} + A_{s1} = 3.57 + 1.99 = 5.56$ in.^{2} (Use six no. 9 bars in two rows.)

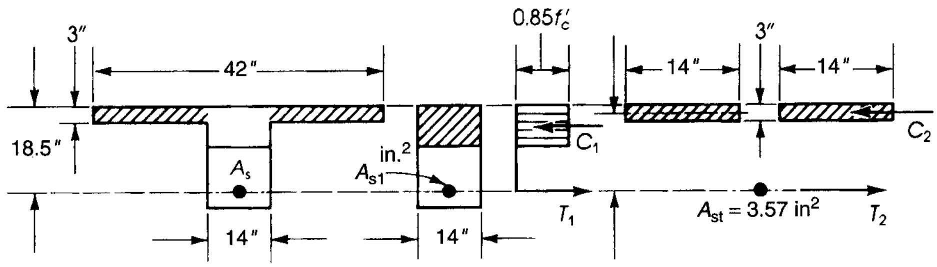


Figure 4.9 Analysis of Example 4.8.

5. Total $h = 18.5 + 3.5 = 22$ in. Calculate $A_{s\,\max}$ for T-sections using Eq. 3.72:

$$\text{Max } A_s = 7.02 \text{ in.}^2 > 5.56 \text{ in.}^2$$

6. Check ε_t: $a = 1.99 \times 60/(0.85 \times 3 \times 14) = 3.34$ in., $c = 3.93$ in., $d_t = 19.5$ in. Then $\varepsilon_t = 0.003(d_t - c)/c = 0.0119 > 0.005$, tension-controlled section ($\phi = 0.9$).

Example 4.9

In a slab-beam system, the flange width was determined to be 48 in., the web width was $b_w = 16$ in., and the slab thickness was $t = 4$ in. (Fig. 4.10). Design a T-section to resist an external factored moment of $M_u = 812$ K·ft. Use $f'_c = 3$ ksi and $f_y = 60$ ksi.

Solution

1. Because the effective depth is not given, let $a = t$ and calculate A_{sft} for the whole flange.

$$A_{\text{sft}} = \frac{0.85 f'_c bt}{f_y} = \frac{0.85(3)(48)(4)}{60} = 8.16 \text{ in.}^2$$

Let $M_u = \phi A_{\text{sft}} f_y (d - t/2)$ and calculate d:

$$812 \times 12 = 0.9(8.16)(60)\left(d - \frac{4}{2}\right) \qquad d = 24.1 \text{ in.}$$

Now, if an effective $d = 24.1$ in. is chosen, then $A_s = A_{\text{sft}} = 8.16$ in.^{2}

2. If a depth $d > 24.1$ in. is chosen, say 26.5 in., then $a < t$ and it is a rectangular analysis. The steel ratio can be calculated from Eq. 4.2 with $\rho = 0.00574$ and $A_s = \rho bd = 0.00574 \times 48 \times 26.5 = 7.3$ in.^{2} (six no. 10 bars in two rows, $A_s = 7.62$ in.^{2}).
3. If a depth $d < 24.1$ in. is chosen, say, 23.5 in., then $a > t$, and the section behaves as a T-section. Calculate

$$A_{\text{sf}} = 0.85 f'_c t(b - b_w)/f_y = 0.85(3)(4)(48 - 16)/60 = 5.44 \text{ in.}^2$$

$$M_{u2} = \phi A_{\text{sf}} f_y \left(d - \frac{t}{2}\right) = 0.9(5.44)(60)\left(23.5 - \frac{4}{2}\right) = 6316 \text{ K·in.}$$

$$M_{u1} = 812 \times 12 - 6316 = 3428 \text{ K·in.}$$

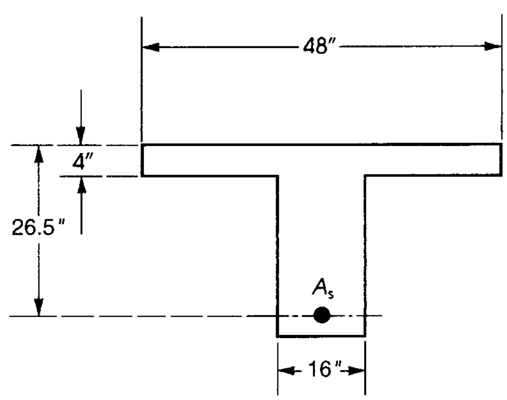


Figure 4.10 Example 4.9.

4. For the basic singly reinforced section, $b_w = 16$ in., $d = 23.5$ in., and $M_{u1} = 3428$ K·in., $R_u = 387$ psi. Calculate ρ_1 from Eq. 4.2 to get $\rho_1 = 0.0079$.

$$A_{s1} = \rho_1 b_w d = 0.0079(16)(23.5) = 2.97 \text{ in.}^2$$

$$\text{Total } A_s = A_{sf} + A_{s1} = 5.44 + 2.97 = 8.41 \text{ in.}^2 \text{(seven no. 10 bars in two rows,}$$
$$A_s = 8.89 \text{ in.}^2\text{)}$$

5. Check ε_t: $a = 2.97 \times 60/(.85 \times 3 \times 16) = 4.368$ in., $c = a/0.85 = 5.14$ in., $d_t = 24.5$ in., and $\varepsilon_t = 0.003\ (d_t - c)/c = 0.0113 > 0.005$, a tension-controlled section.
6. Calculate the total max A_s that can be used for the T-section by Eq. 3.72:

$$\text{Max } A_s =$$
$$= 0.0425[(b - bw)t + 0.319bwd] = 10.54 \text{ in.}^2$$
$$A_s \text{ (used)} \leq \max A_s$$

7. *Note* : If there are no restrictions on the total depth of the beam, it is a common practice to adopt the case when $a \leq t$ (step 2). This is because an increase in d produces a small increase in concrete in the web only while decreasing the quantity of A_s required.

4.6 ADDITIONAL EXAMPLES

The following design examples give some practical applications and combine structural analysis with concrete design of beams and frames.

Example 4.10

For the precast concrete I-section shown in Fig. 4.11, calculate the reinforcement needed to support a factored moment of 360 K·ft. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

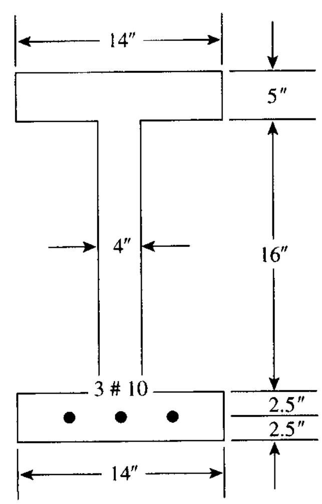


Figure 4.11 Example 4.10.

Solution

Determine if the force in the flange area 14 × 5 in. will be sufficient to resist a factored moment of 360 K·ft. Let $d = 23.5$ in. Force in flange $(C_c) = 0.85 \times f'_c$ (flange area) $= 0.85 \times 4 \times (14 \times 5) =$ 238 K, located at 2.5 in. from the top fibers, and $a = 5$ in.

$$\phi M_n = 0.9 C_c \left(d - \frac{a}{2}\right) = 0.9 \times 238 \frac{(23.5 - 2.5)}{12} = 374.9 \text{ K·ft}$$

which is greater than the applied moment of 360 K·ft. Therefore, a is less than 5 in.

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right), \text{ where } a = \frac{A_s f_y}{(0.85 f'_c b)}$$

$$360 \times 12 = 0.9 A_s (60) \left(\frac{23.5 - 60A_s}{(1.7 \times 4 \times 14)}\right)$$

Solve to get $A_s = 3.79 \text{ in.}^2$ Or use Eq. 4.2 to get $\rho = 0.01152$ and $A_s = 0.01152 \times 14 \times 23.5 =$ 3.79 in.^2 Use three no. 10 bars in one row, as shown in Fig. 4.11.

For similar T-sections or I-sections, it is better to adopt a section with a flange size to accommodate the compression force, C_c. In this case, a is less than or equal to the flange depth. The bottom flange is in tension and not effective.

Example 4.11

The simply supported beam shown in Fig. 4.12 carries a uniform service load of 2.8 K/ft (including self-weight) in addition to a service load of 1.6 K/ft. Also, the beam supports a concentrated dead load of 16 K and a concentrated live load of 7 K at C, 10 ft from support A.

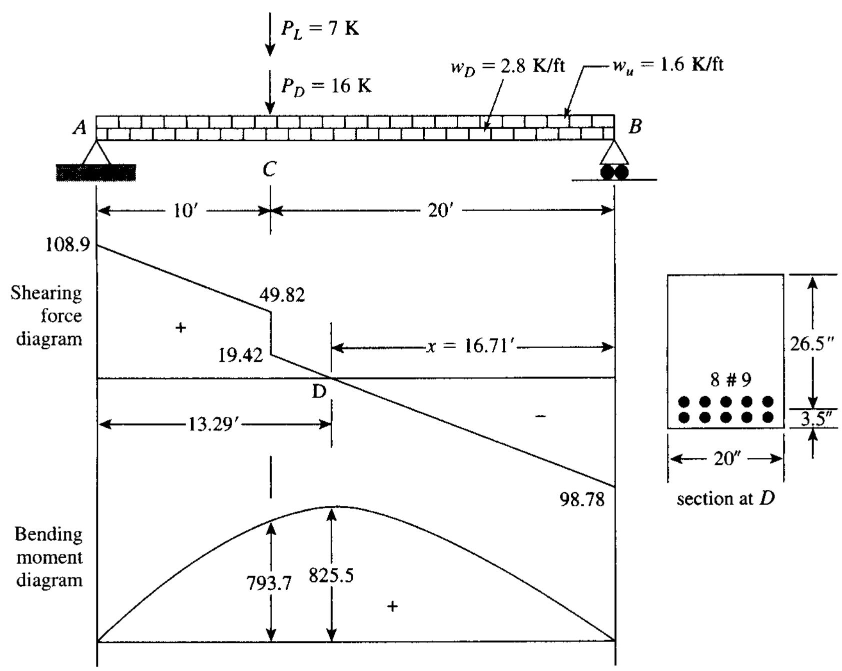


Figure 4.12 Example 4.11.

a. Determine the maximum factored moment and its location on the beam.

b. Design a rectangular section to carry the loads safely using a steel percentage of about 1.5%, $b = 20$ in., $f_c' = 4$ ksi, and $f_y = 60$ ksi.

Solution

a. Calculate the uniform factored load: $w_u = 1.2(2.8) + 1.6(1.6) = 5.91$ K/ft. Calculate the concentrated factored load: $P_u = 1.2(16) + 1.6(7) = 30.4$ K. Calculate the reaction at A by taking moments about B:

$$R_A = 5.91(30)\frac{(30/2)}{30} + \frac{30.4(20)}{30} = 108.92 \text{ K}$$

$$R_B = 5.91(30) + 30.4 - 108.92 = 98.78 \text{ K}$$

Maximum moment in the beam occurs at zero shear. Starting from B,

$$V = 0 = 98.78 - 5.91x \text{ and } x = 16.71 \text{ ft from } B \text{ at } D$$

$$M_u \text{ (at } D) = 98.72(16.71) - 5.91(16.71)\left(\frac{16.71}{2}\right) = 825.5 \text{ K·ft} \quad \text{(design moment)}$$

$$M_u \text{ (at } C) = 98.78(20) - 5.91(20)\left(\frac{20}{2}\right) = 793.6 \text{ K·ft}$$

b. Design of the section at D: For $f_c' = 4$ ksi, and $f_y = 60$ ksi, $\rho_{\max} = 0.01806$ and $\rho_{\min} = 0.00333$, and the design steel ratio of 1.5% is within the limits. For $\rho = 0.015$, $R_u = 700$ psi (from Table A.2) or from Eq. 3.22.

$$M_u = R_u bd^2 \quad \text{or} \quad 825.5 \times 12 = 0.7(20)d^2$$

Solve to get $d = 26.6$ in.

$$A_s = 0.015 \times 20 \times 26.6 = 7.98 \text{ in.}^2$$

Choose eight no. 9 bars in two rows (area $= 8$ in.^{2}), five in the lower row plus three in the upper row. Minimum b for five no. 9 bars in one row is 14 in. (Table A.7). Total depth $(h) = 26.6 + 3.5 = 30.1$ in. Use $h = 30$ in. Actual $d = 30 - 3.5 = 26.5$ in. Check the moment capacity of the section, $a = 8 \times 60/(0.85 \times 4 \times 20) = 7.06$ in.

$$\phi M_n = 0.9 \times 8 \times 60 \frac{\left(26.5 - \frac{7.06}{2}\right)}{12} = 826.9 \text{ K·ft}$$

which is greater than 825.5 K·ft. Check that $A_s = 8$ in.^{2} is less than $A_{s\max}$.

$$A_{s\max} = 0.01806 \times 20 \times 26.5 = 9.57 \text{ in.}^2$$

which exceeds 8 in.^{2} The final section is shown in Fig. 4.12.

Example 4.12

The two-hinged frame shown in Fig. 4.13 carries a uniform service dead load (including estimated self-weight) of 2.33 K/ft and a uniform service live load of 1.5 K/ft on frame beam BC. The moment at the corner B (or C) can be evaluated for this frame dimension, $M_b = M_c = -wL^2/18.4$ and the reaction at A or $D = wL/2$. A typical section of beam BC is shown, the column section is 16 × 21 in. It is required to

a. Draw the bending moment and shear diagrams for the frame $ABCD$ showing all critical values.

b. Design the beam BC for the factored moments, positive and negative, using $f_c' = 4$ ksi and $f_y = 60$ ksi. Show reinforcement details.

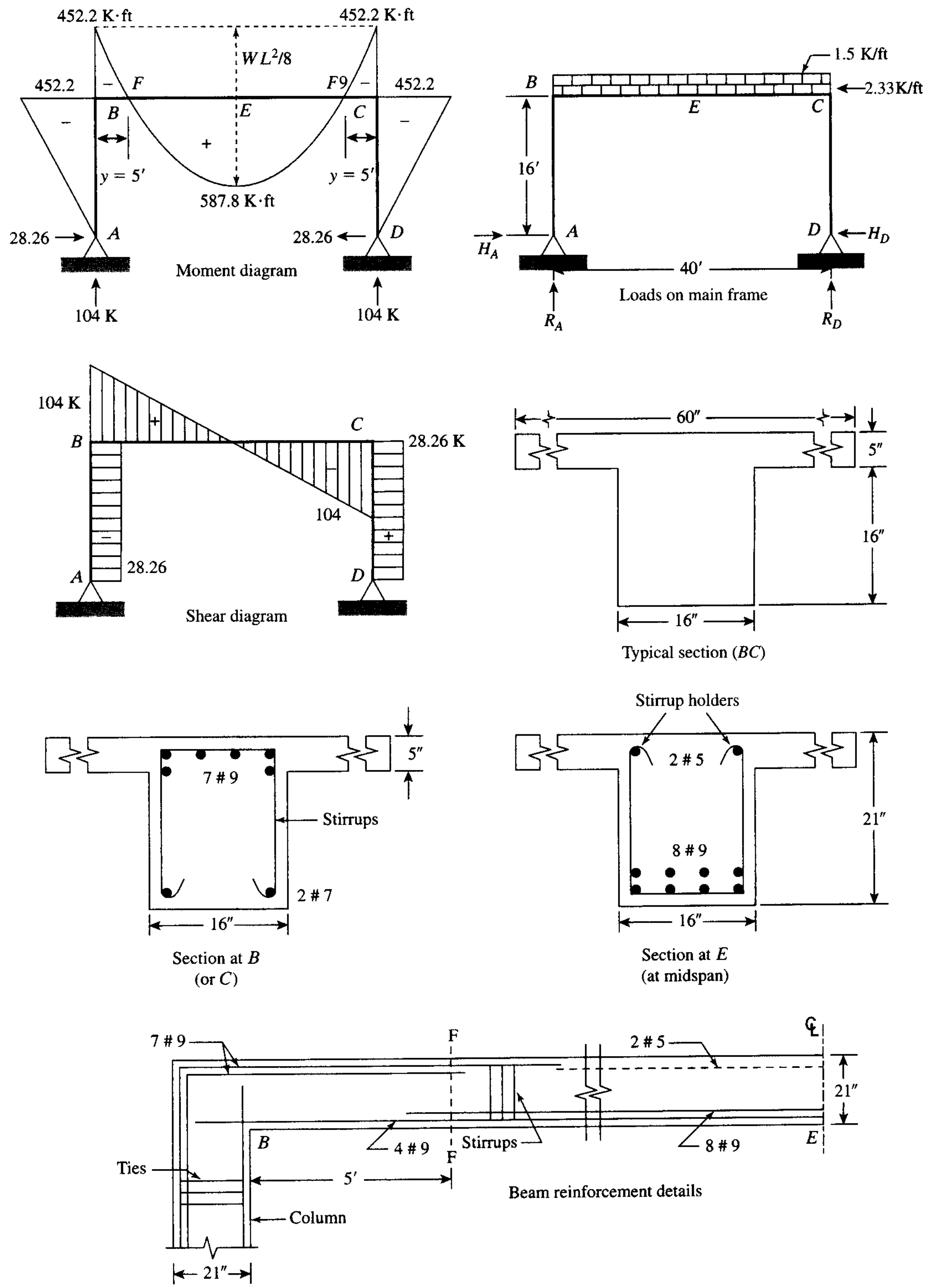


Figure 4.13 Example 4.12.

Solution

a. Calculate the forces acting on the frame using a computer program or the values mentioned previously. Factored load (w_u) = 1.2(2.33) + 1.6(1.5) = 5.2 K/ft. Because of symmetry, $M_B = M_C = -\,wL^2/18.4 = -\,5.2(40)^2/18.4 = -452.2$ K·ft. Positive moment at midspan (E) $= w_u L^2/8 + M_B = 5.2(40)^2/8 - 452.2 = 587.8$ K·ft. Vertical reaction at $A = R_A = R_D = w_u L/2 = 5.2(40)/2 = 104$ K. Horizontal reaction at $A = H_A = M_B/h = 452.2/16 = 28.26$ K. The moment and shear diagrams are shown in Fig. 4.13.

Determine the location of zero moment at section F on beam BC by taking moments about F:

$$104(y) - 28.26(16) - 5.2(y)^2/2 = 0 \qquad y = 4.963 \text{ ft}, \qquad \text{say, 5 ft from joint } B$$

b. Design of beam BC:

1. Design of section E at midspan: $M_u = +\,587.8$ K·ft. Assuming two rows of bars, $d = 21 - 3.5 = 17.5$ in. Calculate the moment capacity of the flange using $a = 5.0$ in.

$$\phi M_n(\text{flange}) = \phi(0.85 f_c')ab\left(d - \frac{5}{2}\right)$$

$$= 0.9(0.85 \times 4) \times (5 \times 60) \times \frac{(17.5 - 2.5)}{12} = 1147.5 \text{ K·ft}$$

which is greater than the applied moment; therefore, a is less than 5.0 in.

Assume $a = 2.0$ in. and calculate A_s.

$$M_u = \phi A_s f_y \left(d - \frac{a}{2}\right)$$

$$587.8 \times 12 = 0.9 \times 60 A_s (17.5 - 1.0) \quad \text{and} \quad A_s = 7.92 \text{ in.}^2$$

Check assumed $a = A_s f_y/(0.85 f_c' b) = 7.92 \times 60/(0.85 \times 4 \times 60) = 2.33$ in. Revised $A_s = 587.8 \times 12/(0.9 \times 60 \times 16.33) = 7.99$ in.^{2} Check revised a: $a = 7.99 \times 2.33/7.92 = 2.35$ in., which is very close to 2.33 in.

Alternatively, Eq. 4.2 can be used to get ρ and A_s. Choose eight no. 9 bars in two rows (area = 8.0 in.^{2}), ($b_{\min} = 11.8$ in.). Extend four no. 9 bars on both sides to the columns. The other four bars can terminate where they are not needed, beyond section F; see the longitudinal section in Fig. 4.13.

2. Design of section at B: $M_u = -452.2$ K·ft. The section acts as a rectangular section, $b = 16$ in. and $d = 17.5$ in. The main tension reinforcement lies in the flange.

$$\rho_{\max} = 0.01806 \quad \text{and} \quad R_{u\max} = 820 \text{ psi (Table 4.1)}$$

Check the maximum moment capacity of the section as singly reinforced.

$$\phi M_{n\max} = R_{u\max} bd^2 = 0.82(16)(17.5)^2/12 = 334.8 \text{ K·ft}$$

which is less than the applied moment. Compression steel is needed.

$$A_{s1} = 0.01806(16)(17.5) = 5.06 \text{ in.}^2$$

$$M_{u2} = 452.2 - 334.8 = 117.4 \text{ K·ft}$$

$$M_{u2} = \phi A_{s2} f_y (d - d'); \text{ assume } d' = 2.5 \text{ in.}$$

$$117.4 \times 12 = 0.9 A_{s2}(60)(17.5 - 2.5) \quad \text{and} \quad A_{s2} = 1.74 \text{ in.}^2$$

Total tension steel = 5.06 + 1.74 = 6.8 in.^{2} Use seven no. 9 bars in two rows (area used = 7.0 in.^{2}, which is adequate). For compression steel, use two no. 9 bars (area = 2.0 in.^{2}), extended from the positive moment reinforcement to the column. Actually, four no. 9 bars are available; see the longitudinal section in Fig. 4.13.

The seven no. 9 bars must extend in the beam BC beyond section F into the compression zone, and also must extend into the column BA to resist the column moment of 452.2 K·ft without any splices at joints B or C.

Check if compression steel yields by using Eq. 3.49 or Table 3.4. K = 0.01552 (d'/d) = 0.1552(2.5)/(17.5) = 0.02217 > ρ_1 = 0.01806. Therefore, compression steel yields, and $f_s' = 60$ ksi as assumed.

Stirrups are shown in the beam to resist shear (refer to Chapter 8), and two no. 5 bars were placed at the top of the beam to hold the stirrups in position. Ties are used in the column to hold the vertical bars (refer to Chapter 10). To determine the extension of the development length of bars in beams or columns, refer to Chapter 7.

4.7 EXAMPLES USING SI UNITS

Example 4.13

Design a singly reinforced rectangular section to resist a factored moment of 280 kN·m using the maximum steel percentage for tension-controlled sections. Given: $f_c' = 20$ N/mm^2, $f_y = 400$ N/mm^2, and $b = 250$ mm.

Solution

$$\rho_b = (0.85)\beta_1 \left[\frac{f_c'}{f_y}\right]\left(\frac{600}{600+f_y}\right)$$

$$= 0.85 \times 0.85 \times \frac{20}{400} \times \left(\frac{600}{600+400}\right) = 0.0217$$

$$\rho_{\max} = \left(\frac{0.003 + \frac{f_y}{E_s}}{0.008}\right)\rho_b \qquad E_s = 200{,}000 \text{ MPa} \qquad \frac{f_y}{E_s} = 0.002$$

$$= 0.625\ \rho_b = 0.01356$$

$$\phi = 0.9$$

$$R_{u\max} = \phi\rho_{\max} f_y \left(1 - \frac{\rho_{\max} f_y}{1.7 f_c'}\right)$$

$$= 0.9 \times 0.01356 \times 400 \left(1 - \frac{0.01356 \times 400}{1.7 \times 20}\right) = 4.1 \text{ N/mm}^2\text{(MPa)}$$

$$M_u = R_u b d^2$$

$$d = \sqrt{\frac{M_u}{R_u b}} = \sqrt{\frac{280 \times 10^6}{4.1 \times 250}} = 523 \text{ mm}$$

$$A_s = \rho b d = 0.01356 \times 250 \times 523 = 1772 \text{ mm}^2 = 17.72 \text{ cm}^2$$

Choose four bars, 25 mm diameter, in two rows.

A_s provided = 4 × 4.9 = 19.6 cm^2. Total depth is

$$h = d + 25 \text{ mm} + 60 \text{ mm}$$

$$= 523 + 25 + 60 = 608 \text{ mm} \qquad \text{say, 610 mm (or 600 mm)}$$

Check minimum width:

$$b_{\min} = 2D + 1S + 95 \text{ mm} = 3 \times 25 + 95 = 170 \text{ mm} < 250 \text{ mm}$$

Bars are placed in two rows.

Example 4.14

Calculate the required reinforcement for a beam that has a section of $b = 300\,\text{mm}$ and a total depth of $h = 600\,\text{mm}$ to resist $M_u = 696$ kN·m. Given: $f_c' = 30$ N/mm^2 and $f_y = 420\,\text{N/mm}^2$.

Solution

1. Determine the design moment strength of the section using $\rho_{\max}$ (for tension-controlled section, $\phi = 0.9$):

$$\rho_b = (0.85)\beta_1 \left[\frac{f_c'}{f_y}\right]\left(\frac{600}{600 + f_y}\right)$$

$$= 0.85 \times 0.85 \times \frac{30}{420} \times \left(\frac{600}{600 + 1020}\right) = 0.0304$$

$$\rho_{\max} = \left(\frac{0.003 + \frac{f_y}{E_s}}{0.008}\right)\rho_b = 0.6375\ \rho_b = 0.01938$$

$$R_{u\max} = \phi\rho_{\max} f_y\left(1 - \frac{\rho_{\max} f_y}{1.7 f_c'}\right)$$

$$= 0.9 \times 0.01938 \times 420\left(1 - \frac{0.01938 \times 420}{1.7 \times 30}\right) = 6.16 \text{ N/mm}^2(\text{MPa})$$

$$d = h - 85 \text{ mm(assuming two rows of bars)}$$

$$= 600 - 85 = 515 \text{ mm}$$

$$\phi M_n = R_u bd^2 = 6.16 \times 300 \times (515)^2 \times 10^{-6} = 490 \text{ kN·m}$$

This is less than the external moment; therefore, compression reinforcement is needed.

2. Calculate A_{s1}, M_{u1}, and M_{u2}:

$$A_{s1} = \rho_{\max} bd = 0.01938 \times 300 \times 515 = 2994 \text{ mm}^2$$

$$M_{u2} = M_u - M_{u1} = 696 - 490 = 206 \text{ kN·m}$$

3. Calculate A_{s2} and A_s' due to M_{u2}. Assume d$' = 60\,\text{mm}$:

$$M_{u2} = \phi A_{s2} f_y (d - d')$$

$$206 \times 10^6 = 0.9 A_{s2} \times 420(515 - 60) \quad A_{s2} = 1198 \text{ mm}^2$$

Total tension steel is $2994 + 1198 = 4192\,\text{mm}^2$.

4. Compression steel yields if

$$(\rho - \rho') = \rho_1 \geq 0.85\beta_1 \times \frac{f_c'}{f_y} \times \frac{d'}{d} \times \frac{600}{600 - f_y} = K$$

$$K = (0.85)^2 \times \frac{30}{420} \times \frac{60}{515} \times \frac{600}{600 - 420} = 0.020$$

Because $(\rho - \rho') = \rho_{\max} = 0.01938 < 0.020$, compression steel does not yield.

5. Calculate

$$a = \frac{A_s f_y}{0.85 f_c' b}$$

$$= \frac{2994(420)}{0.85 \times 30 \times 300} = 164.4 \text{ mm}$$

$$c = \frac{a}{0.85} = 193.4 \text{ mm} \qquad d' = 60 \text{ mm}$$

$$f_c' = 600\left(\frac{c - d'}{c}\right) = 414 \text{ N/mm}^2$$

$$A_s' = A_{s2}(420/414) = 1215 \text{ mm}^2$$

6. Choose steel bars as follows: For tension, choose six bars 30 mm in diameter (30 M). The A_s provided (4200 mm) is greater than A_s, as required. For compression steel, choose three bars 25 mm in diameter (25 M) (Table B.11).

$$A_s' = 1500 \text{ mm}^2 > 1215 \text{ mm}^2$$

SUMMARY

Sections 4.1–4.3 : Design of a Singly Reinforced Rectangular Section

Given: M_u (external factored moment), f_c' (compressive strength of concrete), and f_y (yield stress of steel).

Case 1.
When b, d, and A_s (or ρ) are *not* given:

1. Assume $\rho_{\min} \le \rho \le \rho_{\max}$. Choose $\rho_{\max}$ for a minimum concrete cross-section (smallest) or choose ρ between $\rho_{\max}/2$ and $\rho_b/2$ for larger sections. For example, if $f_y = 60$ ksi, you may choose

$$\rho = 1.2\% \qquad R_n = 618 \text{ psi} \quad \text{for } f_c' = 3 \text{ ksi}$$

$$\rho = 1.4\% \qquad R_n = 736 \text{ psi} \quad \text{for } f_c' = 4 \text{ ksi}$$

$$\rho = 1.4\% \qquad R_n = 757 \text{ psi} \quad \text{for } f_c' = 5 \text{ ksi}$$

For any other value of ρ, $R_n = \rho f_y[1 - (\rho f_y/1.7 f_c')]$, and $R_u = \phi R_n$.

2. Calculate $bd^2 = M_u/\phi R_n (\phi = 0.9)$, for tension-controlled sections.
3. Choose b and d. The ratio of d to b is approximately $1 \to 3$, or $d/b \approx 2.0$.
4. Calculate $A_s = \rho bd$; then choose bars to fit in b either in one row or two rows. (Check $b_{\min}$ from the tables.)
5. Calculate

$$h = d + 2.5 \text{ in. (for one row of bars)}$$

$$h = d + 3.5 \text{ in. (for two rows of bars)}$$

b and h must be to the nearest higher inch. *Note*: If h is increased, calculate new $d = h - 2.5$ (or 3.5) and recalculate A_s to get a smaller value.

Case 2.
When ρ is given, d, b, and A_s are required. Repeat steps (1) through (5) from Case 1.

Case 3.
When b and d (or h) are given, A_s is required.

1. Calculate $R_n = M_u/\phi bd^2$ ($\phi = 0.9$).
2. Calculate

$$\rho = \left(\frac{0.85f'_c}{f_y}\right)\left[1 - \sqrt{1 - \frac{2R_n}{0.85f'_c}}\right]$$

(or get ρ from tables or Eq. 4.2).

3. Calculate $A_s = \rho bd$, choose bars, and check $b_{\min}$.
4. Calculate h to the nearest higher inch (see note, Case 1 (step 5)).

Case 4.
When b and ρ are given, d and A_s are required.

1. Calculate

$$R_n = \rho f_y\left(1 - \frac{\rho f_y}{1.7f'_c}\right) \qquad R_u = \phi R_n (\phi = 0.9)$$

2. Calculate

$$d = \sqrt{\frac{M_u}{\phi R_n b}}$$

3. Calculate $A_s = \rho bd$, choose bars, and check $b_{\min}$.
4. Calculate h to the nearest higher inch (see note, Case 1 (step 5)). *Note*: Equations that may be used to check the moment capacity of the section after the final section is chosen are

$$M_u = \phi M_n = \phi A_s f_y\left(d - \frac{A_s f_y}{1.7f'_c b}\right) = \phi A_s f_y d\left(1 - \frac{\rho f_y}{1.7f'_c}\right)$$

$$= \phi\rho f_y(bd^2)\left(1 - \frac{\rho f_y}{1.7f'_c}\right) = R_u bd^2$$

Section 4.4: Design of Rectangular Sections with Compression Steel

Given: M_u, b, d, d', f'_c, f_y, and $\phi = 0.9$.
Required: A_s and A'_s.

1. *General*
   a. Calculate $\rho_{\max}$ and $\rho_{\min}$ as singly reinforced from equations (or from tables).
   b. Calculate $R_{n\max} = \rho_{\max} f_y\left[1 - \left(\frac{\rho_{\max} f_y}{1.7f'_c}\right)\right]$ (or use tables).
   c. Calculate the maximum capacity of the section as singly reinforced:

$$\phi M_n = \phi R_{n\max} bd^2.$$

   d. If $M_u > \phi M_n$, then compression steel is needed. If $M_u < \phi M_n$, it is a singly reinforced section.

2. If $M_u > \phi M_n$ and compression steel is needed,

   a. Let $M_{u1} = \phi R_{n\,\max} bd^2$.

   b. Calculate $A_{s1} = \rho_{\max} bd$ (basic section).

   c. Calculate $M_{u2} = M_u - M_{u1}$ (for the steel section).

3. Calculate A_{s2} and A'_s as steel section.

   a. $M_{u2} = \phi A_{s2} f_y (d - d')$.

   b. Calculate total tension steel: $A_s = A_{s1} + A_{s2}$.

4. Calculate A'_s (compression steel area):

   a. Calculate $a = (A_{s1} f_y / 0.85 f'_c b)$ and $c = a/\beta_1$.

   b. Calculate $f'_s = 87[(c - d')/c] \leq f_y$.

   If $f'_s \geq f_y$, then $f'_s = f_y$ and $A'_s = A_{s2}$.

   If $f'_s < f_y$, then $A'_s = A_{s2}\left(\frac{f_y}{f'_s}\right)$.

   c. Check that total steel area $(A_s) \geq$ max A_s, or check $\varepsilon_t \geq 0.005$

$$A_s \leq \left[\rho_{\max}(bd) + A'_s\left(\frac{f'_s}{f_y}\right)\right]$$

Section 4.5: Design of T-Sections

Given: M_u, f'_c, f_y, b, t, and b_w.
Required: A_s and d (if not given).
There are two cases:

Case 1.
When d and A_s (or ρ) are *not* given:

1. Let $a \leq t$ (as singly reinforced rectangular section). If $a = t$ is assumed, then

$$M_u = (\text{total flange}) = \phi(0.85 f'_c)bt\left(d - \frac{t}{2}\right) = \phi A_s f_y \left(d - \frac{t}{2}\right)$$

Solve for d and then for A_s.

$$d = \frac{M_u}{\phi(0.85 f'_c)bt} + \frac{t}{2} A_s = \frac{M_u}{\phi f_y\left(d - \frac{t}{2}\right)}$$

2. If a is assumed to be less than t, then

$$d = \frac{M_u}{\phi(0.85 f'_c)ba} + \frac{a}{2} \text{ and } A_s = \frac{M_u}{\phi f_y\left(d - \frac{a}{2}\right)}$$

Case 2.
When d is given and A_s is required (one unknown):

1. Check if a is greater or less than t by considering the moment capacity of the flange (bt).

$$\text{(flange) } \phi M_n = \phi(0.85 f'_c) bt \left(d - \frac{t}{2}\right)$$

If $\phi M_n > M_u$ (external), then $a < t$ (rectangular section).
If $\phi M_n < M_u$ (external), then $a > t$ (T-section).

2. If $a < t$, calculate $R_n = M_u/\phi b d^2$ and then calculate ρ (or determine ρ from tables or Eq. 4.2):

$$\rho = \frac{0.85 f'_c}{f_y}\left(1 - \sqrt{1 - \frac{2R_n}{0.85 f'_c}}\right)$$

Then calculate $A_s = \rho b d$.

3. If $a > t$,

a. Calculate C_f and A_{sf}.

$$A_{sf} = 0.85 f'_c t \frac{(b - b_w)}{f_y} = \frac{C_f}{f_y} \text{(flange)}$$

Then calculate M_{uf} (flange) $= \phi C_f (d - t/2)$.

b. Calculate M_{uw} (web) $= M_u - M_{uf}$. Calculate R_n (web) $= M_{uw}/(\phi b_w d^2)$; then find ρ_w (use the equation or tables). Calculate A_{sw} (web) $= \rho_w b_w d$.

c. Total $A_s = A_{sf}$ (flange) $+ A_{sw}$ (web). Total A_s must be less than or equal to $A_{s\,max}$ and greater than or equal to $A_{s\,min}$.

d. $\rho_w = \left(\frac{0.8 f'_c}{f_y}\right)\left(1 - \sqrt{1 - \frac{2R_n}{0.85 f'_c}}\right)$

e. Check that $\rho_w = A_s/b_w d \geq \rho_{min}$ (ρ_w = steel ratio in web) or $A_s > A_{s\,min}$, where $A_{s\,min} = \rho_{min}(b_w d)$. Check that $A_s \leq \max A_s$; or check $\varepsilon_t = (d_t - c)/c \geq 0.005$

PROBLEMS

4.1 Based on the information given in the accompanying table and for each assigned problem, design a singly reinforced concrete section to resist the factored moment shown in boldface. Use $f'_c = 4$ ksi and $f_y = 60$ ksi, and draw a detailed, neat section.

No.	M_u (K·ft)	b (in.)	d (in.)	ρ %
a	272.7	12	21.5	—
b	969.2	18	32.0	—
c	816.0	16	—	1.70
d	657.0	16	—	1.50
e	559.4	14	—	1.75
f	254.5	10	21.5	—
g	451.4	14	—	1.80
h	832.0	18	28.0	—
i	345.0	15	—	1.77

(*continues*)

No.	M_u (K·ft)	b (in.)	d (in.)	ρ %
j	510.0	0.5 d	—	ρ_{max}
k	720.0	—	2.5b	1.80
l	605.0	—	1.5b	1.80

For problems in SI units, 1 in. = 25.4 mm, 1 ksi = 6.9 MPa (N/mm2), and 1 M_u (K·ft) = 1.356 kN·m.

4.2 Based on the information given in the following table and for each assigned problem, design a rectangular section with compression reinforcement to resist the factored moment shown. Use $f'_c =$ 4 ksi, $f_y = 60$ ksi, and $d' = 2.5$ in. Draw detailed, neat sections.

No.	M_u (K·ft)	b (in.)	d (in.)
a	554	14	20.5
b	790	16	24.5
c	448	12	18.5
d	520	12	20.5
e	765	16	20.5
f	855	18	22.0
g	555	16	18.5
h	300	12	16.5
i	400	16	16.5
j	280	12	16.5
k	290	14	14.5
l	400	14	17.5

For problems in SI units, 1 in. = 25.4 mm, 1 ksi = 6.9 MPa (N/mm2), and 1 M_u (K·ft) = 1.356 kN·m.

4.3 Based on the information given in the following table and for each assigned problem, calculate the tension steel and bars required to resist the factored moment shown. Use $f'_c = 3$ ksi and $f_y = 60$ ksi. Draw detailed, neat sections.

No.	M_u (K·ft)	b (in.)	b_w (in.)	t (in.)	d (in.)	Notes
a	394	48	14	3	18.5	
b	800	60	16	4	19.5	
c	250	44	15	3	15.0	
d	327	50	14	3	13.0	
e	577	54	16	4	18.5	
f	559	48	14	4	17.5	
g	388	44	12	3	16.0	
h	380	46	14	3	15.0	
i	537	60	16	3	16.5	
j	515	54	16	3	17.5	
k	361	44	15	3	15.0	
l	405	50	14	3	15.5	
m	378	44	16	3	—	Let $a = t$.
n	440	36	16	4	—	Let $a = t$.
o	567	48	12	3	—	Let $A_s = 6.0$ in^2.
p	507	46	14	3	—	Let $A_s = 7.0$ in^2.

For problems in SI units, 1 in. = 25.4 mm, 1 ksi = 6.9 MPa (N/mm2), and 1 1Mu (K·ft) = 1.356 kN·m.

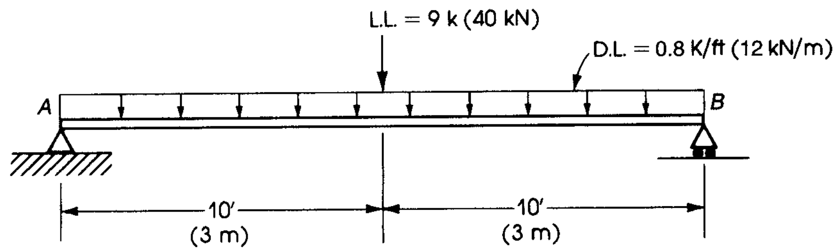


Figure 4.14 Problem 4.4.7.

4.4 Design a singly reinforced rectangular section to resist a factored moment of 232 K·ft (320 kN·m) if $f'_c = 4$ ksi (28 MPa), $f_y = 60$ ksi (420 MPa), and $b = 10$ in. (250 mm), using (a) $\rho_{\max}$, (b) $\rho = 0.016$, and (c) $\rho = 0.012$.

4.5 Design a singly reinforced section to resist a factored moment of 186 K·ft (252 kN·m) if $b = 12$ in. (275 mm), $d = 20$ in. (500 mm), $f'_c = 3$ ksi (20 MPa), and $f_y = 40$ ksi (300 MPa).

4.6 Determine the reinforcement required for the section given in Problem 4.5 when $f'_c = 4$ ksi (30 MPa), and $f_y = 60$ ksi (400 MPa).

4.7 A simply supported beam has a 20-ft (6-m) span and carries a uniform dead load of 800 lb/ft (12 kN/m) and a concentrated live load at midspan of 9 kips (40 kN) (Fig. 4.14). Design the beam if $b = 12$ in. (300 mm), $f'_c = 4$ ksi (30 MPa), and $f_y = 60$ ksi (400 MPa). (Beam self-weight is not included in the dead load.)

4.8 A beam with a span of 24 ft (7.2 m) between supports has an overhanging extended part of 8 ft (2.4 m) on one side only. The beam carries a uniform dead load of 2.3 K/ft (30 kN/m) (including its own weight) and a uniform live load of 1.3 K/ft (18 kN/m) (Fig. 4.15). Design the smallest singly reinforced rectangular section to be used for the entire beam. Select steel for positive and negative moments. Use $f'_c = 4$ ksi (30 MPa), $f_y = 60$ ksi (400 MPa), and $b = 12$ in. (300 mm). (Determine the maximum positive and maximum negative moments by placing the live load once on the span and once on the overhanging part.)

4.9 Design a 15-ft (4.5-m) cantilever beam of uniform depth to carry a uniform dead load of 0.88 K/ft (12 kN/m) and a live load of 1.1 K/ft (15 kN/m). Assume a beam width $b = 14$ in. (350 mm), $f'_c = 4$ ksi (30 MPa), and $f_y = 60$ ksi (400 MPa).

4.10 10-ft (3-m) cantilever beam carries a uniform dead load of 1.50 K/ft (20 kN/m) (including its own weight) and a live load of 0.77 K/ft (10 kN/m) (Fig. 4.16). Design the beam using a variable depth. Draw all details of the beam and reinforcement. Given: $f'_c = 3$ ksi (20 MPa), $f_y = 40$ ksi (300 MPa), and $b = 12$ in. (300 mm). Assume h at the free end is 10 in. (250 mm).

4.11 Determine the necessary reinforcement for a concrete beam to resist an external factored moment of 290 K·ft (400 kN·m) if $b = 12$ in. (300 mm), $d = 19$ in. (475 mm), $d' = 2.5$ in. (65 mm), $f'_c = 3$ ksi (20 MPa), and $f_y = 60$ ksi (400 MPa).

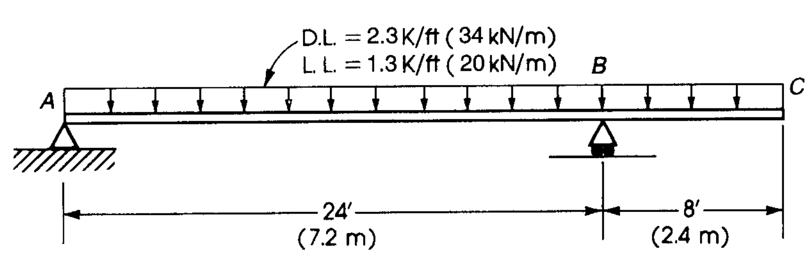


Figure 4.15 Problem 4.4.8.

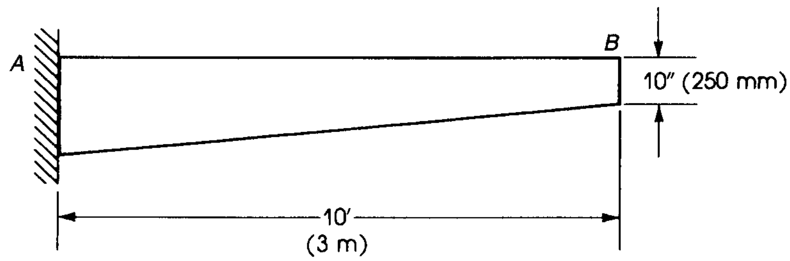


Figure 4.16 Problem 4.4.10.

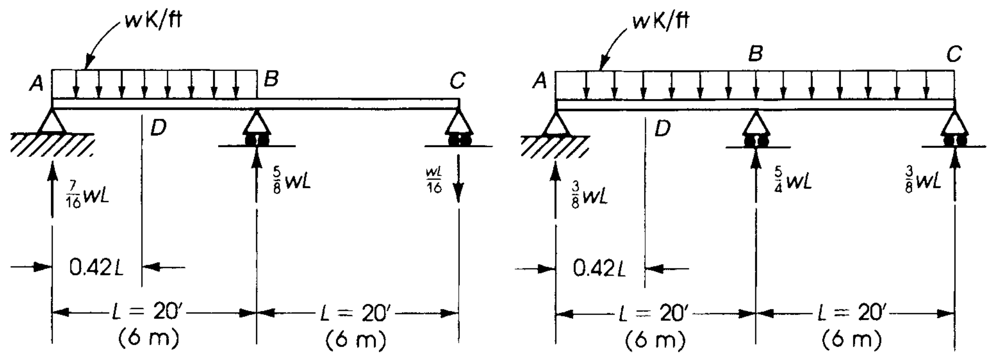


Figure 4.17 Problem 4.4.14.

4.12 Design a reinforced concrete section that can carry a factored moment of 260 K·ft (360 kN·m) as

a. Singly reinforced, $b = 10$ in. (250 mm)

b. Doubly reinforced, 25% of the moment to be resisted by compression steel, $b = 10$ in. (250 mm)

c. T-section, which has a flange thickness of 3 in. (75 mm), flange width of 20 in. (500 mm), and web width of 10 in. (250 mm)

$$f'_c = 3 \text{ ksi (20 MPa), and } f_y = 60 \text{ ksi (400 MPa), for all problems.}$$

Determine the quantities of concrete and steel designed per foot length (meter length) of beams and then determine the cost of each design if the price of the concrete equals \$50/yd^3 (67/m^3) and that of steel is \$0.30/lb (\$0.66/kg). Tabulate and compare results.

4.13 Determine the necessary reinforcement for a T-section that has a flange width of $b = 40$ in. (1000 mm), flange thickness of $t = 4$ in. (100 mm), and web width of $b_w = 10$ in. (250 mm) to carry a factored moment of 545 K·ft (750 kN·m). Given: $f'_c = 3$ ksi (20 MPa) and $f_y = 60$ ksi (400 MPa).

4.14 The two-span continuous beam shown in Fig. 4.17 is subjected to a uniform dead load of 2.6 K/ft (including its own weight) and a uniform live load of 3 K/ft. The reactions due to two different loadings are also shown. Calculate the maximum negative factored moment at the intermediate support B and the maximum positive factored moment within the span AB (at $0.42L$ from support A), design the critical section at B and D, and draw the reinforcement details for the entire beam ABC. Given: $L = 20$ ft, $b = 12$ in., $h = 24$ in. Use $d = 18$ in. for one row of bars and $d = 17$ in. for two rows. $f'_c = 4$ ksi, and $f_y = 60$ ksi.

4.15 The two-hinged frame shown in Fig. 4.18 carries a uniform dead load of 3.93 K/ft and a uniform live load of 2.4 K/ft on BC. The reactions at A and D can be evaluated as follows: $HA = HD = wL/9$ and $RA = RD = wL/2$, where w = uniform load on BC. A typical cross-section of the beam BC is also shown. It is required:

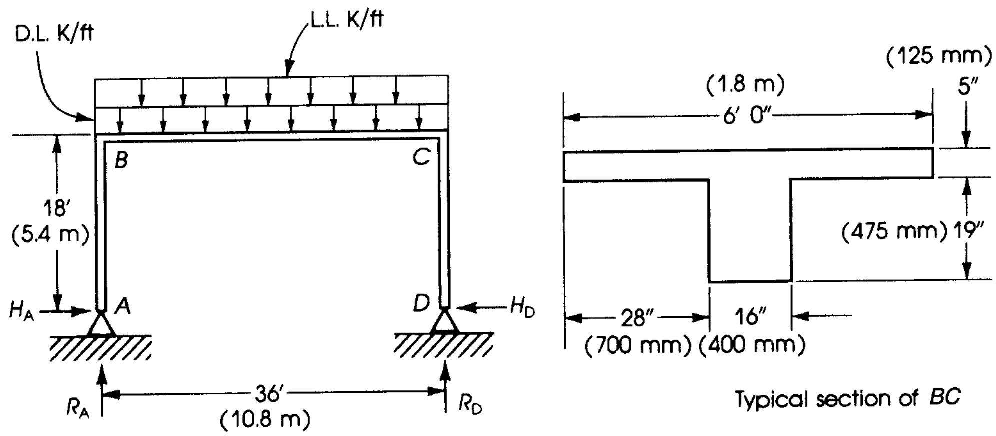


Figure 4.18 Problem 4.4.15.

a. Draw the bending moment diagram for the frame $ABCD$.

b. Design the beam BC for the applied factored moments (positive and negative).

c. Draw the reinforcement details of BC.

Given: $f'_c = 4$ ksi and $f_y = 60$ ksi.

CHAPTER 5

ALTERNATIVE DESIGN METHODS

Office building, Minneapolis, Minnesota.

5.1 INTRODUCTION

In the previous chapters, 3 and 4, the analysis and design of flexural reinforced concrete members were explained based on the provisions of the ACI Code 318-08. An alternative design approach is presented in Appendix B of the ACI Code according to the load factors given in Appendix C. This alternative design method was the basis of analysis and design in the ACI Code 318-99. It is to some extent similar to the method explained earlier except that it uses different load factors and strength reduction, ϕ. The basic analysis and design equations of the previous chapters will be used here. When Appendix B provisions are used in the design, they should replace all other corresponding provisions in the body of the Code.

5.2 LOAD FACTORS

If the required strength is denoted by U and those due to wind and seismic forces are W and E, respectively, then according to the ACI Code, Appendix C, the required strength U, shall be the most critical of the following:

1. In the case of dead, live, and wind loads,

$$U = 1.4D + 1.7L \tag{5.1a}$$

$$U = 0.75(1.4D + 1.7L) + (1.6W \text{ or } 1.0E) \tag{5.1b}$$

$$U = 0.9D + (1.6W \text{ or } 1.0E) \tag{5.1c}$$

2. When wind load, W, has not been reduced by a directionality factor, $1.3W$ can be used in place of $1.6W$. When seismic load is based on service forces, $1.4E$ can be used in place of $1.0E$.
3. In cases when earth pressure load, H, must be included in the design,

$$U = 1.4D + 1.7L + 1.7H \tag{5.2a}$$

Where dead load, D, and live load, L, reduce the effect of H, U shall be checked for

$$U = 0.9D + 1.7H \tag{5.2b}$$

For any combination of D, L, or H,

$$U = 1.4D + 1.7L$$

4. If weight and pressure loads from liquids, F, must be included in the design,

$$U = 1.4D + 1.7L + 1.4F \tag{5.3a}$$

Where dead load, D, and live load, L, reduce the effort of F,

$$U = 0.9D + 1.4F \tag{5.3b}$$

For any combination of D, L, or F,

$$U = 1.4D + 1.7L$$

The vertical pressure of liquids shall be considered as dead load with due regard to variation in liquid depth.
5. When impact effects are taken into account, they shall be included in the live load.
6. Where the structural effects, T, of differential settlement, creep, shrinkage, or temperature change may be significant, they shall be included with the most critical of

$$U = 0.75(1.4D + 1.4T + 1.7L) \tag{5.4a}$$

$$U = 1.4D + 1.4T \tag{5.4b}$$

Equation 5.1a is most generally used. The dead load factor is equal to 1.4, whereas the live load factor is equal to 1.7.

For applied concentrated dead and live loads, P_D and P_L, the factored concentrated load is $P_U = 1.4P_D + 1.7P_L$; also $M_U = 1.4M_D + 1.7M_L$, where M_D and M_L are the actual dead load and live load moments, respectively.

5.3 STRENGTH-REDUCTION FACTOR, ϕ

The nominal strength of a section is reduced by a factor ϕ to account for small adverse variations in material strengths, artisanry, dimensions, control, and degree of supervision. The factor ϕ constitutes a portion of the factor of safety, as discussed in Section 1.8.

The ACI Code, Section C.9.3 (Appendix C), specifies the following values to be used:

- Tension-controlled sections: $\phi = 0.90$
- Compression-controlled sections
  - Members with spiral reinforcement: $\phi = 0.75$
  - Other reinforced members: $\phi = 0.70$
- Shear and torsion: $\phi = 0.85$
- Bearing on concrete: $\phi = 0.70$
- Bending in plain concrete or in concrete with minimum reinforcement of $200/f_y$: $\phi = 0.65$

For sections that lie in the transition region between tension- and compression-controlled sections, ϕ may be increased linearly to 0.9.

Also, the strength eduction factor ϕ to be used for columns (or sections with $\varepsilon_t < 0.005$) may vary according to the following cases:

1. When $P_u = \phi P_n \geq 0.1 f_c' A_g$, then ϕ is 0.70 for tied columns and 0.75 for spirally reinforced columns. This case occurs generally when compression controls. A_g is the gross area of the concrete region.
2. Between values of $0.1 f_c' A_g$ or ϕP_b (whichever is smaller) and 0, P_u lies in the tension control zone and ϕ is larger than 0.7 (or 0.75). The ACI Code, Section C.9.3.2, specifies that for members in which f_y does not exceed 60 ksi, with symmetrical reinforcement and with the distance between compression and tension steel $(d - d')$ not less than $0.7h$ (h = total depth of section) and $d = h - d_s$, the value of ϕ is increased linearly to 0.9.

For this transition region, ϕ may be determined by linear interpolation between 0.7 (or 0.75) and 0.9. Figure 5.1 shows the variation of ϕ for grade 60 steel. The linear equations are as follows:

$$\phi = 0.57 + 67\varepsilon_t \quad \text{(for tied sections)} \tag{5.5}$$

$$\phi = 0.65 + 50\varepsilon_t \quad \text{(for spiral sections)} \tag{5.6}$$

Alternatively, ϕ in the transition region can be determined as a function of (d_t/c) for grade 60 steel as follows:

$$\phi = 0.37 + 0.20\left(\frac{d_t}{c}\right) \quad \text{(for tied sections)} \tag{5.7}$$

$$\phi = 0.50 + 0.15\left(\frac{d_t}{c}\right) \quad \text{(for spiral sections)} \tag{5.8}$$

where c is the depth of the neutral axis at nominal strength.

5.4 RECTANGULAR SECTIONS WITH TENSION REINFORCEMENT

From the analysis of rectangular singly reinforced section (Section 3.9), the following equations were derived, where f_c' and f_y are in ksi:

$$\rho_b = 0.85\beta_1 \frac{f_c'}{f_y}\left(\frac{87}{87 + f_y}\right) \tag{3.18}$$

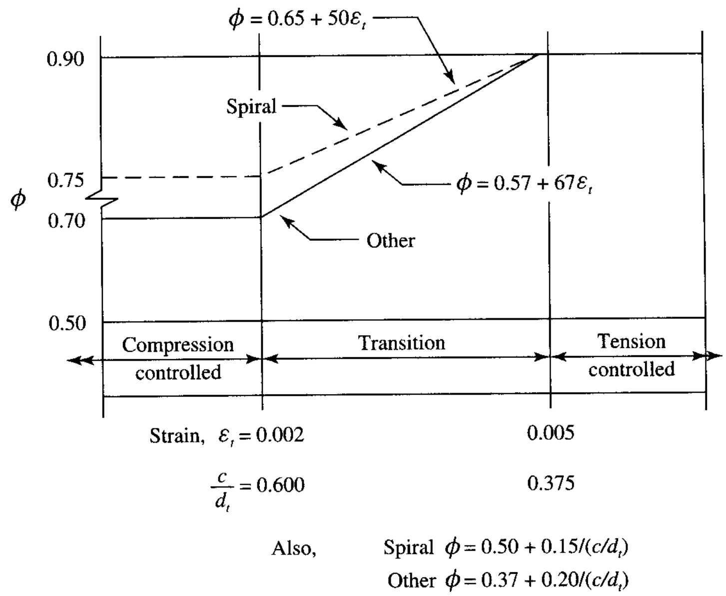


Figure 5.1 Variation of ϕ with the net tensile strain for grade 60 steel [1]. Courtesy of ACI 318-08.

If the maximum percentage of reinforcement is limited to $0.75\rho_b$, then

$$\rho_b = 0.75\rho_b = 0.6375\beta_1 \frac{f'_c}{f_y}\left(\frac{87}{87 + f_y}\right) \tag{5.9}$$

It is to be noted that $\rho_{\max} = 0.75\rho_b$ is greater than that of $0.634\rho_b$ as given earlier in Chapter 3, (Eq. 3.30 for $f_y = 60$ ksi).

For $f'_c \leq 4000$ psi,

$$\rho_{\max} = 0.542 \frac{f'_c}{f_y}\left(\frac{87}{87 + f_y}\right) \tag{5.10}$$

The value of β_1 is 0.85 when $f'_c \leq 4000$ psi (30 N/mm^2) and decreases by 0.05 for every increase of 1000 psi (7 N/mm^2) in concrete strength, or $\beta_1 = 0.85 - 0.05(f'_c - 4) \geq 0.65$.

The steel percentage of a balanced section, ρ_b, and the maximum allowable steel percentage, $\rho_{\max}$, can be calculated for different values of f'_c and f_y, as shown in Table 5.1. Suggested design steel ratios for $\rho \leq \rho_{\max}$ are also shown in Table 5.1.

The design moment equations were derived in the previous chapter in the following forms:

$$\phi M_n = M_u = R_u b d^2 \tag{3.21}$$

where

$$R_u = \phi \rho f_y \left(1 - \frac{\rho f_y}{1.7 f'_c}\right) = \phi R_n \tag{3.22}$$

Table 5.1 Suggested Design Steel Ratios ρ_s

f'_c (ksi)	f_y (ksi)	% ρ_b	% ρ_{max}	% ρ_s	Ratio ρ_s/ρ_b	Ratio ρ_s/ρ_{max}	R_u (psi)	$R_{u\,max}$ (psi)
3	40	3.71	2.78	1.4	0.38	0.50	450	783
	60	2.15	1.61	1.2	0.56	0.75	556	702
4	60	2.85	2.14	1.4	0.49	0.65	662	936
	75	2.07	1.55	1.2	0.58	0.77	702	867
5	60	3.36	2.52	1.4	0.42	0.56	681	1120
	75	2.44	1.83	1.2	0.49	0.66	722	1033

and $\phi = 0.9$. For tension-controlled sections, $\varepsilon_t \geq 0.005$:

$$\phi M_n = M_u = \phi A_s f_y \left(d - \frac{A_s f_y}{1.7 f'_c b}\right) \tag{3.19a}$$

Also,

$$\phi M_n = M_u = \phi \rho f_y b d^2 \left(1 - \frac{\rho f_y}{1.7 f'_c}\right) \tag{3.20}$$

We can see that for a given factored moment and known f'_c and f_y, there are three unknowns in these equations: the width, b, the effective depth of the section, d, and the steel ratio, ρ. A unique solution is not possible unless values of two of these three unknowns are assumed. Usually ρ is assumed (using ρ_{max}, for instance), and b can also be assumed.

Based on the preceding discussion, the following cases may develop for a given M_u, f'_c and f_y:

1. If ρ is assumed, then R_u can be calculated from Eq. 3.19, giving $bd^2 = M_u/R_u$. The designer may use ρ up to ρ_{max}, which produces the minimum size of the singly reinforced concrete section. Using ρ_{min} will produce the maximum concrete section. If b is assumed in addition to ρ, then d can be determined as follows:

$$d = \sqrt{\frac{M_u}{R_u b}} \tag{5.11}$$

If $d/b = 2$, then $d = \sqrt[3]{(2M_u/R_u)}$ and $b = d/2$, rounded to the nearest higher inch.

2. If b and d are given, the required reinforcement ratio, ρ, can be determined by rearranging Eq. 3.20 to obtain

$$\rho = \frac{0.85 f'_c}{f_y}\left[1 - \sqrt{1 - \frac{4M_u}{1.7\phi f'_c b d^2}}\right] \tag{5.12}$$

$$= \frac{0.85 f'_c}{f_y}\left[1 - \sqrt{1 - \frac{2R_n}{0.85 f'_c}}\right]$$

and

$$A_s = \rho b d$$

where all units are in kips (or pounds) and inches. For example, if $M_u = 2440$ K·in., $b = 12$ in., $d = 18$ in., $f'_c = 3$ ksi, and $f_y = 60$ ksi, then $\rho = 0.01389$ (from Eq. 5.22) and $A_s = \rho bd = 0.01389(12)(18) = 3.0\,\text{in}^2$. When b and d are given, it is better to check if compression steel is or is not required because of a small d. This can be achieved as follows:

a. Calculate $\rho_{\max}$ and $R_{u\max} = \phi\rho_{\max} f_y[1 - (\rho_{\max} f_y/1.7f'_c)]$.

b. Calculate $\phi M_{n\max} = R_{u\max} bd^2 =$ the maximum moment strength of a singly reinforced concrete section.

c. If $M_u < \phi M_{n\max}$, then no compression reinforcement is needed. Calculate ρ and A_s from the preceding equations.

d. If $M_u > \phi M_{n\max}$, then compression steel is needed.

3. If ρ and b are given, calculate R_u:

$$R_u = \phi\rho f_y\left(1 - \frac{\rho f_y}{1.7f'_c}\right)$$

The calculate d from Eq. 5.21:

$$d = \sqrt{\frac{M_u}{R_u b}} \quad \text{and} \quad A_s = \rho bd$$

Example 5.1

Find the necessary reinforcement for a given section 10 in. wide and 28 in. total depth (Fig. 5.2) if it is subjected to an external factored moment of 245 K·ft. Given: $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. Assuming one layer of no. 8 steel bars (to be checked later), $d = 28 - 2.5$ in. $= 25.5$ in.
2. Check if the section is adequate without compression reinforcement. Compare design moment strength of the section (using $\rho_{\max}$) with the design moment. For $f'_c = 4$ ksi and $f_y = 60$ ksi, $\rho_{\max} = 0.02138$.

$$R_u = \phi\rho_{\max} f_y\left(1 - \frac{\rho_{\max} f_y}{1.7f'_c}\right) = 937 \text{ psi}$$

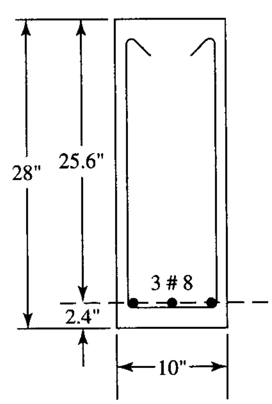


Figure 5.2 Example 5.1.

The design moment strength of a singly reinforced basic section is

$$\phi M_{n\,\max} = R_{u\,\max} bd^2 = 0.937(10)(25.5)^2$$
$$= 6093 \text{ K·in.} > 245 \times 12 = 2940 \text{ K·in.}$$

Therefore, $\rho < \rho_{\max}$ and the section is singly reinforced.

3. Calculate ρ from Eq. 5.12 to get $\rho = 0.009$. $A_s = \rho bd = 0.009(10)(25.5) = 2.30 \text{ in.}^2$ Use three no. 8 bars ($A_s = 2.35 \text{ in.}^2$) in one row, $b_{\min} < 10$ in. The final section is shown in Fig. 5.2.
4. Check ε_t:

$$a = \frac{2.35(60)}{0.85(4)(10)} = 4.15 \text{ in.}$$
$$c = \frac{a}{0.85} = 4.88 \text{ in.}$$
$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) 0.003 = 0.0127 > 0.005 \quad \phi = 0.9$$

5.5 RECTANGULAR SECTIONS WITH COMPRESSION REINFORCEMENT

A singly reinforced section has a maximum design moment strength when $\rho_{\max}$ of steel is used. If the applied factored moment is greater than the internal moment strength, as in the case of a limited cross-section, a doubly reinforced section may be used, adding steel bars in both the compression and tension zones.

The procedure for designing a rectangular section with compressive steel when M_u, f'_c, b, d, and d' are given was summarized in Section 4.4. The only difference is that $\rho_{\max} = 0.75\,\rho_b$ is used in this design approach here.

$$\rho_{\max} = 0.6375\beta_1 \frac{f'_c}{f_y}\left(\frac{87}{87 + f_y}\right) \tag{5.9}$$

Also, check that $\varepsilon_t \geq 0.005$ for $\phi = 0.9$.

Example 5.2

A beam section is limited to $b = 12$ in. and to a total depth of $h = 20$ in. and is subjected to a factored moment of $M_u = 330$ K·ft. Determine the necessary reinforcement using $f'_c = 4$ ksi and $f_y = 60$ ksi. (Refer to Fig. 5.3)

Solution

1. Determine the design moment strength of the section as singly reinforced. Assume $\rho = 0.018$. Therefore, $R_u = 818$ psi (Table A2). For two rows of bars, $d = 20 - 3.5 = 16.5$ in.

$$M_{u1} = R_u bd^2 = 0.818(12)(16.5)^2 = 2672 \text{ K·in.}$$

The design moment is $M_u = 330 \times 12 = 3960$ K·in. $> M_{u1}$; therefore, compression steel is needed.

2. Calculate A_{s1}, M_{u2}, A_{s2}, and total A_s.

$$A_{s1} = \rho bd = 0.018(12)(16.5) = 3.56 \text{ in.}$$
$$M_{u2} = M_u - M_{u1} = 3960 - 2672 = 1288 \text{ K·in.}$$

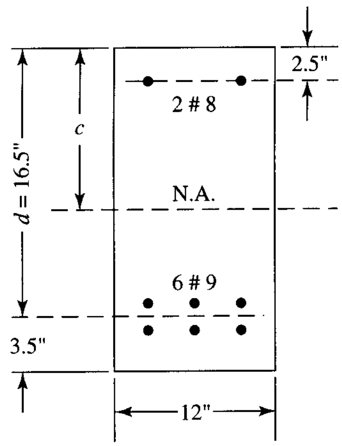


Figure 5.3 Example 5.2.

$$M_{u2} = \phi A_{s2} f_y (d - d'), \text{ assume } d' = 2.5 \text{ in.}$$

$$1288 = 0.9 A_{s2}(60)(16.5 - 2.5) \quad A_{s2} = 1.7 \text{ in.}^2$$

$$\text{Total } A_s = A_{s1} + A_{s2} = 3.56 + 1.7 = 5.26 \text{ in.}^2 \text{ (six no. 9 bars)}$$

3. Check if compression steel yields by Eq. 3.49. Compression steel yields if

$$\rho - \rho' \geq K = 0.85\beta_1 \frac{f'_c}{f_y}\left(\frac{d'}{d}\right)\left(\frac{87}{87 - f_y}\right)$$

$$K = (0.85)^2 \left(\frac{4}{60}\right)\left(\frac{2.5}{16.5}\right)\left(\frac{87}{27}\right) = 0.0235$$

$$\rho - \rho' = \frac{A_{s1}}{bd} = \frac{3.56}{(12)(16.5)} = 0.018 \leq K$$

Therefore, compression steel does not yield: $f'_s < f_y$.

4. Calculate f'_s : $f'_s = 87[(c - d')/c] \leq f_y$. Determine c from A_{s1}: $A_{s1} = 3.56 \text{ in.}^2$

$$a = \frac{A_{s1} f_y}{0.85 f'_c b}$$

$$= \frac{3.56 \times 60}{0.85 \times 4 \times 12} = 5.24 \text{ in.}$$

$$c = \frac{a}{\beta_1} = \frac{5.24}{0.85} = 6.16 \text{ in.}$$

$$f'_s = 87 \times \left(\frac{6.16 - 2.5}{6.16}\right) = 52 \text{ ksi} < 60 \text{ ksi}$$

5. Calculate A'_s from $M_{u2} = \phi A'_s f'_s (d - d')$:

$$1288 = 0.9 A'_s (52)(16.5 - 2.5)$$

Thus, $A'_s = 1.97 \text{ in.}^2$, or calculate A'_s from $A'_s = A_{s2}(f_y/f'_s) = 1.97 \text{ in.}^2$ (two no. 9 bars).

6. Check

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) 0.003$$

$$d_t = h - 2.5 \text{ in.} = 17.5 \text{ in.}$$

$$\varepsilon_t = \left(\frac{17.5 - 6.16}{6.16}\right) 0.003$$

$$= 0.0055 > 0.005 \quad \phi = 0.9$$

or

$$\frac{c}{dt} = \frac{6.16}{17.5} = 0.352 < 0.375 \quad \text{(o.k.)}$$

7. Check final ϕM_n. $A_s = 6.0\,\text{in.}^2$, $A'_s = 2.0\text{ in.}^2$, $A_{s1} = 4.0\,\text{in.}^2$, $a = 5.88\,\text{in.}$, and $c = 6.92\,\text{in.}$

$$M_n = 4 \times 60\left(16.5 - \frac{5.88}{2}\right) + 2 \times 52(16.5 - 2.5) = 4710 \text{ K·in.}$$

Check ε_t, $d_t = 17.5\,\text{in.}$

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) 0.003 = 0.0459 < 0.005$$

$$\phi = 0.57 + 67(0.0459) = 0.88$$

$$\phi M_n = 4145 \text{ K·in.} > M_u = 3960 \text{ K·in.}$$

5.6 DESIGN OF T-SECTIONS

In the design of a T-section for a given factored moment, M_u, the flange thickness t and width b would have been already established from the design of the slab and the ACI Code limitations for the effective flange width b, as given in Section 3.14. The web thickness, b_w, can be assumed to vary between 8 in. and 20 in., with a practical width of 12 to 16 in. Two more unknowns still need to be determined, d and A_s. The design procedure was summarized in Section 4.5.

Example 5.3

The T-beam section shown in Fig. 5.4 has a web width, b_w, of 10 in., a flange width, b, of 40 in., a flange thickness of 4 in., and an effective depth, d, of 14.5 in. Determine the necessary reinforcement if the applied factored moment is 3800 K·in. Given: $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. Check the position of the neutral axis; the section may be rectangular. Assume the depth of compression block a is 4 in.; that is, $a = t = 4$ in. Then

$$\phi M_n = \phi(0.85 f'_c)bt\left(d - \frac{t}{2}\right) = 6120 \text{ K·in.} > M_u = 3800 \text{ K·in.}$$

The design moment that the concrete flange can resist is greater than the factored moment applied. Therefore, the section behaves as a rectangular section.

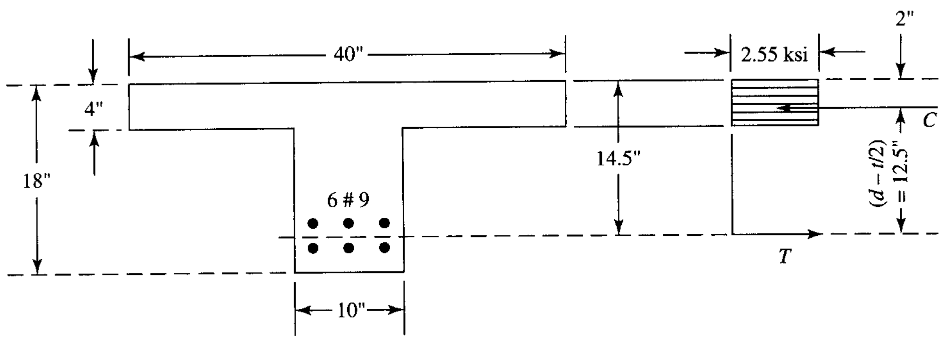


Figure 5.4 Example 5.3: T-section.

2. Determine the area of tension steel, considering a rectangular section, $b = 40$ in.

$$R_u = \frac{M_u}{(bd^2)} = \frac{3,800,000}{40 \times 14.5^2} = 452 \text{ psi}$$

From Eq. 5.22, for $R_u = 452$ psi, $\rho = 0.0091$.

$$A_s = \rho bd = 0.0091 \times 40 \times 14.5 = 5.28 \text{ in.}^2$$

Use six no. 9 bars, $A_s = 6.00 \text{ in.}^2$ (in two rows).

3. Check that $\rho_w = A_s/b_w\, d \geq \rho_{\min}$; $\rho_w = 5.28/(10 \times 14.5) = 0.0364 > \rho_{\min} = 0.00333$.
4. Check

$$\varepsilon_t = \left(\frac{d_t - c}{c}\right)0.003 \qquad d_t = 14.5$$

$$a = \frac{5.28(60)}{0.85 \times 4 \times 40} = 2.33 \text{ in.} \qquad c = 2.74 \text{ in.}$$

$$\varepsilon_t = 0.0129 > 0.005 \qquad \phi = 0.9$$

Note that other examples will be similar to those in Chapters 3 and 4.

5.7 STRUT AND TIE METHOD

5.7.1 Introduction

The ACI Code, Appendix A, introduces an alternative approach to the method explained earlier in Chapter 3, called the strut and tie models. This alternative method can be applied effectively in regions of discontinuity in the structural member, such as support areas, zones of load application, or areas with sudden change in the geometrical dimensions as brackets and portal frames. In these regions, the plane sections do not remain plane after bending (as was assumed in Chapter 3, Section 3.2), and they are called D-regions (Fig. 5.5*a*). The other regions of a standard beam, the basic beam theory, and a linear strain relationship apply. These regions are called B-regions (Fig. 5.5*a*).

The discontinuity in the stress distribution in region *D* (due to geometry or loading condition), based on St. Venants principles, indicates that the stresses due to axial load and bending approach a linear distribution at a distance approximately equal to the height of the member, *h*,

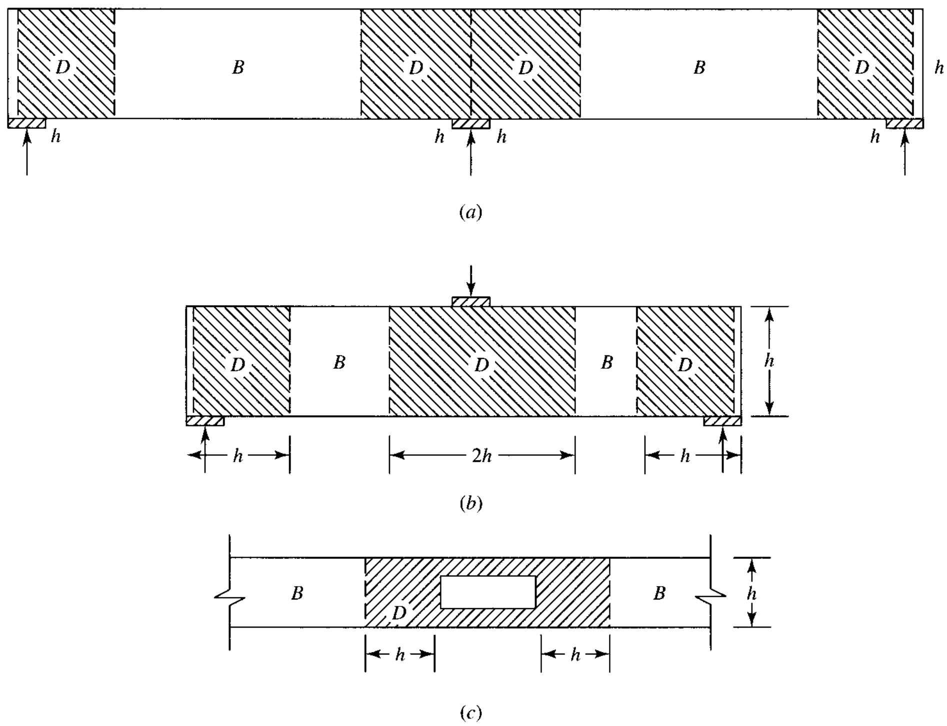


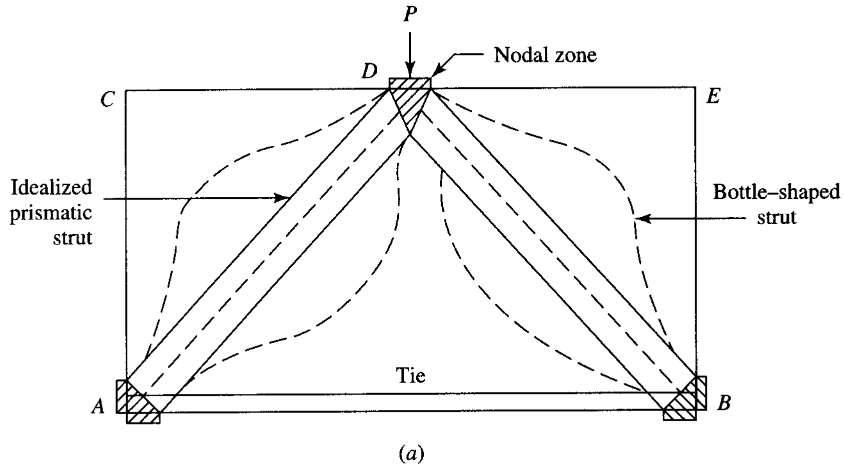
Figure 5.5 D- and B-regions in beam. (*a*) Continuous beam, (*b*) beam with concentrated load, (*c*) beam with an opening [1]. Courtesy of ACI 318-05.

away from the discontinuity (Fig. 5.5*b* and *c*) [1]. If two D-regions overlap or meet, they can be considered a single D-region. The maximum length to depth ratio would be equal to two, producing a minimum angle of 26.5° (tan $\frac{1}{2}$) between the strut and tie (or approximately 25°).

In a strut and tie model (Fig. 5.6), the point where the three forces meet at joint D is called a node, and the volume of concrete around a node is called a nodal zone. Forces at a node can vary between different combinations of compression and tensile forces, C-C-C, C-C-T, C-T-T, or T-T-T (Fig. 5.7). Figure 5.8 shows typical nodal zones for different load applications, while Fig. 5.9 shows extended nodal zones for one or more layers of reinforcing bars [6].

5.7.2 Strut and Tie Models

A strut and tie model can be represented by an idealized truss model with forces acting at the different nodes. Now consider the steel truss shown in Fig. 5.10. Due to symmetry, the reactions at A and B are equal, $R_A = R_B$ 20 K, and from the equilibrium of joints A and D, the tensile force in $AB = 20$ K, while the compressive force in AD or $BD = 28.3$ K. Member AB is considered a tie, while AD and BD act as struts. The forces in the other members are equal to 0. Comparing this truss with concrete beam in Fig. 5.6*a*, it can be seen that most of the areas ACD and BED and below the nodal zone D are not effective and act as fillers. The forces in the struts, for this loading condition, are greater than the force in the tie. In this case, adequate concrete areas are available to act as idealized struts (Fig. 5.6*a*). Steel reinforcement is needed to act as a tie for AB. Proper anchoring of the ties are essential for a safe design and should be anchored in a nodal zone.



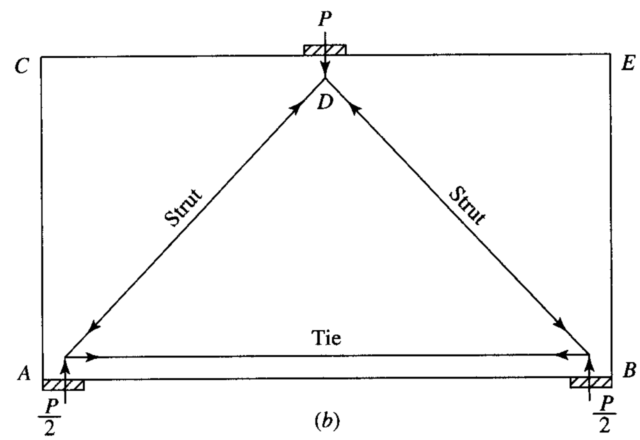


Figure 5.6 (*a*) Strut and tie model, (*b*) idealized model [1]. Courtesy of ACI 318-05.

5.7.3 ACI Design Procedure

Based on the ACI Code, Section A.2, the design of a D-region includes the following steps [1]:

1. Define and isolate each region.
2. Determine the resultant forces acting on each D-region boundary.
3. Select a truss model to transfer the resultant forces across the D-region. The axes of the struts and ties should coincide, approximately, with the compression and tension fields.
4. Determine the effective widths of the struts and nodal zones based on the concrete and steel strengths and the truss model chosen.
5. Check serviceability conditions according to the ACI Code requirements. Deflections of deep beams can be estimated using an elastic analysis. Crack control conditions of the ACI Code, Section 10.6.4, should be checked assuming the tie is encased in a prism of concrete according to RA.4.2.

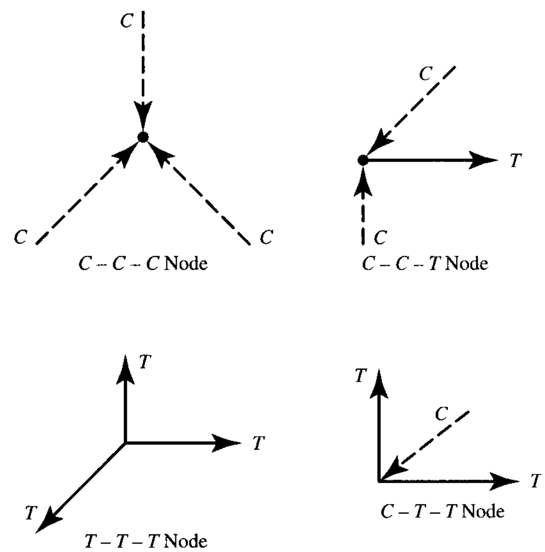


Figure 5.7 Classification of nodes.

5.7.4 Design Requirements

The design requirements for struts and ties can be summarized as follows:

1. Design of struts, ties, and nodal zones:

$$\phi F_n \geq F_u \tag{5.13}$$

where

F_u = force in a strut, tie, or nodal zone due to factored loads
F_n = nominal strength of a strut, tie, or nodal zone
ϕ = 0.75 for both struts and ties

2. Strength of struts: The nominal compressive strengths of a strut without longitudinal reinforcement, F_{ns}, shall be the smaller value of F_{ns} at the two ends of the strut such that:

$$F_{ns} = f_{ce} A_{cs} \tag{5.14}$$

where

A_{cs} = cross-sectional area at one end of a strut
f_{ce} = the smaller effective compressive strength of concrete in a strut or nodal zone

$$f_{ce} = 0.85\beta_s f'_c \tag{5.15}$$

where

a. $\beta_s = 1.0$ for a strut of uniform cross section (ACI A.3.2.1)
b. $\beta_s = 0.4$ for struts in tension members, or the tension flanges of members (ACI A.3.2.3)

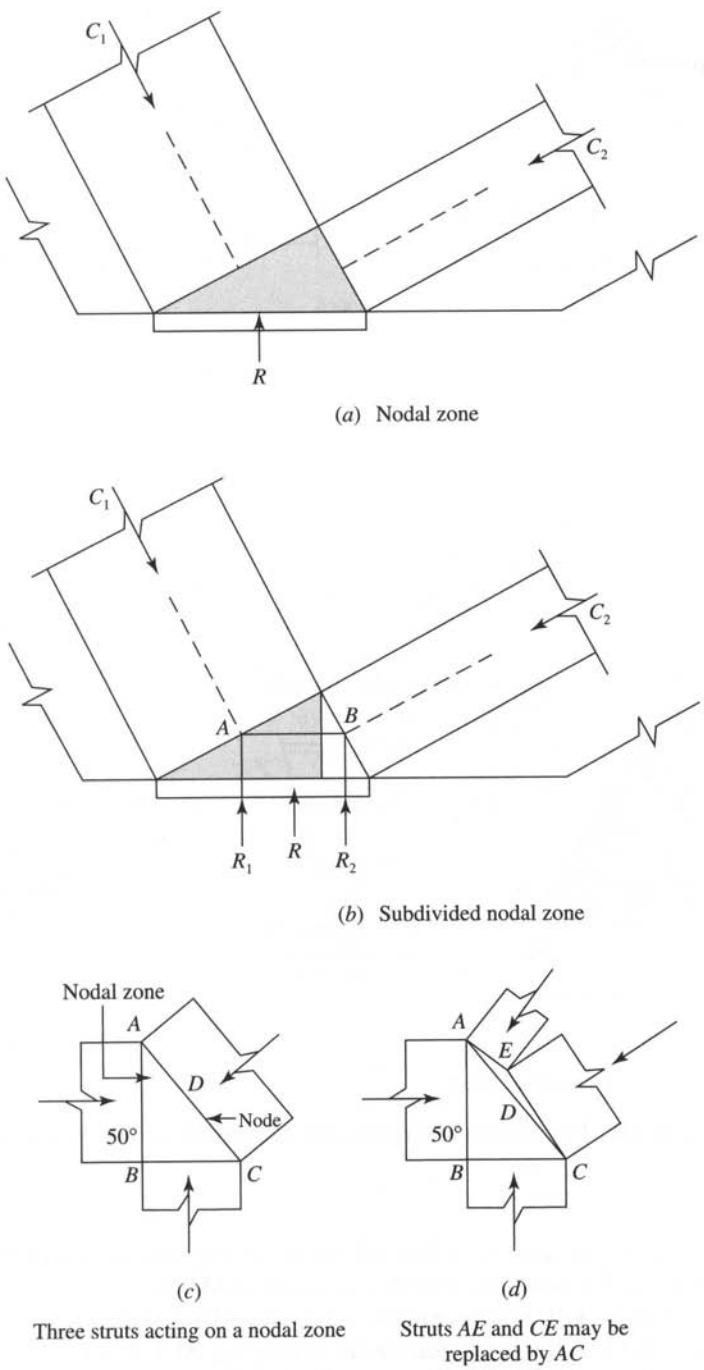


Figure 5.8 Nodal zones [1]. (*a*, *b*) Subdivision of nodal zone, (*c*, *d*) resolution of forces on a nodal zone.

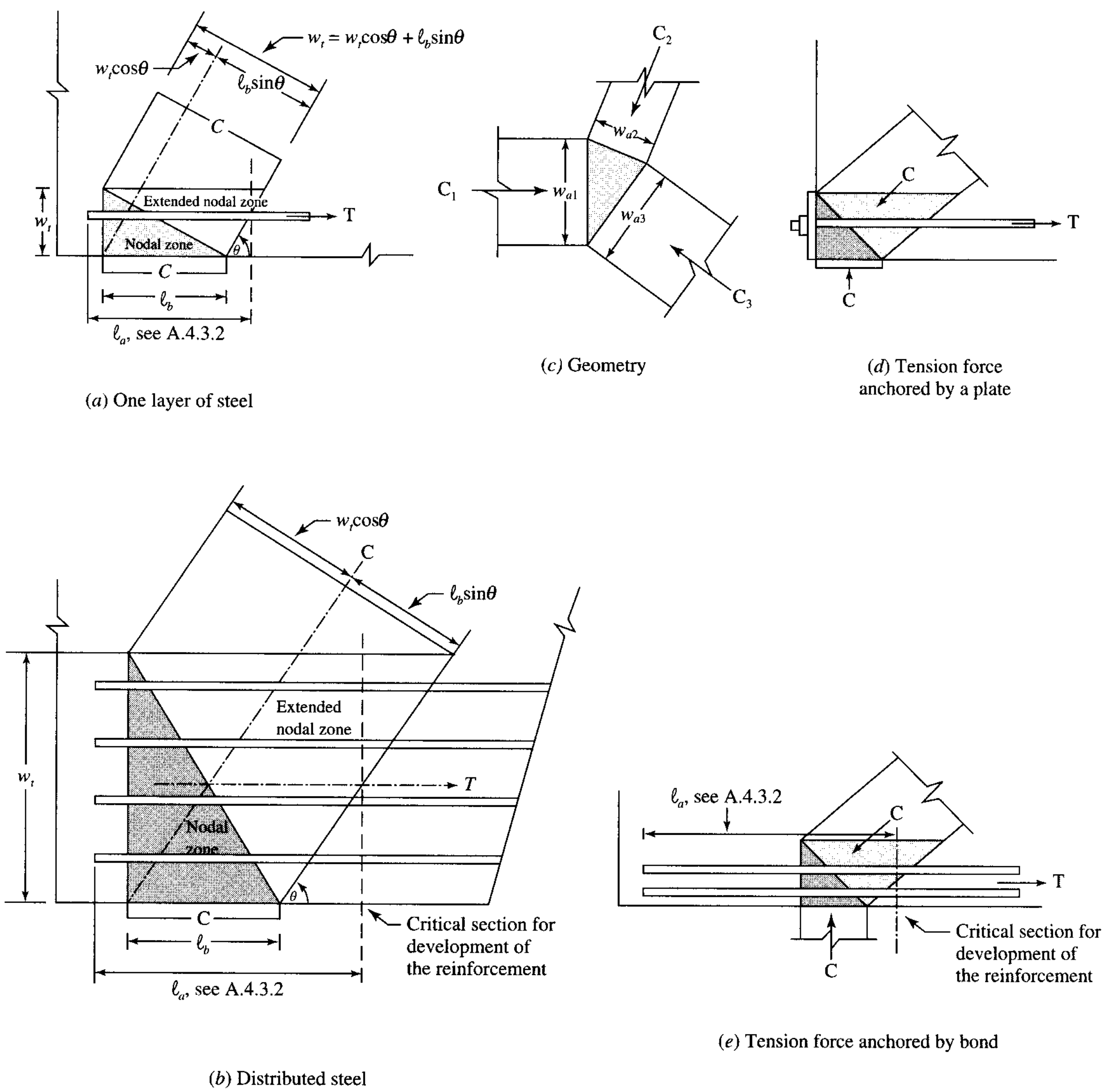


Figure 5.9 Extended nodal zones and hydrostatic nodes [1]. Courtesy of ACI 318-05.

c. For struts located such that the width of the midsection of the strut is larger than the width of the nodes (bottle-shaped struts) (ACI 3.2.2):
$\beta_s = 0.75$ with reinforcement satisfying ACI A.3.3
$\beta_s = 0.6\lambda$ without reinforcement satisfying ACI A.3.3

d. $\beta_s = 0.6\lambda$ for all other cases (ACI 3.2.4)

$$\begin{aligned}\lambda &= 1.00 \text{ normal-weight concrete}\\ &= 0.85 \text{ sand lightweight concrete}\\ &= 0.75 \text{ for all other lightweight concrete}\end{aligned}$$

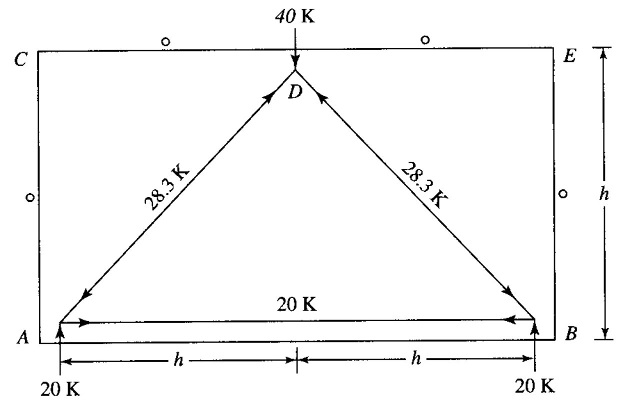


Figure 5.10 Example of a steel truss.

Linear interpolation between 0.75 and 0.85 shall be permitted, on the basis of volumetric fraction, when a portion of the lightweight fine aggregate is replaced with normal-weight fine aggregate. Linear interpolation between 0.85 and 1.0 shall be permitted for concrete containing normal-weight fine aggregate and a blend of light- and normal-weight coarse aggregate.

3. Reinforcement crossing struts (Fig. 5.11): For $f_c' \le 6$ ksi, the value $\beta_s = 0.75$ can be used if the axis of the strut is crossed by layers of bars such that

$$\sum \frac{A_{si}}{b_s s_i} \sin \gamma_i \ge 0.003 \tag{5.16}$$

where

A_{si} = total area of surface reinforcement at a spacing s_i in the ith layer crossing a strut with reinforcement at an angle α_i to the axis of the strut

s_i = spacing of reinforcement in the ith layer crossing a strut at an angle α_i to the axis of the strut member

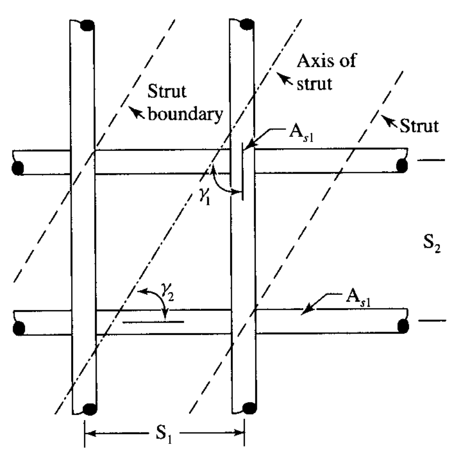


Figure 5.11 Reinforcing bars crossing a strut [1]. Courtesy of ACI 318-08.

b_s = width of member

α_i = angle between the axis of the strut and the bars in ith layer of bars crossing the strut

4. Compression reinforcement in struts: Compression reinforcement can be used to increase the strength of a strut such that

$$F_{ns} = f_{ce}A_{cs} + A'_s f'_s \tag{5.17}$$

where

F_{ns} = strength of a longitudinal reinforced strut

A'_s = area of the compression reinforcement in a strut

f'_s = stress in A'_s ($f'_s = f_y$ for grades 40 to 60)

5. Strength of ties: The nominal strength of a tie, F_{nt} is:

$$F_{nt} = A_{ts}f_y + A_{tp}(f_{se} + \Delta f_p) \tag{5.18}$$

where

A_{ts} = area of nonprestressed reinforcement in the tie

A_{tp} = area of prestressing reinforcement

f_{se} = effective stress after losses in prestressed reinforcement

Δf_p = increase in prestressing stress due to factored loads

A_{tp} = 0 for nonprestressed members

$$(f_{se} + \Delta f_p) \leq f_{py} \tag{5.19}$$

It is permitted to take Δf_p = 60 ksi for bonded prestressed reinforcement or 10 ksi for unbonded prestressed reinforcement. Also, a practical upper limit of the tie width, $w_{t,\max}$ can be taken as follows:

$$w_{t,\max} = F_{nt}/(f_{ce}b_s) \tag{5.20}$$

6. Strength of nodal zones: The nominal compression strength of a nodal zone, F_{nn}, shall be

$$F_{nn} = f_{ce}A_{nz} \tag{5.21}$$

where A_{nz} = the area of the face of the nodal zone or a section through a nodal zone perpendicular to the resultant force on the section.

7. Confinement in nodal zones: Unless confining reinforcement is provided within the nodal zone and its effect is supported by tests and analysis, the calculated effective compressive stress on a face of a nodal zone due to the strut and tie forces should not exceed the following:

$$f_{ce} = 0.85\beta_n f'_c \tag{5.22}$$

where

β_n = 1.0 in nodal zones bounded by struts or bearing areas, or both, C-C-C node.

β_n = 0.80 in nodal zones anchoring one tie, C-C-T node.

β_n = 0.60 in nodal zones anchoring two or more ties, C-T-T node.

The application of the strut and tie method to a deep beam is given in Example 8.6, Section 8.11.

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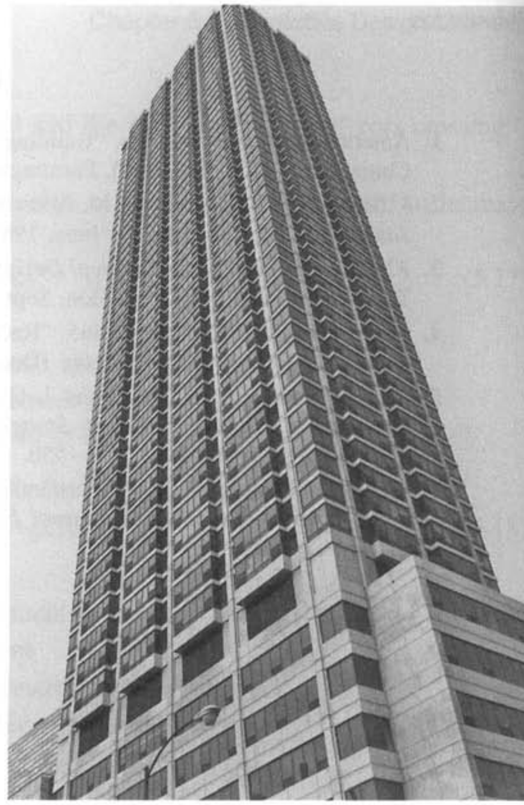
CHAPTER 6

DEFLECTION AND CONTROL OF CRACKING


High-rise building, Chicago, Illinois.

6.1 DEFLECTION OF STRUCTURAL CONCRETE MEMBERS

Flexural concrete members must be designed for safety and serviceability. The members will be safe if they are designed according to the ACI Code equations and limitations. Consequently, as explained in previous chapters, the size of each member is determined as well as the reinforcement required to maintain an internal moment capacity equal to or greater than that of the external moment. Once the final dimensions are determined, the beam must be checked for serviceability: cracks and deflection. Adequate stiffness of the member is necessary to prevent excessive cracks and deflection.

The use of the ACI Code provisions, taking into consideration the nonlinear relationship between stress and strain in concrete, has resulted in smaller sections than those designed by the elastic theory. The ACI Code, Section 9.4, recognizes the use of steel up to a yield strength of 80 ksi (560 MPa) and the use of high-strength concrete. The use of high-strength steel and concrete results in smaller sections and a reduction in the stiffness of the flexural member and consequently increases its deflection.

The permissible deflection is governed by many factors, such as the type of the building, the appearance of the structure, the presence of plastered ceilings and partitions, the damage expected due to excessive deflection, and the type and magnitude of live load.

The ACI Code, Section 9.5, specifies minimum thickness for one-way flexural members and one-way slabs, as shown in Table 6.1 in this chapter. The values are for members not supporting or attached to partitions or other constructions likely to be damaged by large deflections.

Table 6.1 Minimum Thickness of Beams and One-Way Slabs (L = Span Length)

Member	**Yield Strength f_y (ksi)**	**Simply Supported**	**One End Continuous**	**Both Ends Continuous**	**Cantilever**
Solid one-way slabs	40	$L/25$	$L/30$	$L/35$	$L/12.5$
	50	$L/22$	$L/27$	$L/31$	$L/11$
	60*	$L/20$	$L/24$	$L/28$	$L/10$
Beams or ribbed one-way slabs	40	$L/20$	$L/23$	$L/26$	$L/10$
	50	$L/18$	$L/20.5$	$L/23.5$	$L/9$
	60*	$L/16$	$L/18.5$	$L/21$	$L/8$

*Values reported in ACI Table 9.5(a).

The minimum thicknesses indicated in Table 6.1 are used for members made of normal-weight concrete, and for steel reinforcement with yield strengths as mentioned in the table. The values are modified for cases of lightweight concrete or a steel yield strength different from 60 ksi as follows:

- For lightweight concrete having unit weights in the range of 90 to 115 pcf, the values in the tables for $f_y = 60$ ksi (420 MPa) shall be multiplied by the greater of (1.65 − 0.005 W_c) but not less than 1.09, where W_c is the unit weight of concrete in pounds per cubic foot.
- For yield strength of steel different from 60 ksi (420 MPa), the values in the tables for 60 ksi shall be multiplied by $(0.4 + f_y/100)$, where f_y is in ksi.

6.2 INSTANTANEOUS DEFLECTION

The deflection of structural members is due mainly to the dead load plus a fraction of or all the live load. The deflection that occurs immediately upon the application of the load is called the *immediate*, or *instantaneous, deflection*. Under sustained loads, the deflection increases appreciably with time. Various methods are available for computing deflections in statically determinate and indeterminate structures. The instantaneous deflection calculations are based on the elastic behavior of the flexural members. The elastic deflection, Δ, is a function of the load, W, span, L, moment of inertia, I, and the modulus of elasticity of the material, E:

$$\Delta = f\left(\frac{WL}{EI}\right) = \alpha\left(\frac{WL^3}{EI}\right) = K\left(\frac{ML^2}{EI}\right) \tag{6.1}$$

where W = total load on the span and α and K are coefficients that depend on the degree of fixity at the supports, the variation of moment of inertia along the span, and the distribution of load. For example, the maximum deflection on a uniformly loaded simply supported beam is

$$\Delta = \frac{5WL^3}{384EI} = \frac{5wL^4}{384EI} \tag{6.2}$$

where W = the total load on the span = wL (uniform load per unit length × span). Deflections of beams with different loadings and different end conditions as a function of the load, span, and EI are given in Appendix C and in books of structural analysis.

Because W and L are known, the problem is to calculate the modulus of elasticity, E, and the moment of inertia, I, of the concrete member or the flexural stiffness of the member EI.

6.2.1 Modulus of Elasticity

The ACI Code, Section 8.5,specifies that the modulus of elasticity of concrete, E_c, may be taken as

$$E_c = 33W_c^{1.5}\sqrt{f_c'} \text{ psi} \tag{6.3}$$

for values of W_c between 90 and 160 pcf. For normal-weight concrete ($W_c = 145$ pcf),

$$E_c = 57{,}400\sqrt{f_c'} \text{ psi} \quad (\text{or } 57{,}000\sqrt{f_c'})$$

The modulus of elasticity is usually determined by the short-term loading of a concrete cylinder. In actual members, creep due to sustained loading, at least for the dead load, affects the modulus on the compression side of the member. For the tension side, the modulus in tension is assumed to be the same as in compression when the stress magnitude is low. At high stresses the modulus decreases appreciably. Furthermore, the modulus varies along the span due to the variation of moments and shear forces.

6.2.2 Modular Ratio

The modular ratio, $n = Es/E_c$, which is used in the transformed area concept was explained in Section 2.10. It may be used to the nearest whole number but may not be less than 6. For example,

$$\begin{aligned} &\text{when } f_c' = 2500 \text{ psi (17.5 MPa)}, && n = 10 \\ &\text{when } f_c' = 3000 \text{ psi (20 MPa)}, && n = 9 \\ &\text{when } f_c' = 4000 \text{ psi (30 MPa)}, && n = 8 \\ &\text{when } f_c' = 5000 \text{ psi (17.5 MPa)}, && n = 7 \end{aligned}$$

For normal-weight concrete, n may be taken as $500/\sqrt{f_c'}$, (psi units).

6.2.3 Cracking Moment

The behavior of a simply supported structural concrete beam loaded to failure was explained in Section 3.3. At a low load, a small bending moment develops, and the stress at the extreme tension fibers will be less than the modulus of rupture of concrete, $f_r = 7.5\lambda\sqrt{f_c'}$. If the load is increased until the tensile stress reaches an average stress of the modulus of rupture, f_r, cracks will develop. If the tensile stress is higher than f_r, the section will crack, and a cracked section case will develop. This means that there are three cases to be considered:

1. When the tensile stress, f_t, is less than f_r, the whole-uncracked section is considered to calculate the properties of the section. In this case, the gross moment of inertia, I_g, is used: $I_g = bh^3/12$, where bh = the whole concrete section.
2. When the tensile stress, f_t, is equal to the modulus of rupture, $f_r = 7.5\lambda\sqrt{f_c'}$, a crack may start to develop, and the moment that causes this stress is called the cracking moment. Using the flexural formula;

$$f_r = M_{cr}\frac{c}{I_g} \quad \text{or} \quad M_{cr} = f_r \cdot \frac{I_g}{c} \tag{6.4}$$

where $f_r = 7.5\lambda\sqrt{f'_c}$, I_g = the gross moment of inertia, and c = the distance from the neutral axis to the extreme tension fibers. For example, for a rectangular section, $I_g = bh^3/12$ and $c = h/2$, and where

λ is a modification factor for type of concrete (ACI 8.6.1)
$\lambda = 1.0$ Normal-weight concrete
$\lambda = 0.85$ Sand-lightweight concrete
$\lambda = 0.75$ For all-lightweight concrete

Linear interpolation shall be permitted between 0.85 and 1.0 on the basis of volumetric fractions, for concrete containing normal-weight fine aggregate and a blend of lightweight and normal-weight coarse aggregate.

3. When the applied external moment exceeds the cracking moment, M_{cr}, a cracked section case is developed, and the concrete in the tension zone is neglected. A transformed cracked section is used to calculate the cracking moment of inertia, I_{cr}, using the concrete area in compression and the transformed steel area nA_s.

Example 6.1

A rectangular concrete section is reinforced with three no. 9 bars in one row and has a width of 12 in., a total depth of 25 in., and $d = 22.5$. (Fig. 6.1. Calculate the modulus of rupture, f_r, the gross moment of inertia, I_g, and the cracking moment, M_{cr}. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. The modulus of rupture is $f_r = 7.5\lambda\sqrt{f'c} = 7.5 \times 1 \times \sqrt{4000} = 474$ psi. ($\lambda = 1$ normal-weight concrete)
2. The gross moment of inertia for a rectangular section is

$$bh^3/12 = \frac{12(25)^3}{12} = 15{,}625 \text{ in.}^4$$

3. The cracking moment is $M_{cr} = f_r \cdot I_g/c$

$$f_r = 474 \text{ psi} \qquad I_g = 15{,}625 \text{ in.}^4 \qquad c = h/2 = 12.5 \text{ in.}$$

Therefore, $M_{cr} = 474 \times 15{,}625/(12.5 \times 1000) = 592.5$ K·in. = 49.38 K·ft

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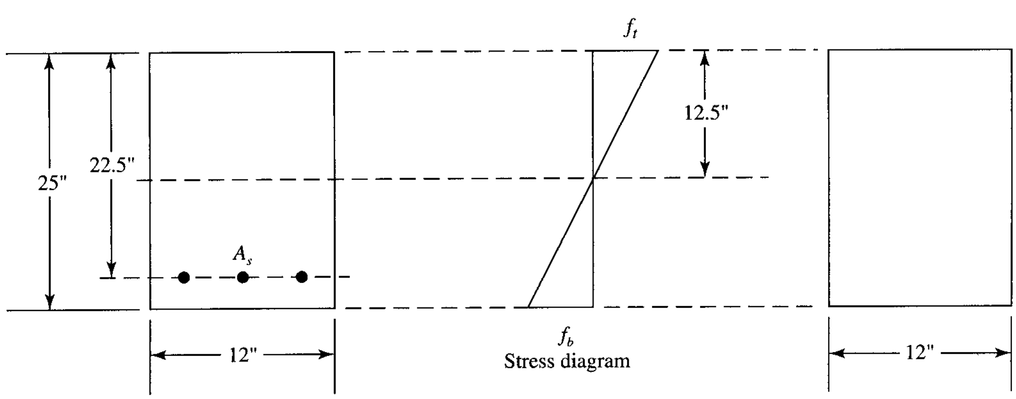


Figure 6.1 Example 6.1.

6.2.4 Moment of Inertia

The moment of inertia, in addition to the modulus of elasticity, determines the stiffness of the flexural member. Under small loads, the produced maximum moment will be small, and the tension stresses at the extreme tension fibers will be less than the modulus of rupture of concrete; in this case, the gross transformed cracked section will be effective in providing the rigidity. At working loads or higher, flexural tension cracks are formed. At the cracked section, the position of the neutral axis is high, whereas at sections midway between cracks along the beam, the position of the neutral axis is lower (nearer to the tension steel). In both locations only the transformed cracked sections are effective in determining the stiffness of the member; therefore, the effective moment of inertia varies considerably along the span. At maximum bending moment, the concrete is cracked, and its portion in the tension zone is neglected in the calculations of moment of inertia. Near the points of inflection the stresses are low, and the entire section may be uncracked. For this situation and in the case of beams with variable depth, exact solutions are complicated.

Figure 6.2*a* shows the load–deflection curve of a concrete beam tested to failure. The beam is a simply supported 17-ft span and loaded by two concentrated loads 5 ft apart, symmetrical about the centerline. The beam was subjected to two cycles of loading: In the first (curve *cy* 1), the load–deflection curve was a straight line up to a load $P = 1.7$ K when cracks started to occur in the beam. Line *a* represents the load–deflection relationship using a moment of inertia for the uncracked transformed section. It can be seen that the actual deflection of the beam under loads less than the cracking load, based on a homogeneous uncracked section, is very close to the calculated deflection (line *a*). Curve *cy* 1 represents the actual deflection curve when the load is increased to about one-half the ultimate load. The slope of the curve, at any level of load, is less than the slope of line *a* because cracks developed, and the cracked part of the concrete section reduces the stiffness of the beam. The load was then released, and a residual deflection was observed at midspan. Once cracks developed, the assumption of uncracked section behavior under small loads did not hold.

In the second cycle of loading, the deflection (curve *c*) increased at a rate greater than that of line *a*, because the resistance of the concrete tension fibers was lost. When the load was increased, the load–deflection relationship was represented by curve *cy* 2. If the load in the first cycle is increased up to the ultimate load, curve *cy* 1 will take the path *cy* 2 at about 0.6 of the ultimate load. Curve *c* represents the actual behavior of the beam for any additional loading or unloading cycles.

Line *b* represents the load–deflection relationship based on a cracked transformed section; it can be seen that the deflection calculated on that basis differs from the actual deflection. Figure 6.2*c* shows the variation of the beam stiffness EI with an increase in moment. ACI Code, Section 9.5, presents an equation to determine the effective moment of inertia used in calculating deflection in flexural members. The effective moment of inertia given by the ACI Code (Eq. 9.8) is based on the expression proposed by Branson [3] and calculated as follows:

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right] I_{cr} \le I_g \tag{6.5}$$

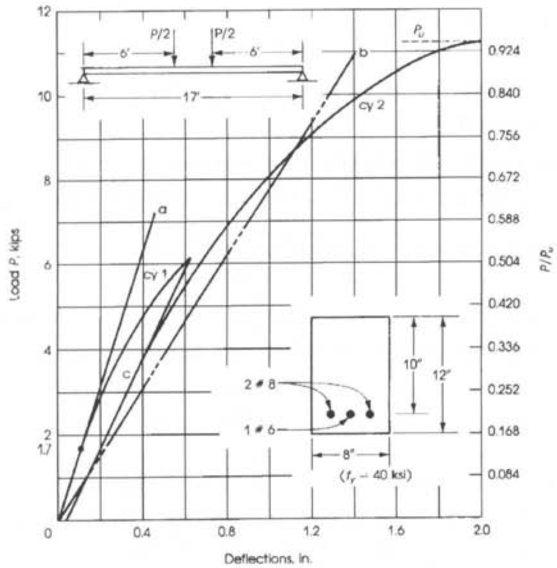
where

I_e = effective moment of inertia

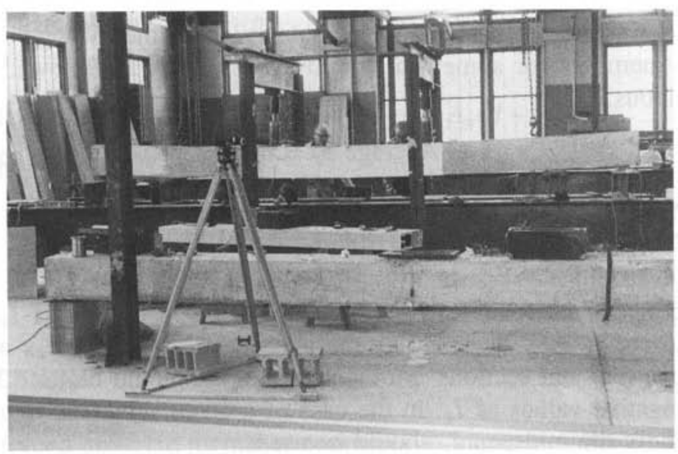
$$M_{cr} = \text{cracking moment}, \left(\frac{f_r I_g}{Y_t}\right) \tag{6.6}$$

f_r = modulus of rupture of concrete

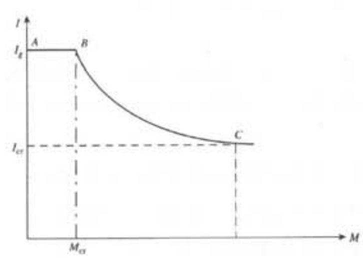
$$= 7.5\lambda\sqrt{f'_c}\ \text{psi},\ (0.623\lambda\sqrt{f'_c}\ \text{MPa}) \tag{6.7}$$



(*a*)



(*b*)



(*c*)

Figure 6.2 (*a*) Experimental and theoretical load – deflection curves for a beam of the section and load illustrated, (*b*) deflection of a reinforced concrete beam, and (*c*) variation of beam moment of inertia, *I*, with an increase in moment (E = constant). *BC* is a transition curve between I_g and I_{cr}.

M_a = maximum unfactored moment in member at stage for which deflection is being computed

I_g = moment of inertia of gross concrete section about the centroidal axis, neglecting the reinforcement

I_{cr} = moment of inertia of cracked transformed section

Y_t = distance from centroidal axis of cross-section, neglecting steel, to the tension face

The following limitations are specified by the code:

1. For continuous spans, the effective moment of inertia may be taken as the average of the moment of inertia of the critical positive and negative moment sections.
2. For prismatic members, I_e may be taken as the value obtained from Eq. 6.5 at midspan for simple and continuous spans and at the support section for cantilevers (ACI Code, Section 9.5.2).

   Note that I_e, as computed by Eq. 6.5, provides a transition between the upper and lower bounds of the gross moment of inertia, I_g, and the cracked moment of inertia, I_{cr}, as a function of the level of M_{cr}/M_a. Heavily reinforced concrete members may have an effective moment of inertia, I_e, very close to that of a cracked section, I_{cr}, whereas flanged members may have an effective moment of inertia close to the gross moment of inertia, I_g.
3. For continuous beams, an approximate value of the average I_e for prismatic or nonprismatic members for somewhat improved results is as follows: For beams with both ends continuous,

$$\text{Average } I_e = 0.70I_m + 0.15(I_{e1} + I_{e2}) \tag{6.8}$$

For beams with one end continuous,

$$\text{Average } I_e = 0.85I_m + 0.15(I_{\text{con}}) \tag{6.9}$$

where I_m = midspan I_e, I_{e1}, I_{e2} = I_e at beam ends, and I_{con} = I_e at the continuous end. Also, I_e may be taken as the average value of the I_es at the critical positive- and negative-moment sections. Moment envelopes should be used in computing both positive and negative values of I_e. In the case of a beam subjected to a single heavy concentrated load, only the midspan I_e should be used.

6.2.5 Properties of Sections

To determine the moment of inertia of the gross and cracked sections, it is necessary to calculate the distance from the compression fibers to the neutral axis (x or kd).

1. Gross moment of inertia, I_g (neglect all steel in the section)
   a. For a rectangular section of width b and a total depth h, $I_g = bh^3/12$.
   b. For a T-section, flange width b, web width b_w, and flange thickness t, calculate y, the distance to the centroidal axis from top of flange:

$$y = \frac{\left(\frac{bt^2}{2}\right) + b_w(h-t)\left[\frac{(h-t)}{2+t}\right]}{bt + b_w(h-t)} \tag{6.10}$$

Then calculate I_g:

$$I_g = \left[\frac{bt^3}{12} + bt\left(y - \frac{t}{2}\right)^2\right] + \left[b_w\frac{(y-t)^3}{3}\right] + \left[b_w\frac{(h-y)}{3}\right] \quad (6.10a)$$

2. Cracked moment of inertia, I_{cr}: Let x = the distance of the neutral axis from the extreme compression fibers ($x = kd$).

a. Rectangular section with tension steel, A_s, only

i. Calculate x from the following equation:

$$\frac{bx^2}{2} - nA_s(d - x) = 0 \quad (6.11)$$

ii. Calculate $I_{cr} = bx^3/3 + nA_s\ (d - x)^2$ (6.11a)

b. Rectangular section with tension steel A_s and compression steel A'_s

i. Calculate x:

$$\frac{bx^2}{2} + (n-1)A'_s(x - d') - nA_s(d - x) = 0 \quad (6.12)$$

ii. Calculate $I_{cr} = (bx^3/3) + (n-1)A'_s(x - d')^2 + nA_s(d - x)^2$. (6.12a)

c. T-sections with tension steel A_s

i. Calculate x : $bt\left(x - \frac{t}{2}\right) + b_w\frac{(x-t)^2}{2} - nA_s(d - x) = 0$ (6.13)

ii. Calculate I_{cr}:

$$I_{cr} = \left[\frac{bt^3}{12} + bt\left(x - \frac{t}{2}\right)^2\right] + \left[b_w\frac{(x-t)^3}{3}\right] + nA_s(d - x)^2 \quad (6.13a)$$

6.3 LONG-TIME DEFLECTION

Deflection of reinforced concrete members continues to increase under sustained load, although more slowly with time. Shrinkage and creep are the cause of this additional deflection, which is called long-time deflection [1]. It is influenced mainly by temperature, humidity, age at time of loading, curing, quantity of compression reinforcement, and magnitude of the sustained load. The ACI Code, Section 9.5.2.5, suggests that unless values are obtained by a more comprehensive analysis, the additional long-term deflection for both normal and lightweight concrete flexural members shall be obtained by multiplying the immediate deflection by the factor

$$\lambda_\Delta = \frac{\zeta}{1 + 50\rho'} \quad (6.14)$$

where

λ_Δ = multiplier for additional deflection due to long-term effect.

ρ' = A'_s/bd for the section at midspan of a simply supported or continuous beam or at the support of a cantilever beam

ζ = time-dependent factor for sustained loads that may be taken as shown in Table 6.2.

Table 6.2 Multipliers for Long-time Deflections

Period (months)	1	3	6	12	24	36	48	60 &over
ζ	0.5	1.0	1.2	1.4	1.7	1.8	1.9	2.0

The factor λ_Δ is used to compute deflection caused by the dead load and the portion of the live load that will be sustained for a sufficient period to cause significant time-dependent deflections. The factor λ_Δ is a function of the material property, represented by ζ, and the section property, represented by $(1 + 50\rho')$. In Eq. 6.14, the effect of compression reinforcement is related to the area of concrete rather than the ratio of compression to tension steel.

The ACI Code Commentary, Section 9.5.2.5, presents a curve to estimate ζ for periods less than 60 months. These values are estimated as shown in Table 6.2.

The total deflection is equal to the immediate deflection plus the additional long-time deflection. For instance, the total additional long-time deflection of a flexural beam with $\rho' = 0.01$ at a 5-year period is equal to λ_Δ times the immediate deflection, where $\lambda_\Delta = 2/(1 + 50 \times 0.01) = 1.33$.

6.4 ALLOWABLE DEFLECTION

Deflection shall not exceed the following values according to the ACI Code, Section 9.5:

- $L/180$ for immediate deflection due to live load for flat roofs not supporting elements that are likely to be damaged
- $L/360$ for immediate deflection due to live load for floors not supporting elements likely to be damaged
- $L/480$ for the part of the total deflection that occurs after attachment of elements, that is, the sum of the long-time deflection due to all sustained loads and the immediate deflection due to any additional live load, for floors or roofs supporting elements likely to be damaged
- $L/240$ for the part of the total deflection occurring after elements are attached, for floors or roofs not supporting elements likely to be damaged

6.5 DEFLECTION DUE TO COMBINATIONS OF LOADS

If a beam is subjected to different types of loads (uniform, nonuniform, or concentrated loads) or subjected to end moments, the deflection may be calculated for each type of loading or force applied on the beam separately and the total deflection calculated by superposition. This means that all separate deflections are added up algebraically to get the total deflection. The deflections of beams under individual loads are shown in Table 6.3.

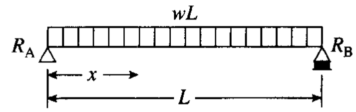
Example 6.2

Calculate the instantaneous midspan deflection for the simply supported beam shown in Fig. 6.3, which carries a uniform dead load of 0.4 K/ft and a live load of 0.6 K/ft in addition to a concentrated dead load of 5 kips at midspan. Given: $f'_c = 4$ ksi normal-weight concrete, $f_y = 60$ ksi, $b = 13$ in., $d = 21$ in., and total depth $= 25$ in. ($n = 8$).

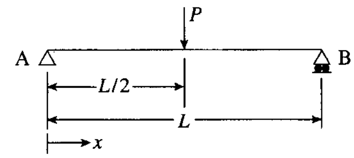
Table 6.3 Deflection of Beams

$$\Delta_{max} = \frac{5}{384} \times \frac{WL^3}{EI} \quad \text{(at center)}$$

$$W = \text{total load} = wL$$



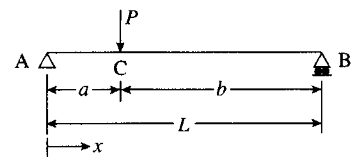
$$\Delta_{max} = \frac{PL^3}{48EI} \quad \text{(at midspan)}$$



$$\Delta_C = \frac{Pa^2b^2}{3EIL} \quad \text{(at point load)}$$

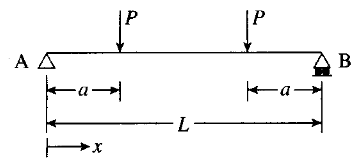
$$\Delta_{max} = \frac{PL^3}{48EI}\left[\frac{3a}{L} - 4\left(\frac{a}{L}\right)^3\right] \quad \text{(when } a \geq b\text{)}$$

$$\text{at } x = \sqrt{a(b+L)/3}$$



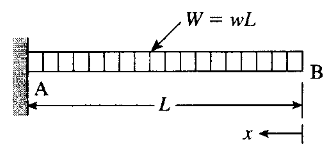
$$\Delta_{max} = \frac{PL^3}{6EI}\left[\frac{3a}{4L} - \left(\frac{a}{L}\right)^3\right] \quad \text{(at midspan)}$$

$$\Delta_{max} = \frac{23PL^3}{648EI} \quad \text{(at midspan) when } a = L/3$$



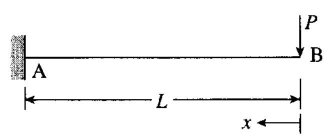
$$\Delta_B\text{max} = \frac{WL^3}{8EI} \quad (W = wL)$$

$$\Delta_x = \frac{w}{24EIL}(x^4 - 4L^3x + 3L^4)$$



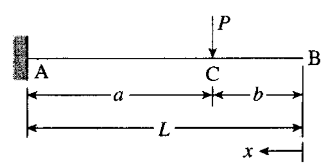
$$\Delta_B\text{max} = \frac{PL^3}{3EI}$$

$$\Delta_x = \frac{P}{6EI}(2L^3 - 3L^2x + x^3)$$



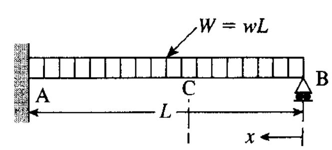
$$\Delta_C = Pa^3/3EI$$

$$\Delta_B\text{max} = \frac{Pa^3}{3EI}\left(1 + \frac{3b}{2a}\right) \quad \text{(at free end)}$$

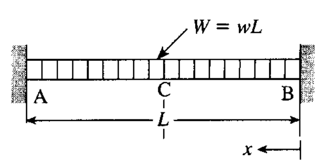


$$\Delta_{max} = \frac{WL^3}{185EI}$$

at a distance $x = 0.4215L$ (from support B)



$$\Delta_{max} = \frac{WL^3}{384EI} \quad \text{(at midspan)}$$



$$\Delta_1 = \frac{5wL^4}{384E_cI_e}$$

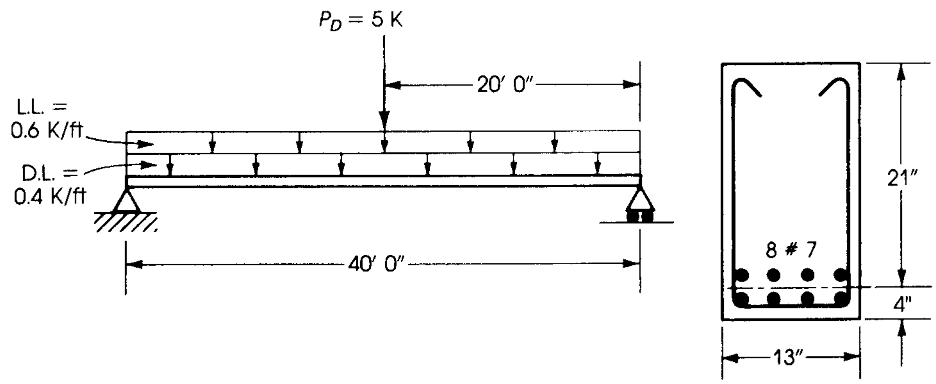


Figure 6.3 Example 6.2.

Solution

1. Check minimum depth according to the ACI Code, Table 6.1.

$$\text{Minimum total depth} = \frac{L}{16} = \frac{40 \times 12}{16} = 30 \text{ in.}$$

The total thickness used in 25 in. < 30 in.; therefore, deflection must be checked.

2. The deflection at midspan due to a distributed load is

$$\Delta_1 = \frac{5wL^4}{384E_cI_e}$$

The deflection at midspan due to a concentrated load is

$$\Delta_2 = \frac{PL^3}{48E_cI_e}$$

Because w, P, and L are known, we must determine the modulus of elasticity, E_c, and the effective moment of inertia, I_e.

3. The modulus of elasticity of concrete is

$$E_c = 57{,}400\sqrt{f_c'} = 57{,}400\sqrt{4000} = 3.63 \times 10^6\text{psi}$$

4. The effective moment of inertia is equal to

$$\Delta_1 = \frac{5wL^4}{384E_cI_e}$$

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right] I_{cr} \le I_g$$

Determine values of all terms on the right-hand side:

$$M_a = \frac{wL^2}{8} + \frac{PL}{4} = \frac{(0.6 + 0.4)}{8}(40)^2 \times 12 + \frac{5 \times 40}{4} \times 12 = 3000 \text{ K·in.}$$

$$I_g = \frac{bh^3}{12} = \frac{13(25)^3}{12} = 16{,}927 \text{ in.}^4$$

$$M_{cr} = \frac{f_r I_g}{Y_t} \qquad Y_t = \frac{h}{2} = 12.5 \text{ in.} \qquad f_r = 7.5\lambda\sqrt{f_c'} = 474 \text{ psi} \quad \lambda = 1 \text{ (normal-weight)}$$

$$M_{cr} = \frac{0.474 \times 16{,}927}{12.5} = 642 \text{ K·in.}$$

The moment of inertia of the cracked transformed area, I_{cr}, is calculated as follows: Determine the position of the neutral axis for a cracked section by equating the moments of the transformed area about the neutral axis to 0, letting $x = kd$ = distance to the neutral axis:

$$\frac{bx^2}{2} - nA_s(d - x) = 0 \quad n = \frac{E_s}{E_c} = 8.0 \quad A_s = 4.8 \text{ in.}^2$$

$$\frac{13}{2}x^2 - (8)(4.8)(21 - x) = 0$$

$$x^2 + 5.9x - 124 = 0 \quad x = 8.8 \text{ in.}$$

$$I_{cr} = \frac{bx^3}{3} + nA_s(d - x)^2 = \frac{13(8.8)^3}{3} + 38.4(21 - 8.8)^2 = 8660 \text{ in.}^4$$

With all terms calculated,

$$I_e = \left(\frac{642}{3000}\right)^3 \times 16{,}927 + \left[1 - \left(\frac{642}{3000}\right)^3\right] \times 8660 = 8740 \text{ in.}^4$$

5. Calculate the deflections from the different loads:

$$\Delta_1 \text{(due to distributed load)} = \frac{5wL^4}{384E_cI_e}$$

$$\Delta_1 = \left(\frac{5}{384}\right) \times \left(\frac{1000}{12}\right) \times \frac{(40 \times 12)^4}{3.63 \times 10^6 \times 8740} = 1.82 \text{ in.}$$

$$\Delta_2 \text{(due to concentrated load)} = \frac{PL^3}{48E_cI_e}$$

$$\Delta_2 = \frac{5000 \times (40 \times 12)^3}{48 \times 3.63 \times 10^6 \times 8740} = 0.36 \text{ in.}$$

Total immediate deflection = $\Delta_1 + \Delta_2 = 1.82 + 0.36 = 2.18$ in.

6. Compare the calculated values with the allowable deflection: The immediate deflection due to a uniform live load of 0.6 K/ft is equal to 0.6(1.82) = 1.09 in. If the member is part of a floor construction not supporting or attached to partitions or other elements likely to be damaged by large deflection, the allowable immediate deflection due to live load is equal to

$$\frac{L}{360} = \frac{40 \times 12}{360} = 1.33 \text{ in.} > 1.09 \text{ in.}$$

If the member is part of a flat roof and similar to the preceding, the allowable immediate deflection due to live load is $L/180 = 2.67$ in. Both allowable values are greater than the actual deflection of 1.09 in. due to the uniform applied live load.

Example 6.3

Determine the long-time deflection of the beam in Example 6.2 if the time-dependent factor equals 2.0.

Solution

1. The sustained load causing long-time deflection is that due to dead load, consisting of a distributed uniform dead load of 0.4 K/ft and a concentrated dead load of 5 K at midspan.

$$\text{Deflection due to uniform load} = 0.4 \times 1.82 = 0.728 \text{ in.}$$

Deflection is a linear function of load, w, all other values (L, E_c, I_e) being the same.

$$\text{Deflection due to concentrated load} = 0.36 \text{ in.}$$

$$\text{Total immediate deflection due to sustained loads} = 0.728 + 0.36 = 1.088 \text{ in.}$$

2. For additional long-time deflection, the immediate deflection is multiplied by the factor λ_Δ:

$$\lambda_\Delta = \frac{\zeta}{1 + 50\rho'} = \frac{2}{1+0}$$

In this problem, $A_s' = 0$; therefore, $\lambda_\Delta = 2.0$.

$$\text{Additional long-time deflection} = 2 \times 1.088 = 2.176 \text{ in.}$$

3. Total long-time deflection is the immediate deflection plus additional long-time deflection: 2.18 + 2.176 = 4.356 in.
4. Deflection due to dead load plus additional long-time deflection due to shrinkage and creep is 1.088 + 2.176 = 3.264 in.

Example 6.4

Calculate the instantaneous and 1-year long-time deflection at the free end of the cantilever beam shown in Fig. 6.4. The beam has a 20-ft span and carries a uniform dead load of 0.4 K/ft, a uniform live load of 0.4 K/ft, a concentrated dead load, P_D, of 3 K at the free end, and a concentrated live load, P_L, of 4 K placed at 10 ft from the fixed end. Given: $f_c' = 4$ ksi, $f_y = 60$ ksi, $b = 12$ in., $d = 21.5$ in., and total depth of section = 25 in. (Tension steel is six no. 8 bars and compression steel is two no. 8 bars.). Assume normal-weight concrete.

Solution

1. Minimum depth $= L/8 = \frac{20}{8} = 2.5$ ft = 30 in., which is greater than the 25 in. used. Therefore, deflection must be checked. The maximum deflection of a cantilever beam is at the free end. The deflection at the free end is as follows.

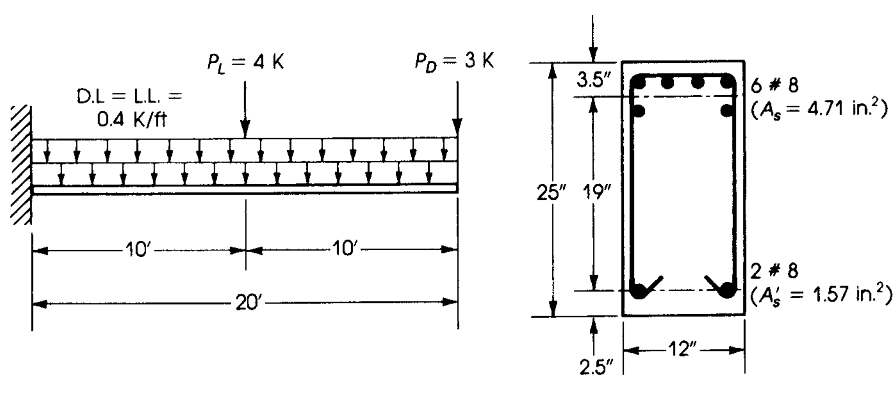


Figure 6.4 Example 6.4.

Deflection due to distributed load:

$$\Delta_1 = \frac{wL^4}{8EI}$$

Deflection due to a concentrated dead load at the free end:

$$\Delta_2 = \frac{P_D L^3}{3EI}$$

Deflection due to concentrated live load at $a = 10\,\text{ft}$ from the fixed end is maximum at the free end:

$$\Delta_3 = \frac{P_L(a)^2}{6EI}(3L - a) \quad \text{or} \quad \frac{Pa^3}{3EI}\left(1 + \frac{3b}{2a}\right)$$

2. The modulus of elasticity of normal-weight concrete is

$$E_c = 57{,}400\sqrt{f'_c} = 57{,}400\sqrt{4000} = 3.63 \times 10^6 \text{ psi}$$

3. Maximum moment at the fixed end is

$$M_a = \frac{wL^2}{2} + P_D \times 20 + P_L \times 10$$

$$= \frac{(0.4 + 0.4)(400)}{2} + 3 \times 20 + 4 \times 10 = 260 \text{ K·ft}$$

4. I_g = gross moment of inertia (concrete only)

$$= \frac{bh^3}{12} = \frac{12 \times (25)^3}{12} = 15{,}625 \text{ in.}^4$$

5. $$M_{cr} = \frac{f_r I_g}{Y_t} = \frac{((7.5)(1)\sqrt{4000}) \times 15{,}625}{\frac{25}{2}} = 592.9 \text{ K·in.} = 49.40 \text{ K·ft}$$

6. Determine the position of the neutral axis; then determine the moment of inertia of the cracked transformed section. Take moments of areas about the neutral axis and equate them to 0. Use $n = 8$ to calculate the transformed area of A_s and use $(n - 1) = 7$ to calculate the transformed area of A'_s. Let $kd = x$.

$$b\frac{(x^2)}{2} + (n-1)A'_s(x - d') - nA_s(d - x) = 0$$

For this section, $x = 8.44\,\text{in.}$

$$I_{cr} = \frac{b}{3}x^3 + (n-1)A'_s(x - d')^2 + nA_s(d - x)^2 = 9220 \text{ in.}^4$$

7. Effective moment of inertia is

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right] I_{cr} \le I_g$$

$$= \left(\frac{49.40}{260}\right)^3 \times 15{,}625 + \left[1 - \left(\frac{49.40}{260}\right)^3\right] \times 9220 = 9264 \text{ in.}^4$$

8. Determine the components of the deflection:

$$\Delta_1 \text{ (due to uniform load of 0.8K/ft)} = \frac{800}{12} \times \frac{(20 \times 12)^4}{8 \times 3.63 \times 10^6 \times 9264} = 0.82 \text{ in.}$$

$$\Delta_1 \text{ (due to dead load)} = 0.82 \times \frac{0.4}{0.8} = 0.41 \text{ in.}$$

$$\Delta_2 \text{ (due to concentrated dead load) at free end} = \frac{3000(20 \times 12)^3}{3 \times 3.63 \times 10^6 \times 9264} = 0.41 \text{ in.}$$

$$\Delta_3 \text{ (due to concentrated live load at 10 ft from fixed end)} = \frac{4000(10 \times 12)^2 \times (3 \times 20 \times 12 - 10 \times 12)}{6 \times 3.63 \times 10^6 \times 9264} = 0.17 \text{ in.}$$

The total immediate deflection is

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3 = 0.82 + 0.41 + 0.17 = 1.40 \text{ in.}$$

9. For additional long-time deflection, the immediated deflection is multiplied by the factor λ_Δ. For a 1-year period, $\zeta = 1.4$.

$$\rho' = \frac{A_s'}{bd} = \frac{1.57}{12 \times 21.5} = 0.0061$$

$$\lambda_\Delta = \frac{1.4}{1 + 50 \times 0.0061} = 1.073$$

Total immediate deflection Δ_s due to sustained load (here only the dead load of 0.4 K/ft and $P_D = 3$ K at free end): $\Delta_s = (0.41 + 0.41) = 0.82$ in. Additional long-time deflection = 1.073 × 0.82 = 0.88 in.

10. Total long-time deflection is the immediate deflection plus long-time deflection due to shrinkage and creep.
Total $\Delta = 1.40 + 0.88 = 2.28$ in.

Example 6.5

Calculate the instantaneous midspan deflection of beam *AB* in Fig. 6.5, which has a span of 32 ft. The beam is continuous over several supports of different span lengths. The absolute bending moment diagram and cross-sections of the beam at midspan and supports are also shown. The beam carries a uniform dead load of 4.2 K/ft and a live load of 3.6 K/ft. Given: $f_c' = 3$ ksi normal-weight concrete, $f_y = 60$ ksi, and $n = 9.2$.

Moment at midspan:	$M_D = 192$ K·ft	$M_{(D+L)} = 480$ K·ft
Moment at left support *A* :	$M_D = 179$ K·ft	$M_{(D+L)} = 420$ K·ft
Moment at right support *B* :	$M_D = 216$ K·ft	$M_{(D+L)} = 542$ K·ft

Solution

1. The beam *AB* is subjected to a positive moment that causes a deflection downward at midspan and negative moments at the two ends, causing a deflection upward at midspan. As was explained earlier, the deflection is a function of the effective moment of inertia, I_e. In a continuous beam, the value of I_e to be used is the average value for the positive and negative moment regions. Therefore, three sections will be considered: the section at midspan and the sections at the two supports.
2. Calculate I_e: For the gross area of all sections, $kd = 13.5$ in. and $I_g = 114{,}300$ in.^{4} Also, $f_r = 7.5\lambda\sqrt{f_c'} = 410$ psi and $E_e = 57{,}400\sqrt{f_c'} = 3.15 \times 10^6$ for all sections. The values of kd, I_{cr}, and M_{cr} for each cracked section, I_e for dead load only (using M_a of dead load), and

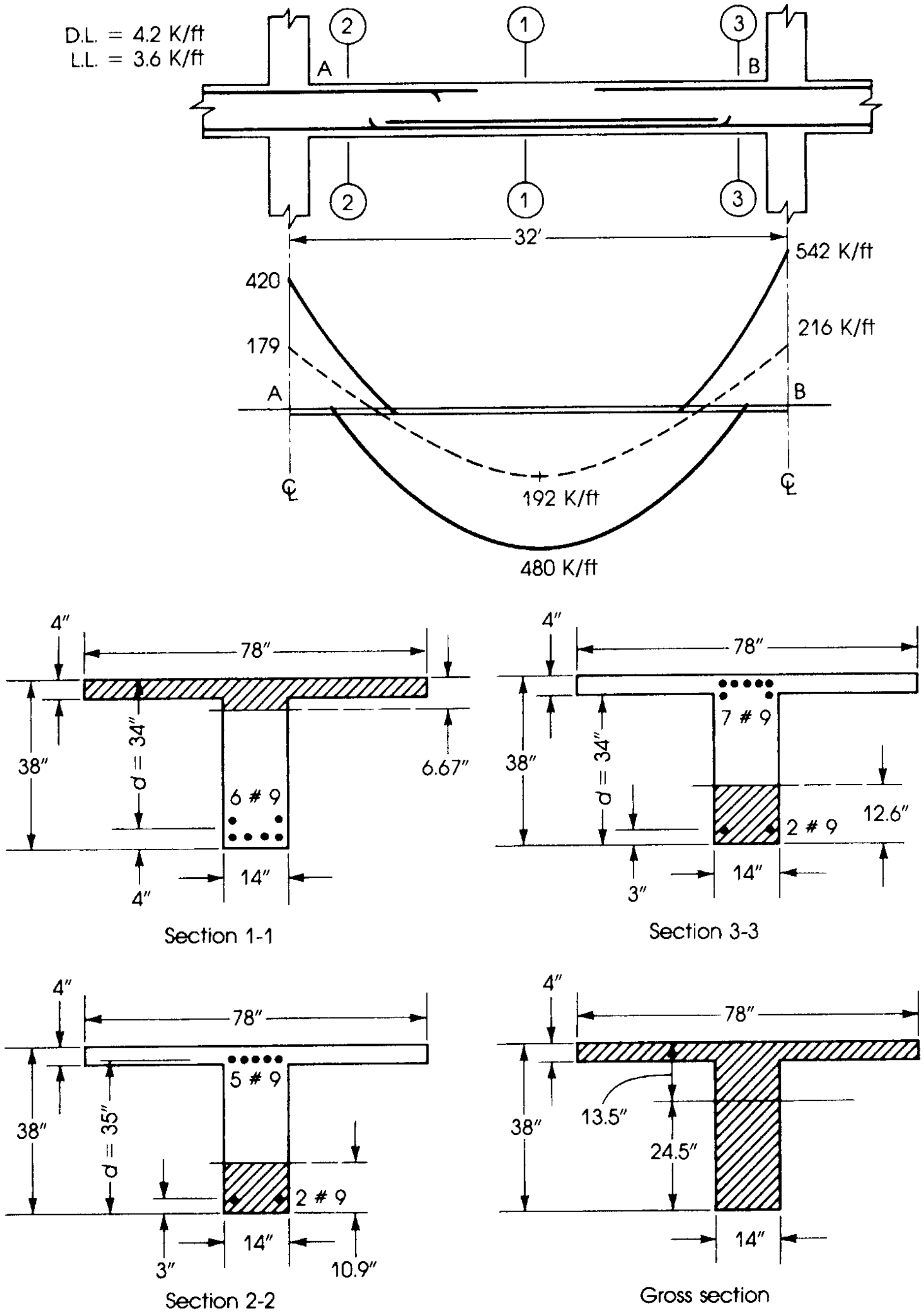


Figure 6.5 Example 6.5: deflection of a continuous beam.

I_e for dead and live loads (using M_a for dead and live loads) are calculated and tabulated as follows.

Section	*kd* (in.)	I_{cr} (in.^{4})	M_{cr} (K·ft)	I_e (in.^{4}) (Dead load)	I_e (in.^{4}) ($D + L$)
Midspan	6.67	48,550	159.4	86,160	50,960
Support *A*	10.9	34,930	289.3	114,300	60,880
Support *B*	12.6	44,860	289.3	114,300	55,415

Note that when the beam is subjected to dead load only and the ratio M_{cr}/M_a is greater than 1.0, I_e is equal to I_g.

3. Calculate average I_e from Eq. 6.8:

$$I_{e1}\,(\text{average}) = 0.7(50{,}960) + 0.15(60{,}880 + 55{,}415)$$
$$= 53{,}116 \text{ in.}^4$$

For dead and live loads,

$$\text{Average } I_e \text{ for end sections} = \tfrac{1}{2}(60{,}880 + 55{,}415)$$
$$= 58{,}150 \text{ in.}^4$$
$$I_{e2}\ (\text{average}) = \tfrac{1}{2}(50{,}960 + 58{,}150) = 54{,}550 \text{ in.}^4$$

For dead loads only,

$$\text{Average } I_e \text{ for end sections} = 114{,}300 \text{ in.}^4$$
$$I_{e3}\ (\text{average}) = \tfrac{1}{2}(86{,}160 + 114{,}300) = 100{,}230 \text{ in.}^4$$

4. Calculate immediate deflection at midspan:

$$\Delta_1 \text{ (due to uniform load)} = \frac{5wL^4}{384EI_e} \quad \text{(downward)}$$
$$\Delta_2 \text{ (due to a moment at } A, M_A) - \frac{M_A L^2}{16EI_e} \quad \text{(upward)}$$
$$\Delta_3 \text{ (due to a moment at } B, M_B) = -\frac{M_B L^2}{16EI_e} \quad \text{(upward)}$$
$$\text{Total deflection } \Delta = \Delta_1 - \Delta_2 - \Delta_3$$

The dead-load deflection for a uniform dead load of 4.2 K/ft, taking M_A(D.L.) = 179 K·ft, M_B (D.L.) = 216 K·ft, and I_{e3} = 100,230 in.^{4} and then substituting in the preceding equations, is

$$\Delta = 0.314 - 0.063 - 0.075 = 0.176 \text{ in.} \quad \text{(downward)}$$

The deflection due to combined dead and live loads is found by taking dead plus live load = 7.8 K/ft, M_A = 420 K·ft, M_B = 542 K·ft, and I_{e2} = 54,550 in.^{4}:

$$\Delta = 1.071 - 0.270 - 0.349 = 0.452 \text{ in.} \quad \text{(downward)}$$

The immediate deflection due to live load only is 0.542 − 0.176 = 0.276 in. (downward). If the limiting permissible deflection is $L/480$ = (32 × 12)/480 = 0.8 in., then the section is adequate.

There are a few points to mention about the results.

a. If I_e of the midspan section only is used ($I_e = 50{,}960\,\text{in.}^4$) then the deflection due to dead plus live loads is calculated by multiplying the obtained value in step 4 by the ratio of the two I_e:

$$\Delta\ (\text{dead} + \text{live}) = 0.452 \times \left(\frac{54{,}550}{50{,}960}\right) = 0.484\ \text{in.}$$

The difference is small, about 7% on the conservative side.

b. If I_e 1 (average) is used ($I_{e1} = 53{,}116\,\text{in.}^4$), then Δ (dead + live) = 0.471 in. The difference is small, about 4% on the conservative side.

c. It is believed that it is more convenient to use I_e at midspan section unless a more rigorous solution is required.

6.6 CRACKS IN FLEXURAL MEMBERS

The study of crack formation, behavior of cracks under increasing load, and control of cracking is necessary for proper design of reinforced concrete structures. In flexural members, cracks develop under working loads, and because concrete is weak in tension, reinforcement is placed in the cracked tension zone to resist the tension force produced by the external loads.

Flexural cracks develop when the stress at the extreme tension fibers exceeds the modulus of rupture of concrete. With the use of high-strength reinforcing bars, excessive cracking may develop in reinforced concrete members. The use of high-tensile steel has many advantages, yet the development of undesirable cracks seems to be inevitable. Wide cracks may allow corrosion of the reinforcement or leakage of water structures and may spoil the appearance of the structure.

A crack is formed in concrete when a narrow opening of indefinite dimension has developed in the concrete beam as the result of internal tensile stresses. These internal stresses may be due to one or more of the following:

- External forces such as direct axial tension, shear, flexure, or torsion
- Shrinkage
- Creep
- Internal expansion resulting from a change of properties of the concrete constituents

In general, cracks may be divided into two main types: secondary cracks and main cracks.

6.6.1 Secondary Cracks

Secondary cracks, very small cracks that develop in the first stage of cracking, are produced by the internal expansion and contraction of the concrete constituents and by low flexural tension stresses due to the self-weight of the member and any other dead loads. There are three types of secondary cracks.

Shrinkage cracks. *Shrinkage cracks* are important cracks, because they affect the pattern of cracking that is produced by loads in flexural members. When they develop, they form a weak path in the concrete. When load is applied, cracks start to appear at the weakest sections, such as along the reinforcing bars. The number of cracks formed is limited by the amount of shrinkage in concrete and the presence of restraints. Shrinkage cracks are difficult to control.

Secondary flexural cracks. Usually *secondary flexural cracks* are widely spaced, and one crack does not influence the formation of others [8]. They are expected to occur under low loads, such as dead loads. When a load is applied gradually on a simple beam, tensile stress develops at the bottom fibers, and when it exceeds the flexural tensile stress of concrete, cracks start to develop. They widen gradually and extend toward the neutral axis. It is difficult to predict the sections at which secondary cracks start because concrete is not a homogeneous, isotropic material.

Salinger [9] and Billing [10] estimated the steel stress just before cracking to be from about 6000 to 7000 psi (42 to 49 MPa). An initial crack width of the order of 0.001 in. (0.025 mm) is expected at the extreme concrete tensile fibers. Once cracks are formed, the tensile stress of concrete at the cracked section decreases to 0, and the steel bars take all the tensile force. At this moment, some slip occurs between the steel bars and the concrete due to the differential elongation of concrete and steel and extends to a section where the concrete and steel strains are equal. Figure 6.6 shows the typical stress distribution between cracks in a member under axial tension.

Corrosion secondary cracks. *Corrosion secondary cracks* form when moisture containing deleterious agents such as sodium chloride, carbon dioxide, and dissolved oxygen penetrates the concrete surface, corroding the steel reinforcement [11]. The oxide compounds formed by deterioration of steel bars occupy a larger volume than the steel and exert mechanical pressure that perpetuates extensive cracking [12,13]. This type of cracking may be severe enough to result in eventual failure of the structure. The failure of a roof in Muskegan, Michigan, in 1955 due to the corrosion of steel bars was reported by Shermer [13]. The extensive cracking and spalling of concrete in the San Mateo–Hayward Bridge in California within 7 years was reported by Stratful [12]. Corrosion cracking may be forestalled by using proper construction methods and high-quality concrete. More details are discussed by Evans [14] and Mozer and others [15].

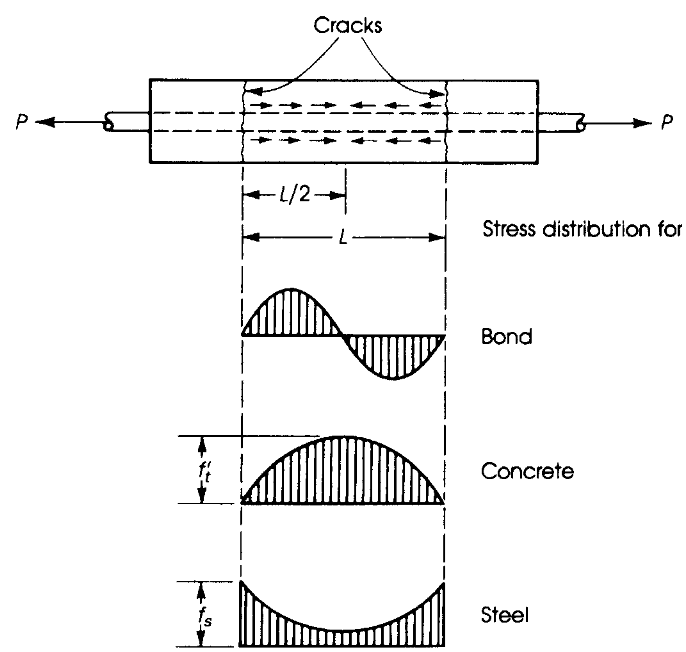


Figure 6.6 Typical stress distribution between cracks.

6.6.2 Main Cracks

Main cracks develop at a later stage than secondary cracks. They are caused by the difference in strains in steel and concrete at the section considered. The behavior of main cracks changes at two different stages. At low tensile stresses in steel bars, the number of cracks increases, whereas the widths of cracks remain small; as tensile stresses are increased, an equilibrium stage is reached. When stresses are further increased, the second stage of cracking develops, and crack widths increase without any significant increase in the number of cracks. Usually one or two cracks start to widen more than the others, forming critical cracks (Fig. 6.7).

Main cracks in beams and axially tensioned members have been studied by many investigators; prediction of the width of cracks and crack control were among the problems studied. These are discussed here, along with the requirements of the ACI Code.

Crack width. *Crack width* and *crack spacing*, according to existing crack theories, depend on many factors, which include steel percentage, its distribution in the concrete section, steel flexural stress at service load, concrete cover, and properties of the concrete constituents. Different equations for predicting the width and spacing of cracks in reinforced concrete members were presented at the Symposium on Bond and Crack Formation in Reinforced Concrete in Stockholm, Sweden, in 1957. Chi and Kirstein [16] presented equations for the crack width and spacing as a function of an effective area of concrete around the steel bar: A concrete circular area of diameter equal to four times the diameter of the bar was used to calculate crack width. Other equations were presented over the next decade [17–23].

Gergely and Lutz [23] presented the following formula for the limiting crack width:

$$W = 0.076\beta f_s \sqrt[3]{Ad_c} \times 10^{-6} \text{ (in.)} \tag{6.15}$$

where β, A, and f_s are as defined previously and d_c = thickness of concrete cover measured from the extreme tension fiber to the center of the closest bar. The value of β can be taken to be approximately equal to 1.2 for beams and 1.35 for slabs. Note that f_s is in psi and W is in inches.

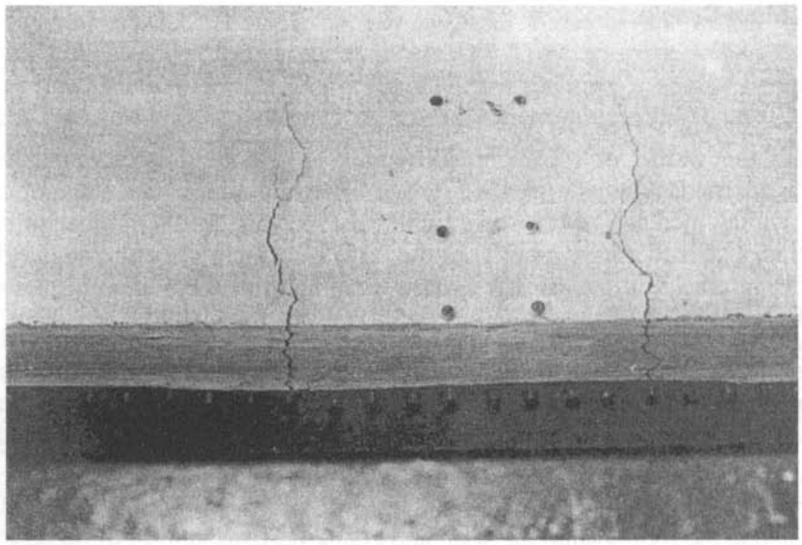
The mean ratio of maximum crack width to average crack width was found to vary between 1.5 and 2.0, as reported by many investigators. An average value of 1.75 may be used.

In SI units (mm and MPa), Eq. 6.15 is

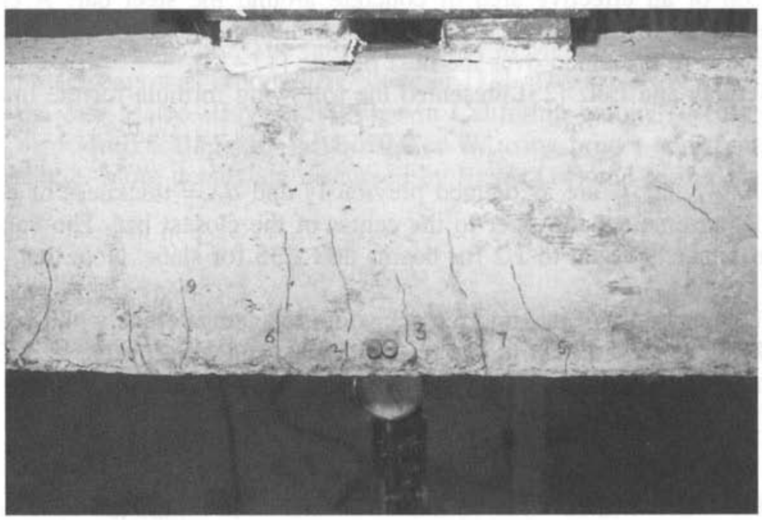
$$W = 11.0\beta f_s \sqrt[3]{Ad_c} \times 10^{-6} \tag{6.16}$$

Tolerable crack width. The formation of cracks in reinforced concrete members is unavoidable. Hairline cracks occur even in carefully designed and constructed structures. Cracks are usually measured at the face of the concrete, but actually they are related to crack width at the steel level, where corrosion is expected. The permissible crack width is also influenced by aesthetic and appearance requirements. The naked eye can detect a crack about 0.006 in. (0.15 mm) wide, depending on the surface texture of concrete. Different values for permissible crack width at the steel level have been suggested by many investigators, ranging from 0.010 to 0.016 in. (0.25–0.40 mm) for interior members and from 0.006 to 0.010 in. (0.15–0.25 mm) for exterior exposed members. A limiting crack width of 0.016 in. (0.40 mm) for interior members and 0.013 in. (0.32 mm) for exterior members under dry conditions can be tolerated.

Crack control. *Control* grows in importance with the use of high-strength steel in reinforced concrete members, as larger cracks develop under working loads because of the high allowable stresses. Control of cracking depends on the permissible crack width: It is always preferable to



(a)



(b)

Figure 6.7 (*a*) Main cracks in a reinforced concrete beam. (*b*) Spacing of cracks in a reinforced concrete beam.

have a large number of fine cracks rather than a small number of large cracks. Secondary cracks are minimized by controlling the total amount of cement paste, water–cement ratio, permeability of aggregate and concrete, rate of curing, shrinkage, and end-restraint conditions.

The factors involved in controlling main cracks are the reinforcement stress, the bond characteristics of reinforcement, the distribution of reinforcement, the diameter of the steel bars used, the steel percentage, the concrete cover, and the properties of concrete constituents. Any improvement in these factors will help in reducing the width of cracks.

6.7 ACI CODE REQUIREMENTS

To control cracks in reinforced concrete members, the ACI Code, Chapter 10, specifies the following:

1. Only deformed bars are permitted as main reinforcement.
2. Tension reinforcement should be well distributed in the zones of maximum tension (Section 10.6.3).
3. When the flange of the section is under tension, part of the main reinforcement should be distributed over the effective flange width or one-tenth of the span, whichever is smaller. Some longitudinal reinforcement has to be provided in the outer portion of the flange (Section 10.6.6).
4. The design yield strength of reinforcement should not exceed 80 ksi (560 MPa) (Section 9.4).
5. The maximum spacing **s** of reinforcement closest to a concrete surface in tension in reinforced concrete beams and one-way slabs is limited to

$$\mathbf{s}\ (\text{in.}) = \left[15\left(\frac{40}{f_s}\right) - 2.5C_c\right] \tag{6.17}$$

but not greater than 12 $(40/f_s)$, where

f_s = calculated stress (ksi) in reinforcement at service load computed as the unfactored moment divided by the product of steel area and the internal moment arm, $f_s = M/(A_s jd)$. (Alternatively, $f_s = \frac{2}{3} f_y$ may be used; an approximate value of $jd = 0.87d$ may be used.)

C_c = clear cover from the nearest surface in tension to the surface of the flexural tension reinforcement (in.).

s = center to center spacing of flexural tension reinforcement nearest to the extreme concrete tension face (in.).

The preceding limitations are applicable to reinforced concrete beams and one-way slabs subject to normal environmental condition and do not apply to structures subjected to aggressive exposure. The spacing limitation just given is independent of the bar size, which may lead to the use of smaller bar sizes to satisfy the spacing criteria. For the case of concrete beams reinforced with grade 60 steel bars and $C_c = 2$ in., clear cover to the tension face, the maximum spacing is calculated as follows: Assume $f_s = 2/3\ f_y = (2/3) \times 60 = 40$ ksi and $\mathbf{s} = 15\left(\frac{40}{40}\right) - 2.5 \times 2 = 10$ in. (controls), which is less than 12(40/40) = 12 in.

6. In SI units, Eq. 6.17 becomes

$$\mathbf{s}\ (\text{mm}) = 105{,}000/f_s - 2.5C_c \tag{6.18}$$

but not greater than 300 $(280/f_s)$, where f_s is in MPa and C_c is in mm. For example, if bars with a clear cover equal to 50 mm are used, then the maximum spacing, **s**, is calculated as follows:

$$\mathbf{s} = (105{,}000/280) - 2.5 \times 50 = 250 \text{ mm (controls)},$$

which is less than 300(280/280) = 300 mm in this example. This is assuming that $f_s = \frac{2}{3} \times 420 = 280$ MPa.

7. In the previous Codes, control of cracking was based on a factor Z defined as follows:

$$Z = f_s\sqrt[3]{Ad_c} \leq 175 \text{ K/in.} \quad (31 \text{ kN/mm}) \quad \text{for interior members}$$

$$Z \leq 140 \text{ K/in.} \ (26 \text{ kN/mm}) \quad \text{for exterior members.} \tag{6.19}$$

where f_s = flexural stress at service load (ksi) and may be taken as 0.6 f_y. A and d_c are the effective tension area of concrete and thickness of concrete cover, respectively. This expression is based on Eq. 6.15 assuming a limiting crack width of 0.016 in. for interior members and 0.013 in. for exterior members. It encouraged a decrease in the reinforcement cover to achieve a smaller Z, while unfortunately it penalized structures with concrete cover that exceeded 2 in.

8. *Skin reinforcement*: For relatively deep girders, with a total depth, h, equal to or greater than 36 in. (900 mm), light reinforcement should be added near the vertical faces in the tension zone to control cracking in the web above the main reinforcement. The ACI Code, Section 10.6.7, referred to this additional steel as skin reinforcement. The skin reinforcement should be uniformly distributed along both side faces of the member for a distance $h/2$ from the tension face.

The spacing S between the longitudinal bars or wires of the skin reinforcement shall be as provided in Eq. 6.17 where C_c is the least distance from the skin reinforcement to the side face.

Referring to Figure 6.8, if b = 16 in., h = 40 in., fy = 60 ksi and choosing no. 3 bars spaced at 6.0 in. as skin reinforcement (3 spaces on each side), then the height covered = 3 × 6 + 2.5 = 20.5 in., which is greater than $h/2$ = 40/2 = 20 in.

Checking the spacing S by Eq. 6.18 and assuming f_s = 2/3, f_y = 2/3 × 60 = 40 ksi, and C_c = 2 in., then S = 15(40/40) − 2.5 × 2 = 10 in., which is less than 12(40/40) = 12 in. The spacing used is adequate. Note that C_c = 1.5 in. may be used for the skin reinforcement concrete cover.

It is recommended to use smaller spacing to control the propagation of tensile cracks along the side of the tension zone with the first side bar to be placed at 4 to 6 in. from the main tensile steel.

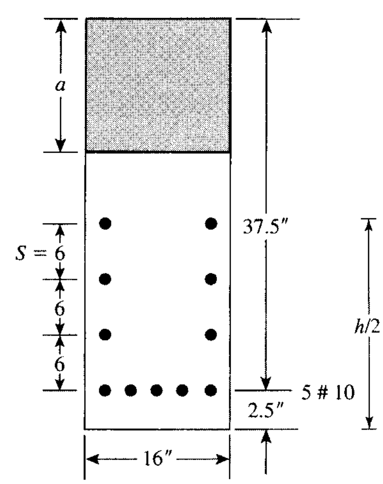


Figure 6.8 Skin reinforcement (6 no. 3 bars).

Example 6.6

The sections of a simply supported beam are shown in Fig. 6.9.

a. Check if the bar arrangement satisfies the ACI Code requirements.
b. Determine the expected crack width.
c. Check the Z-factor based on Eq. 6.19.

Given: $f'_c = 4$ ksi, $f_y = 60$ ksi, and no. 3 stirrups.

Solution

1. Fig. 6.9, section a:

a. For three no. 8 bars, $A_s = 2.35\,\text{in.}^2$, clear cover, C_c, $= 2.5 - 8/16 = 2.0$ in. Assume $f_s = \frac{2}{3} f_y = 2/3 \times 60 = 40$ ksi. Maximum spacing $\mathbf{s} = 600/40 - 2.5 \times 2 = 10$ in., which is less than $12(40/40) = 12$ in. Spacing provided $= 0.5(12 - 2.5 - 2.5) = 3.5$ in., center to center of bars, which is less than 10 in.

b. For this section, $d_c = 2.5$ in. The effective tension area of concrete for one bar is

$$A = 12(2 \times 2.5)/3 = 20\ \text{in.}^2$$

Estimated crack width using Eq. 6.16 is

$$W = 0.076(1.2)(36{,}000)\sqrt[3]{20 \times 2.5} \times 10^{-6} = 0.0121\ \text{in.}$$

This is assuming $\beta = 1.2$ for beams and $f_s = 36$ ksi. The crack width above is less than 0.016 in. and 0.013 in. for interior and exterior members.

2. Fig. 6.9, section b:

a. Calculations of spacing of bars are similar to those in section a.

b. For this section, $d_c = 2.5$ in., and the steel bars are placed in two layers. The centroid of the steel bars is 3.5 in. from the bottom fibers. The effective tension concrete area is $A = 12(2 \times 3.5)/6 = 14\,\text{in.}^2$

$$W = .076 \times 1.2 \times 36{,}000\sqrt[3]{14 \times 2.5} \times 10^{-6} = 0.0107\ \text{in.}$$

which is adequate.

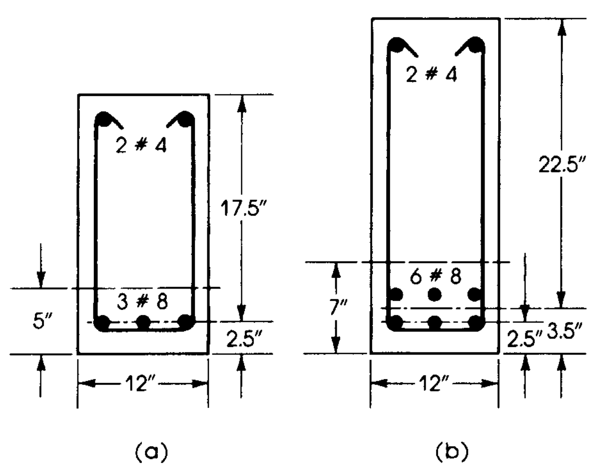


Figure 6.9 Two sections for Example 6.6.

Discussion

It can be seen that the spacing, **s**, in Eq. 6.17 is a function of the stress in the tension bars or, indirectly, is a function of the strain in the tension steel, $f_s = E_s \times \varepsilon_s$, and E_s for steel is equal to 29,000 ksi. The spacing also depends on the concrete cover, C_c. An increase in the concrete cover will reduce the limited spacing **s**, which is independent on the bar size used in the section.

In this example, the expected crack width was calculated by Eq. 6.17 to give the student or the engineer a physical feeling for the crack width and crack control requirement. The crack width is usually measured in beams when tested in the laboratory or else in actual structures under loading when serious cracks develop in beams or slabs and testing is needed. If the crack width measured before and after loading is greater than the yield strain of steel, then the main reinforcement is in the plastic range and ineffective. Sheets with lines of different thickness representing crack widths are available in the market for easy comparisons with actual crack widths. In addition to the steel stress and the concrete cover, W depends on a third factor, A, representing the tension area of concrete surrounding one bar in tension.

Example 6.7

Design a simply supported beam with a span of 24 ft to carry a uniform dead load of 1.5 K/ft and a live load of 1.18 K/ft. Choose adequate bars; then check their spacing arrangement to satisfy the ACI Code. Given: $b = 16$ in., $f_c' = 4$ ksi, $f_y = 60$ ksi, a steel percentage = 0.8%, and a clear concrete cover of 2 in.

Solution

1. For a steel percentage of 0.8%, $R_u = 400$ psi $= 0.4$ ksi ($\phi = 0.9$). The external factored moment is $M_u = w_u \times L^2/8$, and $w_u = 1.2(1.5) + 1.6(1.18) = 3.69$ K/ft.

$$M_u = 3.69(24)^2/8 = 265.68 \text{ K·ft} = 3188.2 \text{ K·in.}$$

$$M_u = R_u \cdot bd^2 \qquad d = 22.32 \qquad A_s = 0.008 \times 16 \times 22.32 = 2.86 \text{ in.}^2$$

Choose three no. 9 bars (area = 3.0 in.^{2}) in one row, and a total depth of $h = 25.0$ in. Actual $d = 25 - 2 - 9/16 = 22.44$ in. (Fig. 6.10).

2. Check spacing of bars using Eq. 6.18. Calculate the service load and moment: $w = 1.5 + 1.18 = 2.68$ K/ft.

$$M = 2.68(24)^2/8 = 193 \text{ K·ft} = 2315 \text{ K·in.}$$

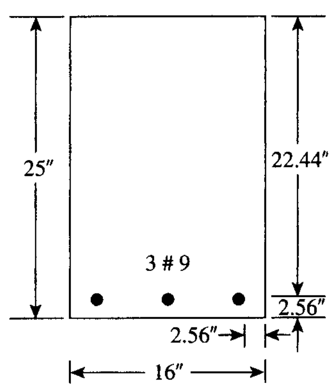


Figure 6.10 Example 6.7.

3. Calculate the neutral axis depth kd and the moment arm jd (Eq. 6.12).

$$b(kd)^2/2 - nA_s(d - kd) = 0 \qquad n = 8 \qquad A_s = 3.0 \qquad d = 22.44 \text{ in.}$$

$$kd = 6.85 \text{ in.} \qquad jd = d - kd/3 = 20.16 \text{ in.} \qquad j = 20.16/22.44 = 0.898$$

Note that an approximate value of $j = 0.87$ may be used if kd is not calculated.

4. Calculate the stress f_s:

$$M = A_s \cdot f_s \cdot jd \quad 2315 = 3(f_s)(20.16) \quad f_s = 38.3 \text{ ksi}$$

5. Calculate the spacing s by Eq. 6.18:

$$s = 600/38.3 - 2.5 \times 2 = 10.7 \text{ in. (controls)}$$

which is less than 12(40/40) = 12.0 in. Spacing provided, = 0.5 (16 − 2.56 − 2.56) = 5.44 in., which is less than 10.7 in.

Example 6.8: SI Units

Design a simply supported beam of 7.2-m span to carry a uniform dead load of 22.2 kN/m and a live load of 17 kN/m. Choose adequate bars, and check their spacing arrangement to satisfy the ACI Code.

Given: $b = 400$ mm, $f'_c = 30$ MPa, $f_y = 400$ MPa, a steel percentage of 0.8%, and a clear concrete cover of 50 mm.

Solution

1. For a steel percentage of 0.008 and from Eq. 3.22, $R_u = 2.7$ MPa. Factored load $w_u = 1.2(22.2) + 1.6(17) = 53.8$ kN/m. $M_u = w_u \cdot L^2/8 = 53.8(7.2)^2/8 = 348.6$ kN·m. $M_u = R_u \cdot bd^2$, or $348.6 \times 10^6 = 2.7 \times 400d^2$ then $d = 568$ mm $A_s = \rho bd = 0.008 \times 400 \times 568 = 1818\text{ mm}^2$. Choose four bars, 25 mm (no. 25 M), $A_s = 2040\text{ mm}^2$, in one row ($b_{\min} = 220$ mm). Let $h = 650$ mm, the actual $d = 650 - 50 - 25/2 = 587.5$ mm, say 585 mm. Final section: $b = 400$ mm, $h = 650$ mm, with four no. 25 mm bars (Fig. 6.11).
2. Check spacing of bars using Eq. 6.17. Calculate the service load moment, $w = 22.2 + 17 = 39.2$ kN/m.

$$M = 39.2(7.2)^2/8 = 254 \text{ kN·m}$$

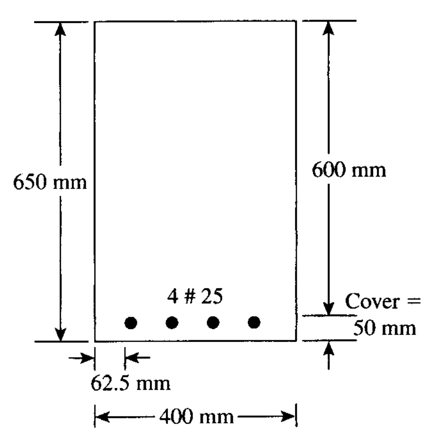


Figure 6.11 Example 6.8.

Calculate kd and jd as in the previous example. Alternatively, use a moment arm, $jd = 0.87d = 0.87(585) = 509$ mm and $f_s = M/(A_s \cdot jd) = 254(10)^6/(2040 \times 585) = 213$ MPa. From Eq. 6.19, maximum s = (105,000/213) − 2.5(50) = 368 mm (controls), which is less than $300(280/f_s) = 300(280)/213 = 394$ mm. Note that if $f_s = 0.6\ f_y = 0.6(400) = 240$ MPa is used, then maximum s = 312 mm. It is preferable to calculate f_s from the moment equation to reflect the actual stress in the bars. Spacing provided = (1/3)(400 − 50 − 25) = 92 mm, which is adequate.

SUMMARY

Sections 6.1–6.2

1. Deflection $\Delta = \alpha(WL^3/EI) = 5WL^3/384EI = 5\ wL^4/384EI$ for a simply supported beam subjected to a uniform total load of $W = wL$.

$$E_c = 33w^{1.5}\sqrt{f_c'} = 57{,}400\ f_c'\ \text{psi}$$

   for normal-weight concrete.

2. Effective moment of inertia is

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right] I_{cr} \le I_g$$

$$M_{cr} = f_r \times \frac{I_g}{y_t} \quad \text{and} \quad f_r = 7.5\lambda\sqrt{f_c'} \tag{6.5}$$

Section 6.3

The deflection of reinforced concrete members continues to increase under sustained load.

Additional long-time deflection = ζ_Δ × instantaneous deflection:

$$\zeta_\Delta = \frac{\zeta}{1 + 50\rho'} \tag{6.14}$$

ζ = 1.0, 1.2, 1.4, and 2.0 for periods of 3, 6, 12, and 60 months, respectively.

Sections 6.4–6.5

1. The allowable deflection varies between $L/180$ and $L/480$.
2. Deflections for different types of loads may be calculated for each type of loading separately and then added algebraically to obtain the total deflection.

Section 6.6

1. Cracks are classified as secondary cracks (shrinkage, corrosion, or secondary flexural cracks) and main cracks.
2. Maximum crack width is

$$W = 0.076\beta f_s \sqrt[3]{Ad_c} \times 10^{-6}\ \text{(in.)} \tag{6.15}$$

Approximate values for β, f_s, and d_c are $\beta = 1.2$ for beams and 1.35 for slabs, $d_c = 2.5$ in., and $f_s = (2/3)f_y$.

3. The limiting crack width is 0.016 in. for interior members and 0.013 in. for exterior members.

Section 6.7

The maximum spacing s of bars closest to a concrete surface in tension is limited to

$$s = 600/f_s - 2.5C_c \tag{6.17}$$

but not more than $12(40/f_s)$. Note that f_s may be taken as $2/3 f_y$.

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PROBLEMS

6.1 Determine the instantaneous and long-time deflection of a 20-ft-span simply supported beam for each of the following load conditions. Assume that 10% of the live loads are sustained and the dead loads include the self-weight of the beams. Use $f'_c = 4$ ksi, $f_y = 60$ ksi, $d' = 2.5$ in., and a time limit of 5 years. Refer to Fig. 6.12.

No.	*b* (in.)	*d* (in.)	*h* (in.)	A_s (in.^{2})	A'_s (in.^{2})	W_D (K/ft)	W_L (K/ft)	P_D (K)	P_L (K)
a	14	17.5	20	5 no. 9	—	2.2	1.8	—	—
b	20	27.5	30	6 no. 10	—	7.0	3.6	—	—
c	12	19.5	23	6 no. 8	—	3.0	1.5	—	—
d	18	20.5	24	6 no. 10	2 no. 9	6.0	2.0	—	—
e	16	22.5	26	6 no. 11	2 no. 10	5.0	3.2	12	10
f	14	20.5	24	8 no. 9	2 no. 9	3.8	2.8	8	6

h-d = 2.5 in. indicates one row of bars, whereas h-d = 3.5 in. indicates two rows of bars. Concentrated loads are placed at midspan.

6.2 Determine the instantaneous and long-term deflection of the free end of a 12-ft-span cantilever beam for each of the following load conditions. Assume that only dead loads are sustained, and the dead loads include the self-weight of the beams. Use $f'_c = 4$ ksi, $f_y = 60$ ksi, and a time limit of more than 5 years. Refer to Fig. 6.13.

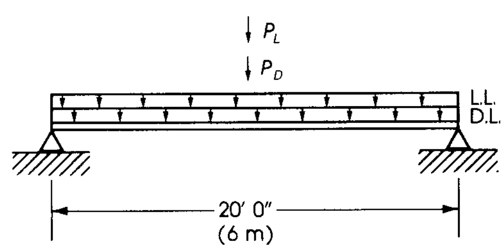


Figure 6.12 Problem 6.1.

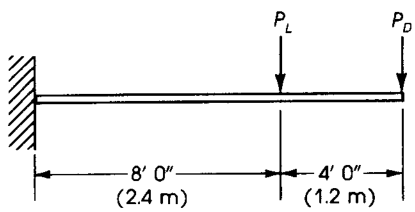


Figure 6.13 Problem 6.2.

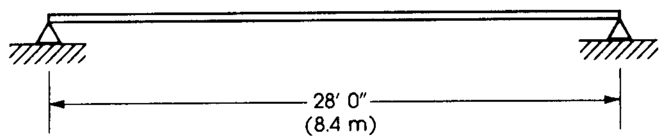


Figure 6.14 Problem 6.3: Dead load = 2 K/ft (30 kN/m) and live load = 1.33 K/ft (20 kN/m).

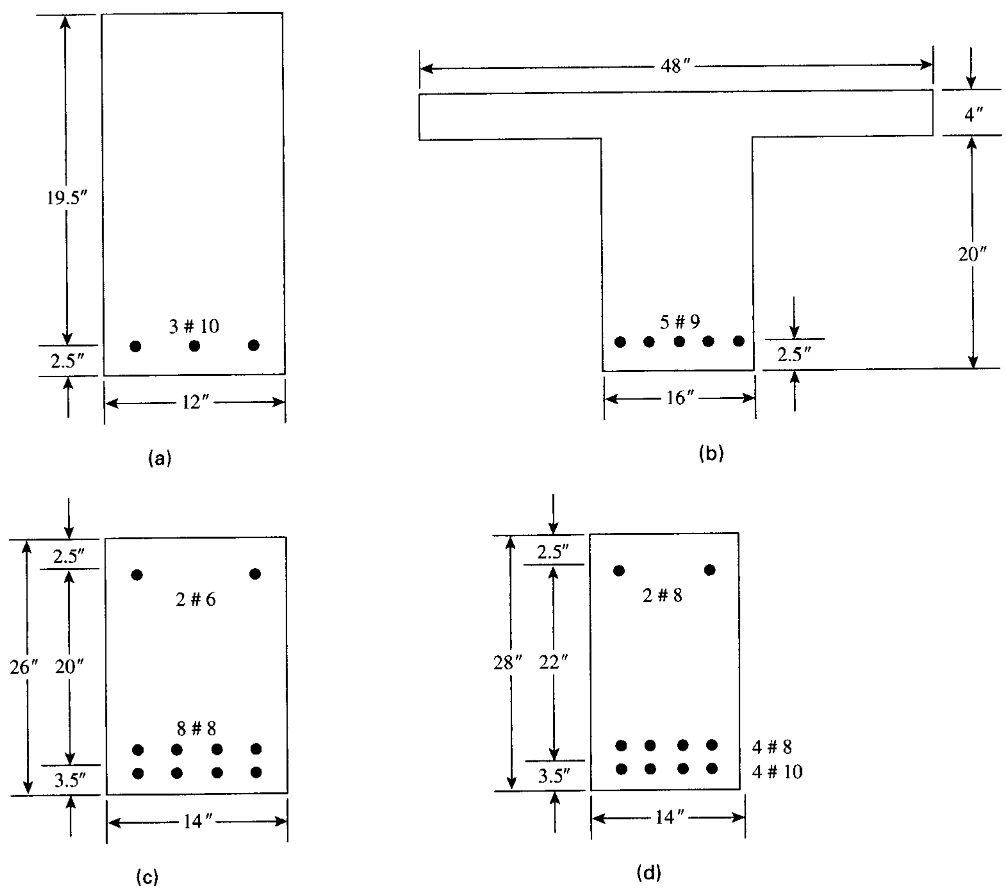


Figure 6.15 Problem 6.5.

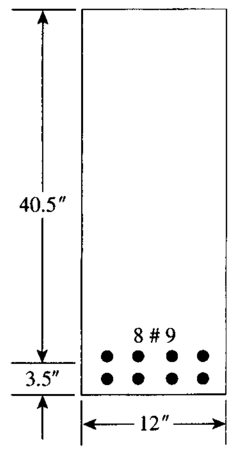


Figure 6.16 Problem 6.6 (skin reinforcement).

No.	*b* (in.)	*d* (in.)	*h* (in.)	A_s (in.^{2})	A'_s (in.^{2})	W_D (K/ft)	W_L (K/ft)	P_D (K)	P_L (K)
a	15	20.5	24	8 no. 9	2 no. 9	3.5	2.0	—	—
b	18	22.5	26	6 no. 10	—	2.0	1.5	7.4	5.0
c	12	19.5	23	8 no. 8	2 no. 8	2.4	1.6	—	—
d	14	20.5	24	8 no. 9	2 no. 9	3.0	1.1	5.5	4.0

h-d = 2.5 in. indicates one row of bars, whereas h-d = 3.5 in. indicates two rows of bars. Concentrated loads are placed as shown

6.3 A 28-ft simply supported beam carries a uniform dead load of 2 K/ft (including self-weight) and a live load of 1.33 K/ft. Design the critical section at midspan using the maximum steel ratio allowed by the ACI Code and then calculate the instantaneous deflection. Use $f'_c = 4$ ksi, $f_y = 60$ ksi, and $b = 12$ in. See Fig. 6.14.

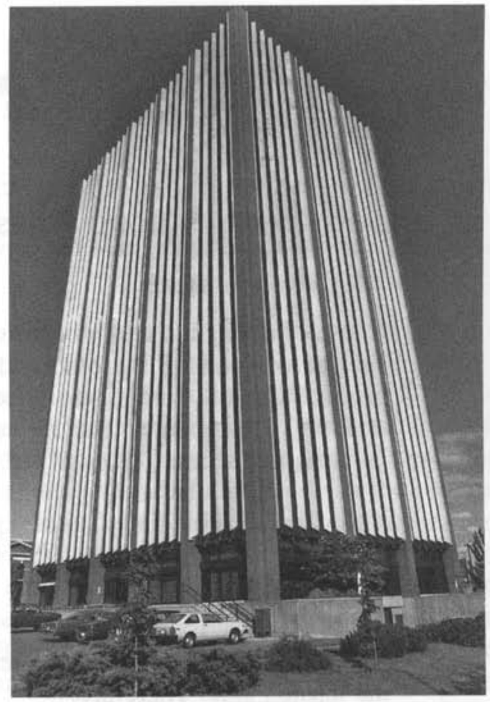
6.4 Design the beam in Problem 6.3asdoubly reinforced, considering that compression steel resists 20% of the maximum bending moment. Then calculate the maximum instantaneous deflection.

6.5 The four cross-sections shown in Fig. 6.15 belong to four different beams with $f'_c = 4$ ksi and $f_y = 60$ ksi. Check the spacing of the bars in each section according to the ACI Code requirement using $f_s = 0.6 f_y$. Then calculate the tolerable crack width, W.

6.6 Determine the necessary skin reinforcement for the beam section shown in Fig. 6.16. Then choose adequate bars and spacings. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

CHAPTER 7

DEVELOPMENT LENGTH OF REINFORCING BARS



Reinforced concrete columns supporting an office building, Toronto, Canada.

7.1 INTRODUCTION

The joint behavior of steel and concrete in a reinforced concrete member is based on the fact that a bond is maintained between the two materials after the concrete hardens. If a straight bar of round section is embedded in concrete, a considerable force is required to pull the bar out of the concrete. If the embedded length of the bar is long enough, the steel bar may yield, leaving some length of the bar in the concrete. The bonding force depends on the friction between steel and concrete. It is influenced mainly by the roughness of the steel surface area, the concrete mix, shrinkage, and the cover of concrete. Deformed bars give a better bond than plain bars. Rich mixes have greater adhesion than weak mixes. An increase in the concrete cover will improve the ultimate bond stress of a steel bar [2].

In general, the bond strength is influenced by the following factors:

1. Yield strength of reinforcing bars, f_y. Longer development length is needed with higher f_y.
2. Quality of concrete and its compressive strength, f'_c. An increase in f'_c reduces the required development length of reinforcing bars.
3. Bar size, spacing, and location in the concrete section. Horizontal bars placed with more than 12 in. of concrete below them have lower bond strength due to the fact that concrete shrinks and settles during the hardening process. Also, wide spacings of bars improve the bond strength, giving adequate effective concrete area around each bar.
4. Concrete cover to reinforcing bars. A small cover may cause the cracking and spalling of the concrete cover.

5. Confinement of bars by lateral ties. Adequate confinement by ties or stirrups prevents the spalling of concrete around bars.

7.2 DEVELOPMENT OF BOND STRESSES

7.2.1 Flexural Bond

Consider a length dx of a beam subjected to uniform loading. Let the moment produced on one side be M_1 and on the other side be M_2 with M_1 being greater than M_2. The moments will produce internal compression and tension forces, as shown in Fig. 7.1. Because M_1 is greater than M_2, T_1 is greater than T_2; consequently, C_1 is greater than C_2.

At any section, $T = M/jd$, where jd is the moment arm:

$$T_1 - T_2 = dT = \frac{dM}{jd}$$

but

$$T_1 = T_2 + u\Sigma O dx$$

where u is the average bond stress and ΣO is the sum of perimeters of bars in the section at the tension side. Therefore,

$$T_1 - T_2 = u\Sigma O dx = \frac{dM}{jd}$$

$$u = \frac{dM}{dx} \times \frac{1}{jd\Sigma O}$$

The rate of change of the moment with respect to x is the shear, or $dM/dx = V$. Therefore,

$$u = \frac{V}{jd\Sigma O} \tag{7.1}$$

The value u is the average bond stress; for practical calculations, j can be taken to be approximately equal to 0.87:

$$u = \frac{V}{0.87\, d\Sigma O}$$

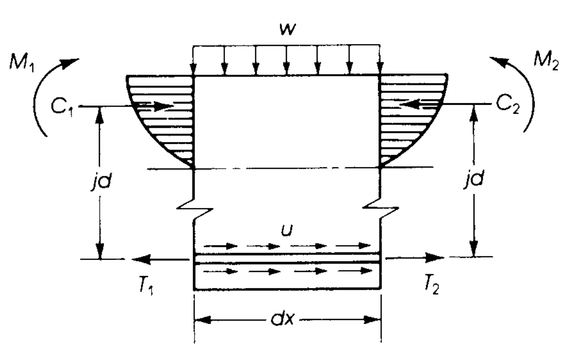


Figure 7.1 Flexural bond.

In the strength design method, the nominal bond strength is reduced by the capacity reduction factor, $\phi = 0.85$. Thus,

$$U_u = \frac{V_u}{\phi(0.87)d\Sigma O} \tag{7.2}$$

Based on the preceding analysis, the bond stress is developed along the surface of the reinforcing bar due to shear stresses and shear interlock.

7.2.2 Tests for Bond Efficiency

Tests to determine the bond stress capacity can be made using the pullout test (Fig. 7.2). This test evaluates the bond capacity of various types of bar surfaces relative to a specific embedded length. The distribution of tensile stresses will be uniform around the reinforcing bar at a specific section and varies along the anchorage length of the bar and at a radial distance from the surface of the bar (Fig. 7.2). However, this test does not represent the effective bond behavior in the surface of the bars in flexural members, because stresses vary along the depth of the concrete section. A second type of test can be performed on an embedded rod (Fig. 7.3). In these tests, the tensile force, P, is increased gradually and the number of cracks and their spacings and widths are recorded. The bond stresses vary along the bar length between the cracks. The strain in the steel bar is maximum at the cracked section and decreases toward the middle section between cracks.

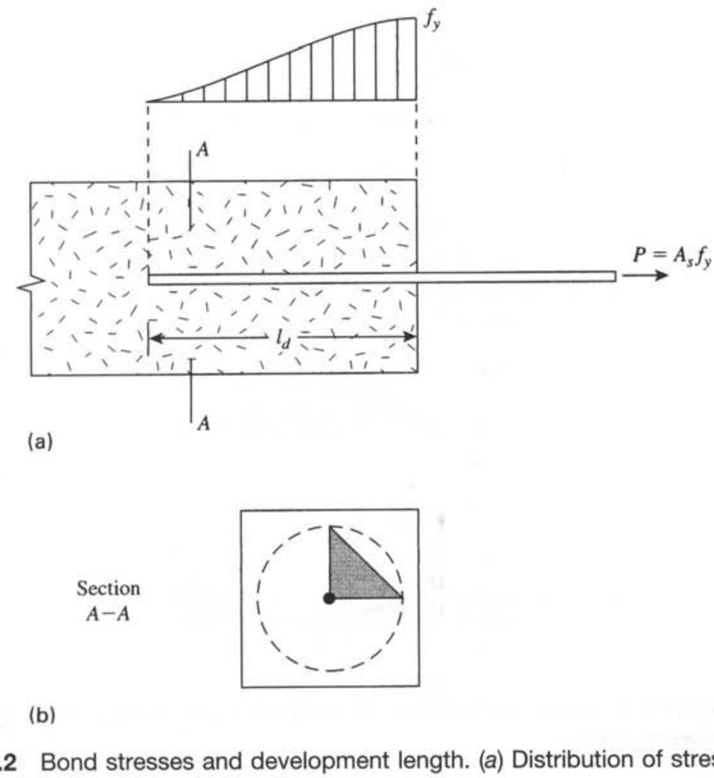


Figure 7.2 Bond stresses and development length. (*a*) Distribution of stress along l_d and (*b*) radial stress in concrete around the bar.

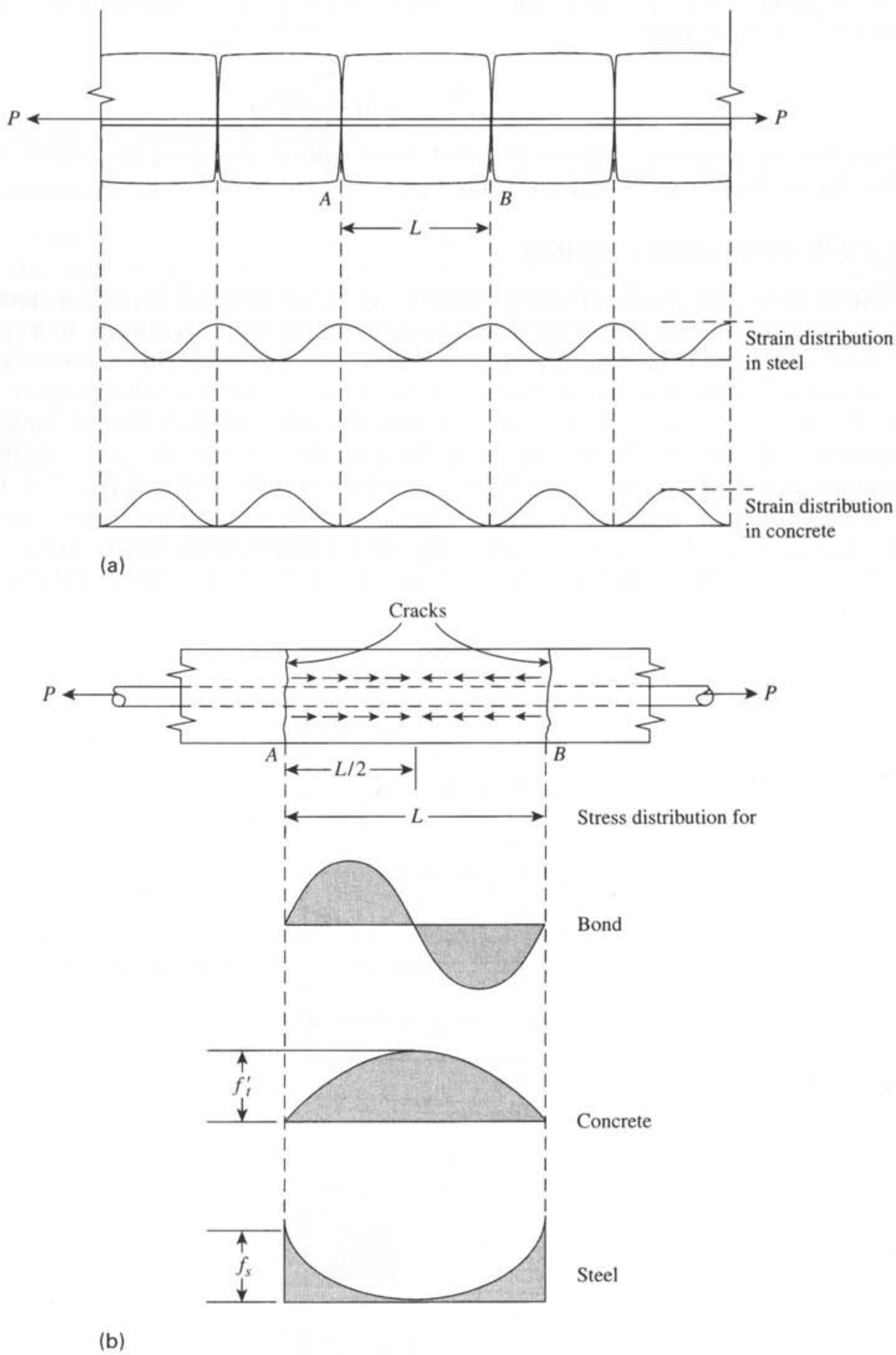


Figure 7.3 Bond mechanism in an embedded bar. Strain (*a*) and stress (*b*) distribution between cracks.

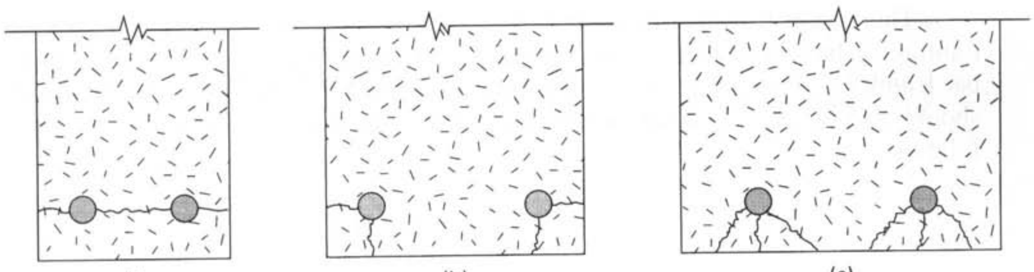


Figure 7.4 Examples of spalling of concrete cover. (*a*) High bottom cover, (*b*) wide spacing, and (*c*) small bottom cover.

Tests on flexural members are also performed to study the bond effectiveness along the surface of the tension bars. The analysis of bond stresses in the bars of these members was explained earlier, and they are represented by Eq. 7.2.

Based on this discussion, it is important to choose an appropriate length in each reinforcing bar to develop its full yield strength without a failure in the bond strength. This length is called the *development length*, l_d. If this length is not provided, the bond stresses in the tension zone of a beam become high enough to cause cracking and splitting in the concrete cover around the tension bars (Fig. 7.4). If the split continues to the end of the bar, the beam will eventually fail. Note that small spacings between tensile bars and a small concrete cover on the sides and bottom will reduce the bond capacity of the reinforcing bars (Fig. 7.4).

7.3 DEVELOPMENT LENGTH IN TENSION

7.3.1 Development Length, l_d

If a steel bar is embedded in concrete, as shown in Fig. 7.2, and is subjected to a tension force T, then this force will be resisted by the bond stress between the steel bar and the concrete. The maximum tension force is equal to $A_s f_y$, where A_s is the area of the steel bar. This force is resisted by another internal force of magnitude $U_u O l_d$, where U_u is the ultimate average bond stress, l_d is the embedded length of the bar, and O is the perimeter of the bar (πD). The two forces must be equal for equilibrium:

$$A_s f_y = U_u O l_d \quad \text{and} \quad l_d = \frac{A_s f_y}{U_u O}$$

For a combination of bars,

$$l_d = \frac{A_s f_y}{U_u \Sigma O} \tag{7.3}$$

The length l_d is the minimum permissible anchorage length and is called the development length.

$$l_d = \frac{\pi d_b^2 f_y}{4U_u(\pi d_b)} = \frac{d_b f_y}{4U_u} \tag{7.4}$$

where d_b = diameter of reinforcing bars.

This means that the development length is a function of the size and yield strength of the reinforcing bars in addition to the ultimate bond stress, which in turn is a function of $\sqrt{f'_c}$. The bar length l_d given in Eq. 7.4 is called the *development length*, l_d. The final development length should also include the other factors mentioned in Section 7.1. Equation 7.4 may be written as follows:

$$\frac{l_d}{d_b} = K\left(\frac{f_y}{\sqrt{f'_c}}\right) \tag{7.5}$$

where K is a general factor that can be obtained from tests to include factors such as the bar characteristics (bar size, spacing, epoxy coated or uncoated, location in concrete section, and bar splicing), amount of transverse reinforcement, and the provision of excess reinforcement compared to that required from design.

The ACI Code, Section 12.2.3, evaluated K as follows:

$$K = \left(\frac{3}{40\lambda}\right)\frac{\psi_t\psi_e\psi_s}{\dfrac{(c_b + K_{tr})}{d_b}} \tag{7.6}$$

and Eq. 7.5 becomes

$$\frac{l_d}{d_b} = \frac{3}{40\lambda}\frac{f_y}{\sqrt{f'_c}}\frac{\psi_t\psi_e\psi_s}{\left(\dfrac{c_b + K_{tr}}{d_b}\right)} \tag{7.7}$$

where

ψ_t = bar location
ψ_e = coating factor
ψ_s = bar-size factor
λ = lightweight aggregate concrete factor (ACI Code, Section 8.6.1)
= 1.0 normal-weight concrete
= shall not exceed 0.75 unless splitting tensile strength is specified, then
$\lambda = f_{ct}/(6.7\sqrt{f'_c}) \le 1$
c_b = spacing or cover dimension (in.), whichever is smaller
K_{tr} = transverse reinforcement index
= $40A_{tr}/sn$
n = number of bars or wires being developed along the plane of splitting
s = maximum spacing of transverse reinforcement within l_d, center to center (in.).
f_{yt} = yield strength of transverse reinforcement (psi)
A_{tr} = total sectional area of all transverse reinforcement within spacing s that crosses the potential plane of splitting through to the reinforcement being developed (in.^{2})

Notes:

1. $(c_b + K_{tr})/d_b$ shall not exceed 2.5 to safeguard against pullout-type failures.
2. The value of $\sqrt{f'_c}$ shall not exceed 100 psi (ACI Code, Section 12.1.2).
3. $K_{tr} = 0$ can be used as a design simplification (ACI Code, Section 12.2.3).

7.3.2 ACI Code Factors for Calculating l_d for Bars in Tension

1. ψ_t = bar location factor
   ψ_t = 1.3 for top bars defined as horizontal reinforcement, placed so that more than 12 in. of fresh concrete is below the development length, or splice
   ψ_t = 1.0 for all other reinforcement
2. ψ_e = coating factor
   ψ_e = 1.5 for epoxy-coated bars or wires with cover less than $3d_b$ or clear spacing less than $6d_b$
   ψ_e = 1.2 for all other epoxy coated bars or wires
   ψ_e = 1.0 for uncoated and zinc-coated (galvanized) reinforcement (However, the value of the product $\psi_t \psi_e$ should not exceed 1.7)
3. ψ_s = bar size factor
   ψ_s = 0.8 for no. 6 bars or smaller bars and deformed wires
   ψ_s = 1.0 for no. 7 bars and larger bars
4. λ = lightweight aggregate concrete factor
   λ = λ shall not exceed 0.75 unless f_{ct} is specified
   λ = 1.0 for normal-weight concrete
5. The ACI Code permits using $K_{tr} = 0$ even if transverse reinforcement is present. In this case,

$$\frac{l_d}{d_b} = \left(\frac{3}{40\lambda}\right)\left(\frac{f_y}{\sqrt{f'_c}}\right)\left(\frac{\psi_t\psi_e\psi_s}{(c_b/d_b}\right) \tag{7.7a}$$

   The value of $\sqrt{f'_c}$ should not exceed 100 psi.
6. R_s is the reduction factor due to excess reinforcement. The ACI Code, Section 12.2.5, permits the reduction of l_d by the factor R_s when the reinforcement in a flexural member exceeds that required by analysis, except where anchorage or development for f_y is specifically required or the reinforcement is designed considering seismic effects.

$$R_s = \frac{A_s \text{ (required)}}{A_s \text{ (provided)}}$$

7. The development length, l_d, in all cases shall not be less than 12 in.

7.3.3 Simplified Expressions for l_d

The ACI Code, Section 12.2.2, permits the use of simplified expressions to calculate the ratio l_d/d_b. This is based on the fact that current practical construction cases utilize spacing and cover values along with confining reinforcement, such as stirrups and ties, that produce a value of $(c_b + K_{tr})/d_b \geq 1.5$. Moreover, tests indicated that the development length, l_d, can be reduced by 20% for no. 6 and smaller bars. Based on these assumptions and assuming $(c_b + K_{tr})/d_b = 1.5$, Eq. 7.7 can be reduced to the following expressions:

1. For no. 7 and larger bars,

$$\frac{l_d}{d_b} = \left(\frac{f_y}{\sqrt{f'_c}}\right)\frac{\psi_t\psi_e}{20\lambda} \tag{7.8}$$

   For no. 6 and smaller bars and deformed wires,

$$\frac{l_d}{d_b} = \left(\frac{f_y}{\sqrt{f'_c}}\right)\frac{\psi_t\psi_e}{25\lambda} \tag{7.9}$$

The ratio l_d/d_b in Eq. 7.9 represents 80% of that in Eq. 7.8. These equations are used when one of the following conditions is met:

a. Clear spacing of bars or wires being developed or spliced not less than d_b, clear cover not less than d_b, and stirrups or ties throughout l_d not less than the code minimum.

b. Clear spacing of bars or wires being developed or spliced not less than $2d_b$ and clear cover not less than d_b.

2. For all other cases, the value of l_d/d_b in Eqs. 7.8 and 7.9 must be multiplied by 1.5 to restore them to equivalence with Eq. 7.7.

These equations are relatively simple to use for the general conditions involved in practical design and construction. For example, in all structures with normal-weight concrete ($\psi_t = 1.0$), uncoated reinforcement ($\psi_e = 1.0$), no. 7 *or larger* bars ($\psi_s = 1.0$), Eq. 7.8 becomes

$$\frac{l_d}{d_b} = \frac{f_y}{(20\lambda\sqrt{f'_c})} \tag{7.10}$$

This equation is used when conditions a and b are met, whereas for all other cases, l_d/d_b is multiplied by 1.5, or

$$\frac{l_d}{d_b} = \frac{3f_y}{(40\lambda\sqrt{f'_c})} \tag{7.11}$$

Similarly, for the same conditions and for no. 6 *or smaller* bars, Eq. 7.9 becomes

$$\frac{l_d}{d_b} = \frac{f_y}{(25\lambda\sqrt{f'_c})} \tag{7.12}$$

This is used when conditions a and b are met; for all other cases, l_d/d_b is multiplied by 1.5, or

$$\frac{l_d}{d_b} = \frac{3f_y}{(50\lambda\sqrt{f'_c})} \tag{7.13}$$

It is quite common to use $f'_c = 4$ ksi and $f_y = 60$ ksi in the design and construction of reinforced concrete buildings. If these values are substituted in the preceding equations, and assuming normal-weight concrete ($\lambda = 1.0$) then

Equation 7.10 becomes	$l_d = 47.5d_b$	($\geq$ no. 7 bars).	(7.10a)
Equation 7.11 becomes	$l_d = 71.2d_b$	($\geq$ no. 7 bars).	(7.11a)
Equation 7.12 becomes	$l_d = 38d_b$	($\leq$ no. 6 bars).	(7.12a)
Equation 7.13 becomes	$l_d = 57d_b$	($\leq$ no. 6 bars).	(7.13a)

Other values of l_d/d_b ratios are shown in Table 7.1. Table 7.2 gives the development length, l_d, for different reinforcing bars (when $f_y = 60$ ksi and $f'_c = 3$ ksi and 4 ksi) for both cases, when conditions a and b are met and for all other cases.

7.4 DEVELOPMENT LENGTH IN COMPRESSION

The development length of deformed bars in compression is generally smaller than that required for tension bars, due to the fact that compression bars do not have the cracks that develop in tension concrete members that cause a reduction in the bond between bars and the surrounding

Table 7.1 Values of l_d/d_b for Various Values of f'_c and f_y (Tension Bars), ($\lambda = 1.0$)

	$f_y = 40$ ksi				$f_y = 60$ ksi			
	No. 6 Bars		≥ No. 7 Bars		No. 6 Bars		≥ No. 7 Bars	
f'_c (ksi)	Conditions met	Other cases	Conditions met	Other cases	Conditions met	Other cases	Conditions met	Other cases
3	29.3	43.9	36.6	54.8	43.9	65.8	54.8	82.2
4	25.3	38.0	31.7	47.5	38.0	57.0	47.5	71.2
5	22.7	34.0	28.3	42.5	34.0	51.0	42.5	63.7
6	20.7	31.0	25.9	38.8	31.0	46.5	38.8	58.1

Table 7.2 Development Length l_d (in.) for Tension Bars and $f_y = 60$ ksi ($\psi_t = \psi_e = \lambda = 1.0$)

		Development Length l_d (in.) — Tension Bars			
		$f'_c = 3$ksi		$f'_c = 4$ ksi	
Bar number	Bar diameter (in.)	Conditions met	Other cases	Conditions met	Other cases
3	0.375	17	25	15	21
4	0.500	22	33	19	29
5	0.625	28	41	24	36
6	0.750	33	50	29	43
7	0.875	48	72	42	63
8	1.000	55	83	48	72
9	1.128	62	93	54	81
10	1.270	70	105	61	92
11	1.410	78	116	68	102

concrete. The ACI Code, Section 12.3.2, gives the basic development length in compression for all bars as follows:

$$l_{dc} = \frac{0.02 d_b f_y}{\lambda \sqrt{f'_c}} \geq 0.0003 d_b f_y \tag{7.14}$$

which must not be less than 8 in. The development length, l_{dc}, may be reduced by multiplying l_{dc} by $R_s = (A_s \text{ required})/(A_s \text{ provided})$. For spirally reinforced concrete compression members with spirals of not less than $\frac{1}{4}$ in. diameter and a spacing of 4 in. or less, the value of l_{dc} in Eq. 7.14 may be multiplied by $R_{sl} = 0.75$. In general, $l_d = l_{dc} \times (R_s \text{ or } R_{sl}$, if applicable) ≥ 8 in. Tables 7.3 and 7.4 give the values of l_{dc}/d_b when $f_y = 60$ ksi.

Table 7.3 Values of l_d/d_b for Various Values of f'_c and f_y (Compression Bars), $\lambda = 1.0$, Minimum $l_{dc} =$ 8 in. $l_{dc}/d_b = 0.02 f_y/\lambda\sqrt{f'_c} \geq 0.0003 f_y$

f'_c(ksi)	3	4	5 or more
$f_y = 40$ ksi	15	13	12
$f_y = 60$ ksi	22	19	18

Table 7.4 Development Length, l_{dc} (in.), for Compression Bars (f_y = 60 ksi), $\lambda = 1.0$

Bar number	Bar diameter (in.)	Development Length, l_{dc} (in.) when f'_c = 3 (ksi)	4 (ksi)	5 (ksi) or more
3	0.375	9	8	8
4	0.500	11	10	9
5	0.625	14	12	12
6	0.750	17	15	14
7	0.875	20	17	16
8	1.000	22	19	18
9	1.128	25	22	21
10	1.270	28	25	23
11	1.410	31	27	26

7.5 SUMMARY FOR THE COMPUTATION OF l_d IN TENSION

Assuming normal construction practices, $(c_b + K_{tr})/d_b = 1.5$.

1. If one of the following two conditions is met:
   a. Clear spacing of bars $\geq d_b$, clear cover $\geq d_b$, and bars are confined with stirrups not less than the code minimum.
   b. Clear spacing of bars $\geq 2d_b$ and clear cover $\geq d_b$; then

$$\text{for no. 7 and larger bars, } \frac{l_d}{d_b} = \frac{\psi_t \psi_e f_y}{20\lambda\sqrt{f'_c}} \tag{7.8}$$

$$\text{for no. 6 or smaller bars, } \frac{l_d}{d_b} = \frac{\psi_t \psi_e f_y}{25\lambda\sqrt{f'_c}} \tag{7.9}$$

2. For all other cases, multiply these ratios by 1.5.
3. Note that $f'_c \leq 100$ psi and $\psi_t \psi_e \leq 1.7$; values of ψ_t, ψ_e, and λ are as explained earlier.
4. For bundled bars, either in tension or compression, l_d should be increased by 20% for three-bar bundles and by 33% for four-bar bundles. A unit of bundled bars is considered a single bar of a diameter and area equivalent to the total area of all bars in the bundle. This equivalent diameter is used to check spacings and concrete cover.

Example 7.1

Figure 7.5 shows the cross-section of a simply supported beam reinforced with four no. 8 bars that are confined with no. 3 stirrups spaced at 6 in. Determine the development length of the bars if the beam is made of normal-weight concrete, bars are not coated, $f'_c = 3$ ksi, and $f_y = 60$ ksi.

Solution

1. Check if conditions for spacing and concrete cover are met:
   a. For no. 8 bars, $d_b = 1.0$ in.
   b. Clear cover $= 2.5 - 0.5 = 2.0$ in. $> d_b$

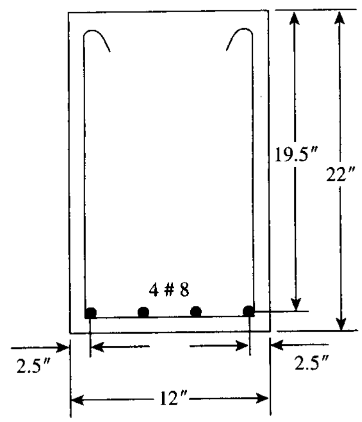


Figure 7.5 Example 7.1.

c. Clear spacing between bars $= \dfrac{12-5}{3} - 1.0 = 1.33$ in. $> d_b$

d. Bars are confined with no. 3 stirrup. The conditions are met. Then

$$\frac{l_d}{d_b} = \frac{\psi_t \psi_e f_y}{20\lambda\sqrt{f_c'}} \quad \text{(for bars > no. 7)} \tag{7.8}$$

2. Determine the multiplication factors: $\psi_t = 1.0$ (bottom bars), $\psi_e = 1.0$ (no coating), and $\lambda =$ 1.0 (normal-weight concrete). Also check that $\sqrt{f_c'} = 54.8$ psi < 100 psi.

$$\frac{l_d}{d_b} = \frac{60,000}{(20 \times 1 \times \sqrt{3000})} = 54.8$$

So, $l_d = 54.8(1.0) = 54.8$ in., say, 55 in. These values can be obtained directly from Tables 7.1 and 7.2. Note that if the general formula for l_d/d_b (Eq. 7.7) is used, assuming $K_{tr} = 0$, then

$$\frac{l_d}{d_b} = \left(\frac{3}{40\lambda}\right)\left(\frac{f_y}{\sqrt{f_c'}}\right)\left(\frac{\psi_t \psi_e}{c_b/d_b}\right) \tag{7.7}$$

In this example, $\psi_t = \psi_e = \lambda = 1$.
Also, $c_b =$ smaller of distance from center of bar to the nearest concrete surface (c_1) or one-half the center-to-center of bars spacing (c_2).

$$c_1 = 2.5 \text{ in.} \quad c_2 = \frac{0.5(12-5)}{3} = 1.17 \text{ in. (controls)}$$

$f_y/(c_b + K_{tr})/d_b = 1.17/1.0 = 1.17 < 1.5$, so use $(c_b + K_{tr})/d_b = 1.5$. Consequently, $l_d/d_b =$ $60{,}000(20\lambda\sqrt{f_c'})$ as in step 2, and $l_d = 55$ in.
Note : If the bars are not confined by stirrups, this value of l_d must be multiplied by 1.5 ($s =$ 1.33 in. $< 2d_b = 2.0$ in.).

Example 7.2

Repeat Example 7.1 if the beam is made of lightweight aggregate concrete, the bars are epoxy coated, and A_s required from analysis is 2.79 in.^{2}

Solution

1. Determine the multiplication factors: $\psi_t = 1.0$ (bottom bars), $\psi_e = 1.5$ (epoxy coated), $\lambda = 0.75$ (lightweight aggregate concrete), and $R_s = (A_s \text{ required})/(A_s \text{ provided}) = 2.79/3.14 = 0.89$. The value of ψ_e is 1.5, because the concrete cover is less than $3d_b = 3$ in. Check that $\psi_t\psi_e = 1.0(1.5) = 1.5 < 1.7$.
2. $$\frac{l_d}{d_b} = \frac{R_s\psi_t\psi_e f_y}{20\lambda\sqrt{f_c'}} \quad \text{(for bars > no. 7)}$$
$$= \frac{0.89(1.0)(1.5)(60,000)}{((20)(0.75)\sqrt{3000})} = 73.1 \text{ in.,} \quad \text{say, 74 in.}$$
3. The development length l_d can be obtained from Table 7.2 ($l_d = 55$ in. for no. 8 bars) and then divided by the factor 0.75.

Example 7.3

A reinforced concrete column is reinforced with eight no. 10 bars, which should extend to the footing. Determine the development length needed for the bars to extend down in the footing. Use normal-weight concrete with $f_c' = 4$ ksi and $f_y = 60$ ksi.

Solution

The development length in compression is

$$l_{dc} = \frac{0.02 d_b f_y}{\lambda\sqrt{f_c'}} \geq 0.0003 d_b f_y$$

$$l_{dc} = \frac{0.02(1.27)(60,000)}{(1)\sqrt{4000}} = 24.1 \text{ in.} \quad \text{(controls)}$$

The minimum l_{dc} is 0.0003(1.27)(60,000) = 22.86 in., but it cannot be less than 8 in. Because there are no other multiplication factors, then $l_d = 24.1$ in., or 25 in. (The same value is shown in Table 7.4.)

7.6 CRITICAL SECTIONS IN FLEXURAL MEMBERS

The critical sections for development of reinforcement in flexural members are

- At points of maximum stress
- At points where tension bars within the span are terminated or bent
- At the face of the support
- At points of inflection at which moment changes signs

The critical sections for a typical uniformly loaded continuous beam are shown in Fig. 7.6. The sections and the relative development lengths are explained as follows:

1. Three sections are critical for the negative moment reinforcement: Section 1 is at the face of the support, where the negative moment as well as stress are at maximum values. Two development lengths, x_1 and x_2, must be checked.

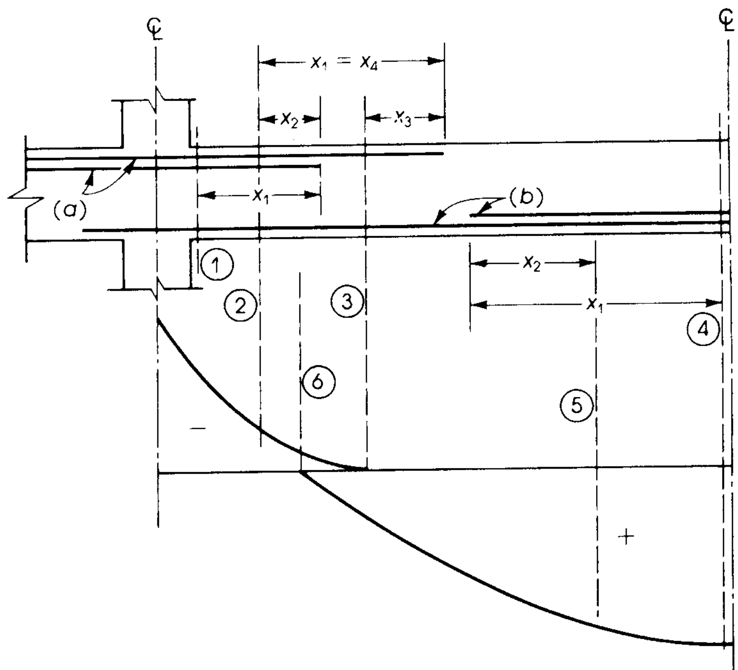


Figure 7.6 Critical sections (circled numbers) and development lengths ($X_1 - X_4$).

Section 2 is the section where part of the negative reinforcement bars can be terminated. To develop full tensile force, the bars should extend a distance x_2 before they can be terminated. Once part of the bars are terminated, the remaining bars develop maximum stress.

Section 3 is at the point of inflection. The bars shall extend a distance x_3 beyond section 3:x_3 must be equal to or greater than the effective depth, d, 12 bar diameters, or $\frac{1}{16}$ clear span, whichever is greater. At least one-third of the total reinforcement provided for negative moment at the support shall be extended a distance x_3 beyond the point of inflection, according to the ACI Code, Section 12.12.3.

2. Three sections are critical for positive moment reinforcement: Section 4 is that of maximum positive moment and maximum stresses. Two development lengths, x_1 and x_2, have to be checked. The length x_1 is the development length l_d specified by the ACI Code, Section 12.11, as mentioned later. The length x_2 is equal to or greater than d or 12 bar diameters.

Section 7.5 is where part of the positive reinforcement bars may be terminated. To develop full tensile force, the bars should extend a distance x_2. The remaining bars will have a maximum stress due to the termination of part of the bars. At the face of support, section 7.1, at least one-fourth of the positive moment reinforcement in continuous members shall extend along the same face of the member into the support, according to the ACI Code, Section 12.11.1. For simple members, at least one-third of the reinforcement shall extend into the support.

At points of inflection, section 7.6, limits are according to Section 12.11.3 of the ACI Code.

Example 7.4

A continuous beam has the bar details shown in Fig. 7.7. The bending moments for maximum positive and negative moments are also shown. We must check the development lengths at all critical sections. Given: $f'_c = 3$ ksi normal-weight concrete, $f_y = 40$ ksi, $b = 12$ in., $d = 18$ in., and span $L = 24$ ft.

Solution

The critical sections are (1) at the face of the support for tension and compression reinforcement (section 1), (7.2) at points where tension bars are terminated within the span (sections 2 and 5), (3) at point of inflection (sections 3 and 6), and (4) at midspan (section 4).

1. Development lengths for negative-moment reinforcement, from Fig. 7.7, are as follows: Three no. 9 bars are terminated at a distance $x_1 = 4.5$ ft from the face of the support, whereas the other three bars extend to a distance of 6 ft 0 in. (72 in.) from the face of the support.

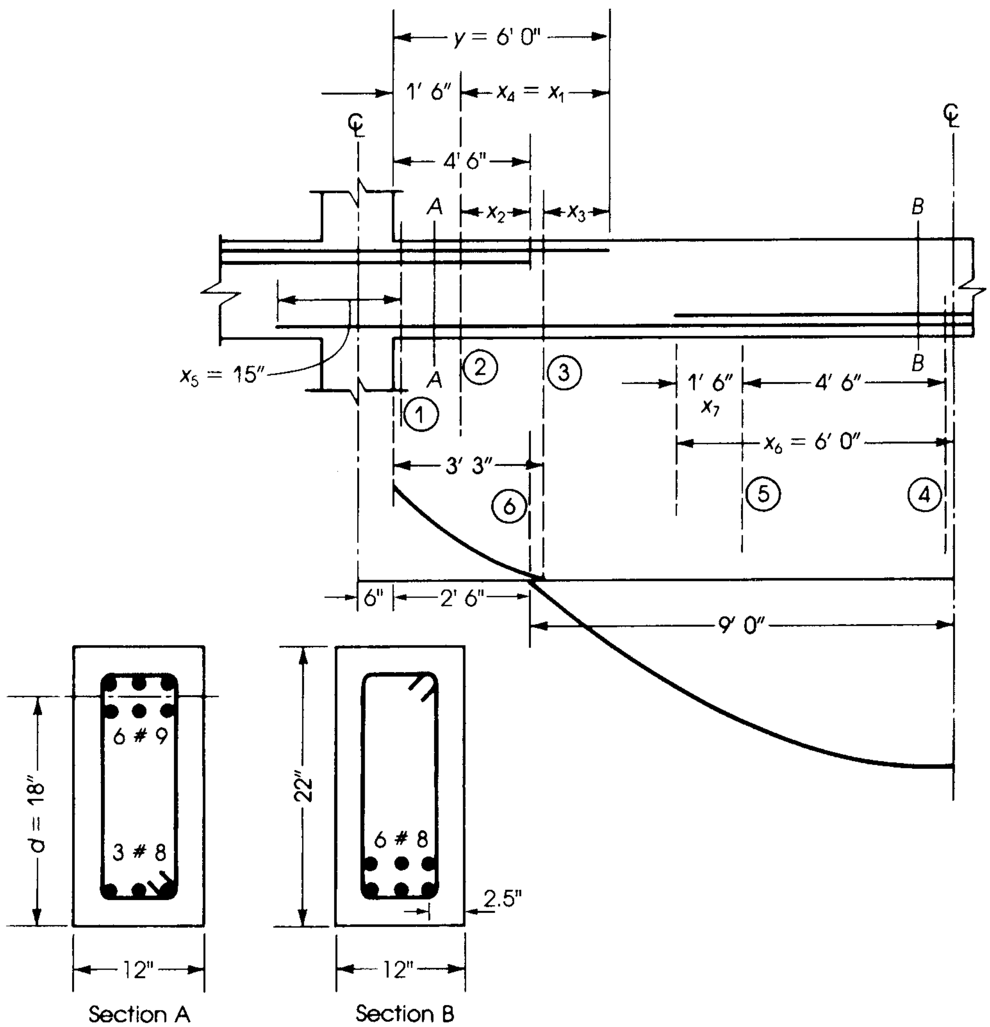


Figure 7.7 Example 7.4: Development length of a continuous beam.

a. The development length of no. 9 tension bars is $36.3d_b$ (Table 7.1) if conditions of spacing and cover are met.

$$\text{For no.9bars, } d_b = 1.128 \text{ in.}$$

$$\text{Cover} = 2.5 - \frac{1.128}{2} = 1.94 \text{ in.} > d_b$$

$$\text{Clear spacing} = \frac{12-5}{2} - 1.128 = 2.37 \text{ in.} > 2d_b$$

Then conditions are met, and $l_d = 36.6(1.128) = 41.3$ in. For top bars, $x_1 = l_d = 1.3(41.3) = 54$ in. $= 4.50$ ft $= x_1 > 12$ in. (minimum).

b. The development length x_2 shall extend beyond the point where three no. 9 bars are not needed, either $d = 18$ in. or $12d_b = 13.6$ in., whichever is greater. Thus, $x_2 = 18$ in. The required development length is $x_4 = 4.50$ ft, similar to x_1. Total length required is $y = x_1 + 1.5$ ft $= 6.0$ ft.

c. Beyond the point of inflection (section 3), three no. 9 bars extend a length $x_3 = y - 39 = 72 - 39 = 33$ in. The ACI Code requires that at least one-third of the bars should extend beyond the inflection point. Three no. 9 bars are provided, which are adequate. The required development length of x_3 is the greatest of $d = 18$ in., $12d_b = 13.6$ in., or $L/16 = 24 \times \frac{12}{16}$ in. $= 18$ in., which is less than x_3 provided.

2. Compressive reinforcement at the face of the support (section 7.1) (no. 8 bars): The development length x_5 is equal to

$$l_{dc} = \frac{0.02 d_b f_y}{\lambda \sqrt{f'_c}} = \frac{0.02 \times 1 \times 40,000}{1 \times \sqrt{3000}} = 14.6 \text{ in.}$$

So, we can use 15 in.

$$\text{Minimum } l_{dc} = 0.0003 d_b f_y = 0.0003 \times 1 \times 40,000 = 12 \text{ in.}$$

but it cannot be less than 8 in. The length 15 in. controls. For no. 8 bars, $d_b = 1$ in.; l_{dc} provided $= 15$ in., which is greater than that required.

3. Development length for positive moment reinforcement: Three no. 8 bars extend 6 ft beyond the centerline, and the other bars extend to the support. The development length x_6 from the centerline is $l_d = 36.6d_b = 37$ in. (Table 7.1), but it cannot be less than 12 in. That is, x_6 provided is 6 ft $= 72$ in. > 37 in.

The length x_7 is equal to d or $12d_b$, that is, 18 in. or $12 \times 1 = 12$ in. The provided value is 18 in., which is adequate.

The actual position of the termination of bars within the span can be determined by the moment-resistance diagram, as will be explained later.

7.7 STANDARD HOOKS (ACI CODE, SECTIONS 12.5 AND 7.1)

A *hook* is used at the end of a bar when its straight embedment length is less than the necessary development length, l_d. Thus the full capacity of the bar can be maintained in the shortest distance of embedment. The minimum diameter of bend, measured on the inside of the main bar of a standard hook D_b, is as follows (Fig. 7.8) [[9]]:

- For no. 3 to no. 8 bars (10–25 mm), $D_b = 6d_b$.
- For no. 9 to no. 11 bars (28, 32, and 36 mm), $D_b = 8d_b$.
- For no. 14 and no. 18 bars (43 and 58 mm), $D_b = 10d_b$.

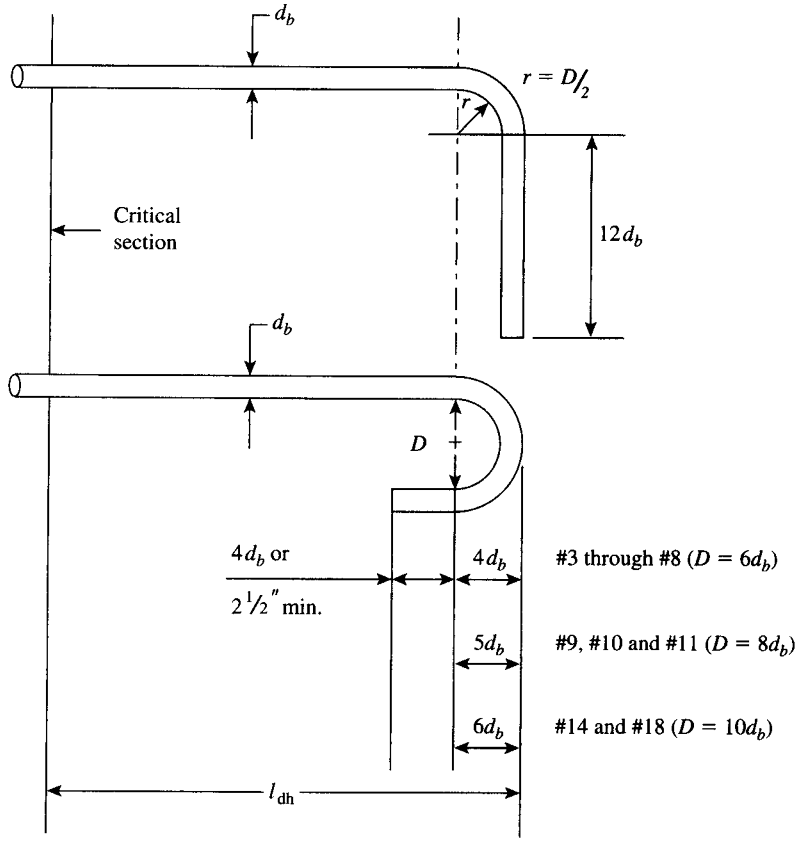


Figure 7.8 Hooked-bar details for the development of standard hooks [9]. Courtesy of ACI.

The ACI Code, Section 12.5.2, specifies a development length l_{dh} for hooked bar as follows:

$$l_{dh} = \left(\frac{0.02\psi_e f_y}{\lambda\sqrt{f_c'}}\right) \text{(Modification Factor) } d_b \tag{7.15}$$

where

$\psi_e = 1.2$ for epoxy-coated bars
$\lambda = 0.75$ for lightweight aggregate concrete unless f_{ct} is specified then $\lambda = f_{ct}/(6.7(\sqrt{f_c'}) \leq 1$
ψ_e and $\lambda = 1.0$ for all other cases

For grade 60 hooked bar ($f_y = 60$ ksi) with $\psi_e = \lambda = 1$, l_{dh} becomes:

$$l_{dh} = \frac{1200 d_b}{\sqrt{f_c'}} \text{(Modification Factor) } d_b \tag{7.15a}$$

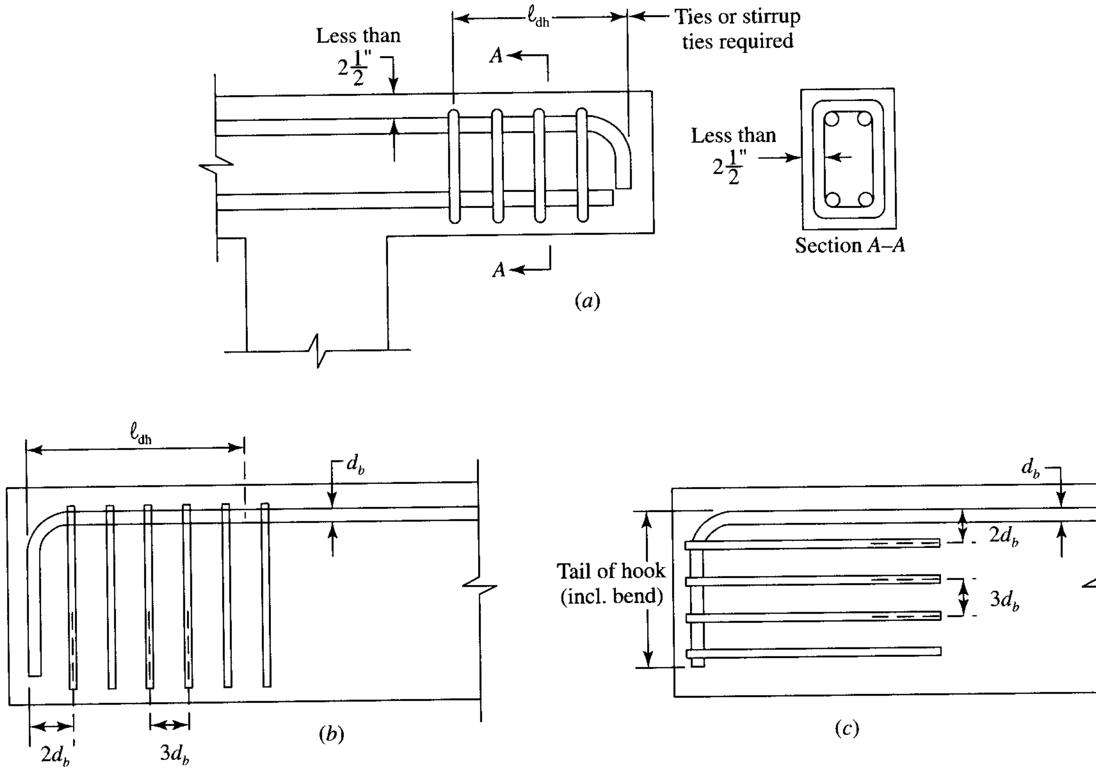


Figure 7.9 (*a*) Concrete cover limitations, (*b* and *c*) stirrups or ties placed perpendicular or parallel to the bar being developed [9]. Courtesy of ACI.

Based on different conditions, the development length, l_{dh}, must be multiplied by one of the following factors:

1. For 90° hooks of no. 11 or smaller bars are used and the hook is enclosed vertically or horizontally within stirrups or ties spaced not greater than three times the diameter of the hooked bar, the basic development length is multiplied by 0.8.
2. When no. 11 or smaller bars are used and the side concrete cover, normal to the plane of the hook, is not less than 2.5 in., the development length is multiplied by 0.7. The same factor applies for a 90° hook when the concrete cover on bar extension beyond the hook is not less than 2 in.
3. For 180° hooks of no. 11 or smaller bars that are enclosed with ties or stirrups perpendicular to the bar and spaced not greater than $3d_b$, the development length is multiplied by 0.8.
4. When a bar anchorage is not required, the basic development length for the reinforcement in excess of that required is multiplied by the ratio

$$\frac{A_s \text{ (required)}}{A_s \text{ (provided)}}$$

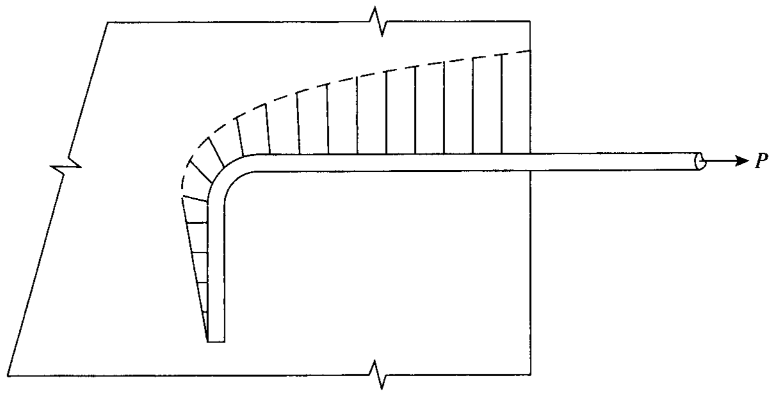


Figure 7.10 Stress distribution in 90° hooked bar.

5. When standard hooks with less than a 2.5-in. concrete cover on the side and top or bottom are used at a discontinuous end of a member, the hooks shall be enclosed by ties or stirrups spaced at no greater than $3d_b$. Moreover, the factor 0.8 given in item 1 shall not be used.

The development length, l_{dh}, of a standard hook for deformed bars in tension must not be less than $8d_b$ or 6 in., whichever is greater. Note that hooks are not effective for reinforcing bars in compression and may be *ignored* (ACI Code, Section 12.5).

Details of standard 90° and 180° hooks are shown in Fig. 7.8 [9]. The dimensions given are needed to protect members against splitting and spalling of concrete cover. Figure 7.9*a* shows details of hooks at a discontinuous end with a concrete cover less than 2.5 in. that may produce concrete spalling [9]. The use of closed stirrups is necessary for proper design. Figures 7.9*b* and *c* show placement of stirrups or ties perpendicular and parallel to the bar being developed, spaced along the development length. Figure 7.10 shows the stress distribution along a 90° hooked bar under a tension force p.

The development length required for deformed welded wire fabric is covered in Section 12.7 in the ACI Code. The basic development length (measured from the critical section) with at least one cross wire within the development length and not less than 2 in. shall be the greater of $(f_y - 35{,}000)/f_y$ (units in psi) or $5d_b/S_w$ but should not be taken greater than 1.0, where S_w = spacing of wire to be developed or spliced (in.).

Example 7.5

Compute the development length required for the top no. 8 bars of the cantilever beam shown in Fig. 7.11 that extend into the column support if the bars are

a. Straight
b. Have a 90° hook at the end
c. Have a 180° hook at the end

The bars are confined by no. 3 stirrups spaced at 6 in. and have a clear cover = 1.5 in. and clear spacings = 2.0 in. Use $f'_c = 4$ ksi normal-weight concrete and $f_y = 60$ ksi.

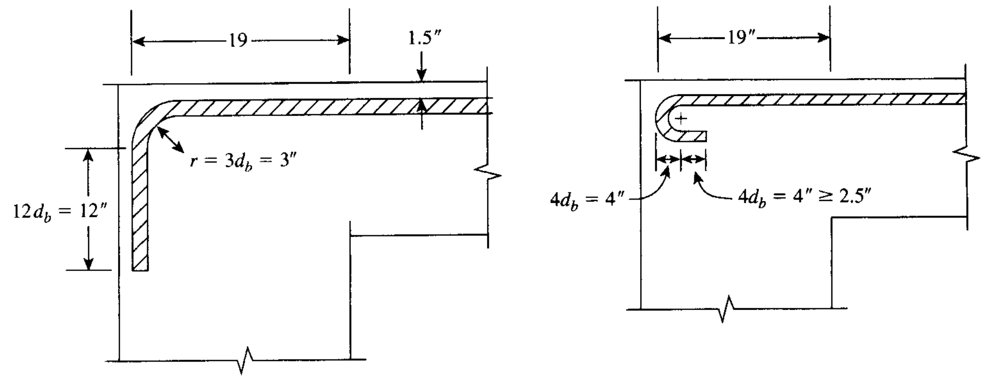


Figure 7.11 Example 7.5.

Solution

a. Straight bars: For no. 8 bars, $d_b = 1.0\,\text{in.}$ Because clear spacing $= 2d_b$ and clear cover is greater than d_b with bars confined by stirrups, then conditions a and b are met. Equation 7.10 can be used to calculate the basic l_d or you can get it directly from Table 7.2: $l_d = 48\,\text{in.}$ For top bars, $\psi_e = 1.3$ and final $l_d = 1.3(48) = 63\,\text{in.}$

b. Bars with 90° hook: For no. 8 bars, $d_b = 1.0\,\text{in.}$ development length for $f_y = 60$ ksi $l_{dh} = 1200d_b/\sqrt{f_c'} = 1200(1.0)/\sqrt{4000} = 19$ in. Because no other modifications apply, then $l_{dh} = 19\,\text{in.} > 8d_b = 8\,\text{in.}$ or 6 in. Other details are shown in Fig. 7.11. The factor $\psi_e = 1.3$ for top bars does not apply to hooks.

c. Bars with 180° hook: $l_{dh} = 19\,\text{in.}$, as calculated before. No other modifications apply; then $l_{dh} = 19\,\text{in.} > 8d_b = 8\,\text{in.}$ Other details are shown in Fig. 7.11.

7.8 SPLICES OF REINFORCEMENT

7.8.1 General

Steel bars that are used as reinforcement in structural members are fabricated in lengths of 20, 40, and 60 ft (6, 12, and 18 m), depending on the bar diameter, transportation facilities, and other reasons. Bars are usually tailored according to the reinforcement details of the structural members. When some bars are short, it is necessary to splice them by lapping the bars a sufficient distance to transfer stress through the bond from one bar to the other.

Splices may be made by lapping or welding or with mechanical devices that provide positive connection between bars. Lap splices should not be used for bars larger than no. 11 (36 mm). For noncontact lap splices in flexural members, bars should not be spaced transversely farther apart than one-fifth the required length or 6 in. (150 mm). An approved welded splice is one in which the bars are butted and welded to develop in tension at least 125% of the specified yield strength of the bar. The ACI Code, Section 12.14, also specifies that full positive mechanical connections must develop in tension or compression at least 125% of the specified yield strength of the bar.

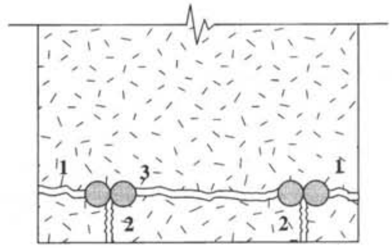


Figure 7.12 Lap splice failure due to the development of one or more cracks.

Splices should not be made at or near sections of maximum moments or stresses. Also, it is recommended that no bars should be spliced at the same location to avoid a weakness in the concrete section and to avoid the congestion of bars at the same location, which may cause difficulty in placing the concrete around the bars.

The stresses developed at the end of a typical lap splice are equal to 0, whereas the lap length, l_d, embedded in concrete is needed to develop the full stress in the bar, f_y. Therefore, a minimum lap splice of l_d is needed to develop a continuity in the spliced tension or compression bars. If adequate splice length is not provided, splitting and spalling occurs in the concrete shell (Fig. 7.12).

Splices in tension and compression are covered by Sections 12.15 and 12.16 of the ACI Code.

7.8.2 Lap Splices in Tension, l_{st}

Depending upon the percentage of bars spliced on the same location and the level of stress in the bars or deformed wires, the ACI Code introduces two classes of splices (with a minimum length of 12 in.):

1. Class A splices: These splices have a minimum length $l_{st} = l_d$ and are used when (a) one-half or less of the total reinforcement is spliced within the required lap length; and (b) the area of reinforcement provided is at least twice that required by analysis over the entire length of the splice. The length l_d is the development length of the bar, as calculated earlier.
2. Class B splices: These splices have a minimum length $l_{st} = 1.3l_d$ and are used for all other cases that are different from the aforementioned conditions. For example, class B splices are required when all bars or deformed wires are spliced at the same location with any ratio of (A_s provided)/(A_s required). Splicing all the bars in one location should be avoided when possible.
3. l_d in class A and B splice is calculated without the 12 in. minimum requirement and without the modification factor of (A_s required)/(A_s provided).

7.8.3 Lap Splice in Compression, l_{sc}

The splice lap length of the reinforcing bars in compression, l_{sc}, should be equal to or greater than the development length of the bar in compression, l_d (including the modifiers), calculated earlier (Eq. 7.14).Moreover, the lap length shall satisfy the following (ACI Code, Section 12.16.1):

Table 7.5 Lap-Splice Length in Compression, l_{sc}(in.), ($f_c' \geq$ 3ksi and Minimum l_{sc} = 12 in.)

Bar number	Bar diameter (in.)	f_y (ksi) 40	60	80
3	0.375	12	12	18
4	0.500	12	15	24
5	0.625	13	19	30
6	0.750	15	23	36
7	0.875	18	27	42
8	1.000	20	30	48
9	1.128	23	34	55
10	1.270	26	39	61
11	1.410	29	43	68

$$l_{sc} \geq (0.0005 f_y d_b) \quad \text{(for } f_y \leq 60{,}000 \text{ psi)} \tag{7.16}$$

$$l_{sc} = (0.0009 f_y - 24) d_b \quad \text{(for } f_y > 60{,}000 \text{ psi)} \tag{7.17}$$

For all cases, the lap length must not be less than 12 in. Table 7.5 gives the lap-splice length for various f_y values. If the concrete strength, f_c', is less than 3000 psi, the lap length, l_{sc}, must be increased by one-third.

In spirally reinforced columns, lap-splice length within a spiral may be multiplied by 0.75 but may not be less than 12 in. In tied columns, with ties within the splice length having a minimum effective area of $0.0015hs$, lap splice may be multiplied by 0.83 but may not be less than 12 in., where h = overall thickness of column and s = spacing of ties (in.).

Example 7.6

Calculate the lap-splice length for six no. 8 tension bottom bars (in two rows) with clear spacing = 2.5 in. and clear cover = 1.5 in. for the following cases:

a. When three bars are spliced and (A_s provided)/(A_s required) > 2
b. When four bars are spliced and (A_s provided)/(A_s required) < 2
c. When all bars are spliced at the same location. Given: $f_c' = 5$ ksi and $f_y = 60$ ksi.

Solution

a. For no. 8 bars, $d_b = 1.0$ in., and $\psi_t = \psi_e = \lambda = 1.0$.: check first for $\sqrt{5000} = 70.7$ psi < 100 psi, and then calculate l_d from Equation 7.8 or Table 7.1, $l_d = 42.5 d_b$, conditions for clear spacings and cover are met. $l_d = 42.5(1.0) = 42.5$ in., or 43 in. For (A_s provided)/(A_s required) > 2, class A splice applies, $l_{st} = 1.0 l_d = 43$ in. > 12 in. (minimum). Bars spliced are less than half the total number.

b. $l_d = 43$ in., as calculated before. Because (A_s provided)/(A_s required) is less than 2, class B splice applies, $l_{st} = 1.3 l_d = 1.3(42.5) = 55.25$ in., say, 56 in., which is greater than 12 in.

c. Class B splice applies and $l_{st} = 56$ in. > 12 in.

Example 7.7

Calculate the lap-splice length for a no. 10 compression bar in a tied column when $f'_c = 5$ ksi and when (a) $f_y = 60$ ksi and (b) $f_y = 80$ ksi.

Solution

a. For no. 10 bars, $d_b = 1.27$ in., and the development length from Table 7.4 or 7.3 is 23 in. Because no modifiers apply, $l_{sc} = 23$ in. > 12 in. Check that $l_{sc} \geq 0.0005 d_b f_y = 0.0005(1.27)(60{,}000) = 38.1$ in. Therefore, $l_{sc} = 39$ in. controls.

b. The basic l_d is 23 in., as calculated before. Check that $l_{sc} \geq (0.0009 f_y - 24) d_b = [0.0009(80{,}000) - 24](1.27) = 61$ in. Therefore, $l_{sc} = 61$ in. controls.

7.9 MOMENT-RESISTANCE DIAGRAM (BAR CUTOFF POINTS)

The moment capacity of a beam is a function of its effective depth, d, width, b, and the steel area for given strengths of concrete and steel. For a given beam, with constant width and depth, the amount of reinforcement can be varied according to the variation of the bending moment along the span. It is a common practice to cut off the steel bars where they are no longer needed to resist the flexural stresses. In some other cases, as in continuous beams, positive-moment steel bars may be bent up, usually at 45°, to provide tensile reinforcement for the negative moments over the supports.

The factored moment capacity of an under-reinforced concrete beam at any section is

$$M_u = \phi A_s f_y \left(d - \frac{a}{2}\right) \tag{7.18}$$

The lever arm $(d - a/2)$ varies for sections along the span as the amount of reinforcement varies; however, the variation in the lever arm along the beam length is small and is never less than the value obtained at the section of maximum bending moment. Thus, it may be assumed that the moment capacity of any section is proportional to the tensile force or the area of the steel reinforcement, assuming proper anchorage lengths are provided.

To determine the position of the cutoff or bent points, the moment diagram due to external loading is drawn first. A moment-resistance diagram is also drawn on the same graph, indicating points where some of the steel bars are no longer required. The factored moment resistance of one bar, M_{ub}, is

$$M_{ub} = \phi A_{sb} f_y \left(d - \frac{a}{2}\right) \tag{7.19}$$

where

$$a = \frac{A_s f_y}{0.85 f'_c b}$$

$$A_{sb} = \text{area of one bar}$$

The intersection of the moment-resistance lines with the external bending moment diagram indicates the theoretical points where each bar can be terminated. To illustrate this discussion, Fig. 7.13 shows a uniformly loaded simple beam, its cross-section, and the bending moment diagram. The bending moment curve is a parabola with a maximum moment at midspan of

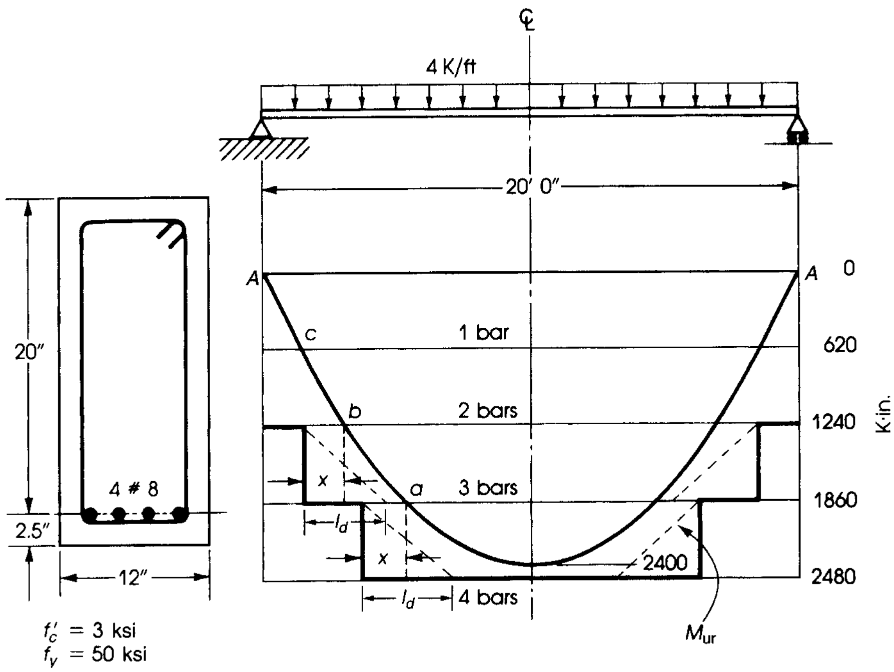


Figure 7.13 Moment-resistance diagram.

2400 K·in. Because the beam is reinforced with four no. 8 bars, the factored moment resistance of one bar is

$$M_{\text{ub}} = \phi A_{\text{sb}} f_y \left(d - \frac{a}{2}\right)$$

$$a = \frac{A_s f_y}{0.85 f_c' b} = \frac{4 \times 0.79 \times 50}{0.85 \times 3 \times 12} = 5.2 \text{ in.}$$

$$M_{\text{ub}} = 0.9 \times 0.79 \times 50 \left(20 - \frac{5.2}{2}\right) = 620 \text{ K·in.}$$

The factored moment resistance of four bars is thus 2480 K·in., which is greater than the external moment of 2400 K·in. If the moment diagram is drawn to scale on the base line A-A, it can be seen that one bar can be terminated at point a, a second bar at point b, the third bar at point c, and the fourth bar at the support end a. These points are the theoretical positions for the termination of the bars. However, it is necessary to develop part of the strength of the bar by bond, as explained earlier. The ACI Code specifies that every bar should be continued at least a distance equal to the effective depth, d, of the beam or 12 bar diameters, whichever is greater, beyond the theoretical points a, b, and c. The Code (Section 12.11.1) also specifies that at least one-third of the positive moment reinforcement must be continued to the support for simple beams. Therefore, for the example discussed here, two bars must extend into the support, and the moment-resistance diagram, M_{ur}, shown in Fig. 7.13, must enclose the external bending moment diagram at all points. Full load capacity of each bar is attained at a distance l_d from its end.

For continuous beams, the bars are bent at the required points and used to resist the negative moments at the supports. At least one-third of the total reinforcement provided for the negative moment at the support must be extended beyond the inflection points a distance not less than the effective depth, 12 bar diameters, or $\frac{1}{6}$ the clear span, whichever is greatest (ACI Code, Section 12.12.3).

Bent bars are also used to resist part of the shear stresses in beams. The moment–resistance diagram for a typical continuous beam is shown in Fig. 7.14.

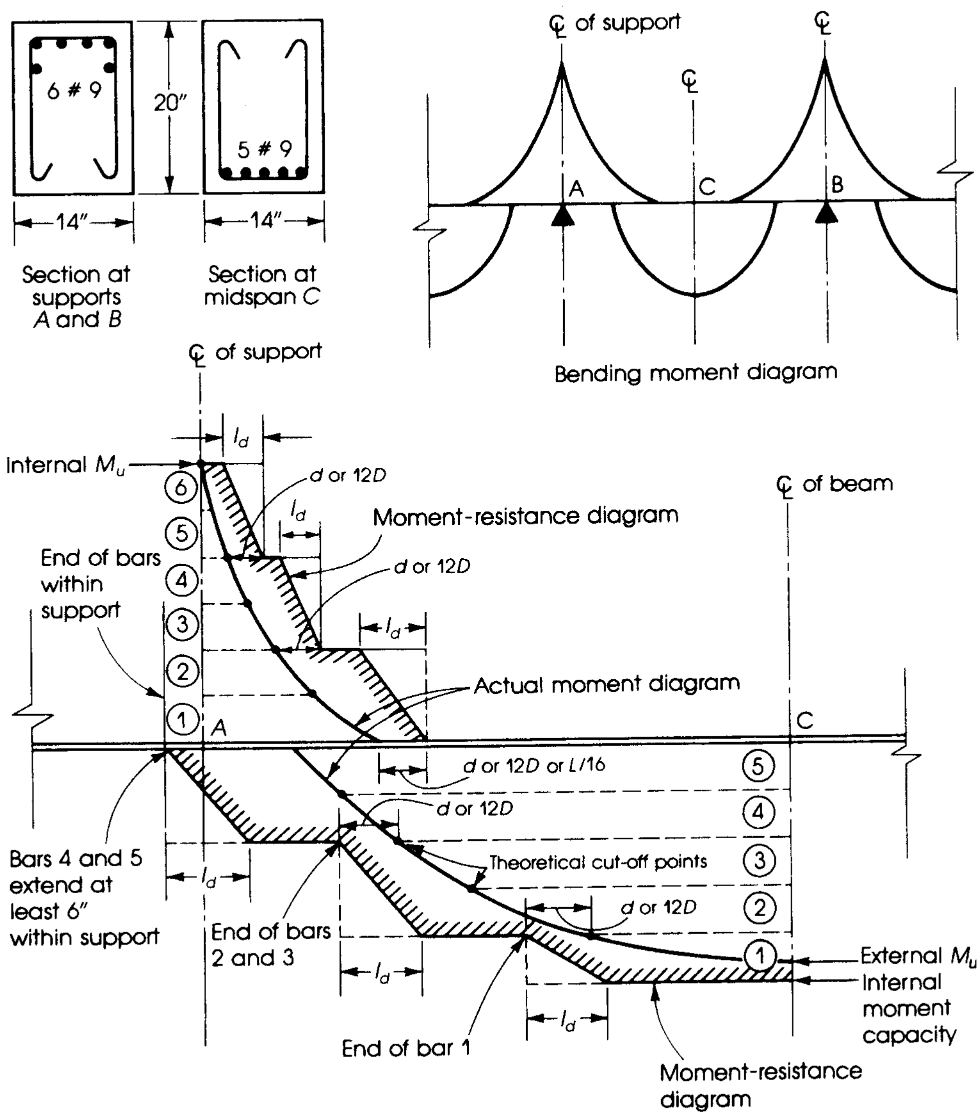


Figure 7.14 Sections and bending moment diagram (*top*) and moment–resistance diagram (*bottom*) of a continuous beam. Bar diameter is signified by *D*.

Example 7.8

For the simply supported beam shown in Fig. 7.15, design the beam for the given factored loads and draw the moment—resistance diagram. Also, show where the reinforcing bars can be terminated. Use $b = 10$ in., a steel ratio of 0.018, $f'_c = 3$ ksi, and $f_y = 40$ ksi.

Solution

For $\rho = 0.018$, $R_u = 556$ psi and $M_u = R_u bd^2$. $M_u = 132.5$ K·ft. Now $132.5(12) = 0.556(10)d^2$, so $d = 17$ in.; let $h = 20$ in. $A_s = 0.018(10)(17) = 3.06$ in.^{2}; use four no. 8 bars ($A_s = 3.14$ in.^{2}). Actual $d = 20 - 2.5 = 17.5$ in.

$$M_{ur} = \phi A_s f_y \left(d - \frac{a}{2}\right) \text{ and } a = \frac{3.14(40)}{0.85(3)(10)} = 4.93 \text{ in.}$$

$$M_{ur} \text{ (for one bar)} = 0.9(0.79)(40)\left(17.5 - \frac{4.93}{2}\right)$$

$$= 427.7 \text{K·in.} = 35.64 \text{ K·ft}$$

$$M_{ur} \text{ (for all four bars)} = 1710.8 \text{ K·in.} = 142.6 \text{ K·ft}$$

For the calculation of 'a', the four no. 8 bars were utilized rather than calculating the 'a' for the extended two bars. This assumption will slightly increase the length of the bars beyond the cutoff point.

Details of the moment–resistance diagram are shown in Fig. 7.15. Note that the bars can be bent or terminated at a distance of 17.5, say, 18 in. (or 12 bar diameters, whichever is greater), beyond the points where (theoretically) the bars are not needed. The development length, l_d, for no. 8 bars is $36.6d_b = 37$ in. (Table 7.1). The cutoff points of the first and second bars are at points A and B, but the actual points are at A' and B', 18 in. beyond A and B. From A', a length $l_d = 37$ in. backward is shown to establish the moment–resistance diagram (the dashed line). The end of the last two bars extending to the support will depend on how far they extend inside the support, say, at C'. Normally, bars are terminated within the span at A' and B' as bent bars are not commonly used to resist shear.

SUMMARY

Sections 7.1–7.2

Bond is influenced mainly by the roughness of the steel surface area, the concrete mix, shrinkage, and the cover of concrete. In general,

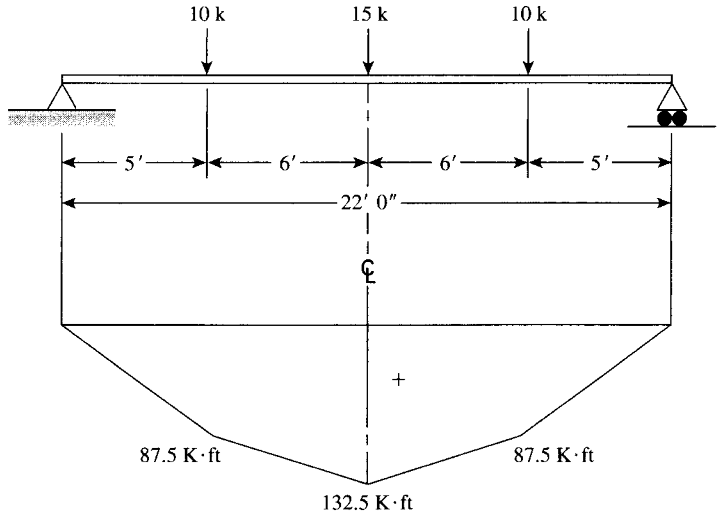
$$l_d = \frac{A_s f_y}{U_u \Sigma O} \tag{7.3}$$

Sections 7.3 and 7.5

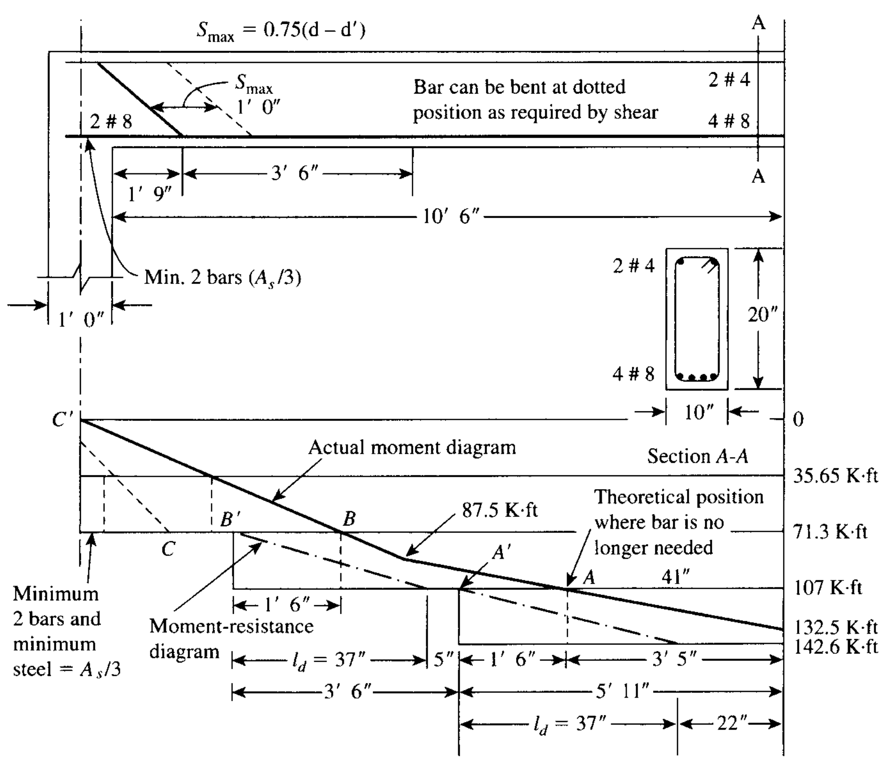
1. The general formula for the development length of deformed bars or wire shall be

$$\frac{l_d}{d_b} = \left(\frac{3}{40}\right)\left(\frac{f_y}{\lambda\sqrt{f'_c}}\right)\frac{\psi_t \psi_e \psi_s}{(c_b + k_{tr})/d_b} \tag{7.7}$$

As design simplification, K_{tr} may be assumed to be zero. Other values of l_d/d_b are given in Tables 7.1 and 7.2. ψ_t, ψ_e, ψ_s, and λ are multipliers defined in Section 7.3.1.



(a)



(b)

Figure 7.15 Example 7.8: Details of reinforcing bars and the moment-resistance diagram.

2. Simplified expressions are used when conditions for concrete cover and spacings are met. For no. 7 and larger bars,

$$\frac{l_d}{d_b} = \left(\frac{f_y}{\sqrt{f'_c}}\right)\left(\frac{\psi_t \psi_e}{20\lambda}\right) = Q \tag{7.8}$$

For no. 6 and smaller bars,

$$\frac{l_d}{d_b} = 0.8Q \tag{7.9}$$

3. For all other cases, multiply the previous Q by 1.5.
4. Minimum length is 12 in.

Section 7.4

Development length in compression for all bars is

$$l_d = \frac{0.02 d_b f_y}{\lambda\sqrt{f'_c}} \geq 0.0003 d_b f_y \geq 8 \text{ in.} \tag{7.14}$$

For specific values, refer to Tables 7.3 and 7.4.

Section 7.6

The critical sections for the development of reinforcement in flexural members are

- At points of maximum stress
- At points where tension bars are terminated within the span
- At the face of the support
- At points of inflection

Section 7.7

The minimum diameter of bends in standard hooks is

- For no. 3 to no. 8 bars, $6d_b$
- For no. 9 to no. 11 bars, $8d_b$

The development length l_{dh} of a standard hook is

$$l_{dh} = \left(\frac{0.02\psi f_y}{\lambda\sqrt{f'_c}}\right) (\text{Modification factor})\, d_b \tag{7.15}$$

Section 7.8

1. For splices in tension, the minimum lap-splice length is 12 in. If (a) one-half or less of the total reinforcement is spliced within the required lap length and (b) the area of reinforcement provided is at least twice that required by analysis over the entire length of the splice, then $l_{st} = 1.0 l_d$ = class A splice.

2. For all other cases, class B has to be used when $l_{st} = 1.3l_d$.
3. For splices in compression, the lap length should be equal to or greater than l_{dc} in compression, but it also should satisfy the following: $l_{sc} \geq 0.0005\ f_y d_b$ (for $f_y \leq 60{,}000\,\text{psi}$).

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PROBLEMS

7.1 For each assigned problem, calculate the development length required for the following tension bars. All bars are bottom bars in normal-weight concrete unless specified otherwise in the notes.

No.	Bar no.	f'_c (ksi)	f_y (ksi)	Clear cover (in.)	Clear spacing (in.)	Notes
a	5	3	60	2.0	2.25	
b	6	4	60	2.0	2.50	Lightweight aggregate concrete
c	7	5	60	2.0	2.13	Epoxy coated
d	8	3	40	2.5	2.30	Top bars, lightweight aggregate concrete
e	9	4	60	1.5	1.5	
f	10	5	60	2.0	2.5	No. 3 stirrups at 6 in.
g	11	5	60	3.0	3.0	
h	9	3	40	2.0	1.5	Epoxy coated
i	8	4	60	2.0	1.75	(A_s provided)/(A_s required) = 1.5
j	6	4	60	1.5	1.65	Top bars, epoxy coated and no. stirrup at 4 in.

7.2 For each assigned problem, calculate the development length required for the following bars in compression.

No.	Bar no.	f'_c (ksi)	f_y (ksi)	Notes
a	8	3	60	
b	9	4	60	
c	10	4	40	
d	11	5	60	(A_s required)/(A_s provided) = 0.8
e	7	6	60	(A_s required)/(A_s provided) = 0.9
f	9	5	60	Column with spiral no. 3 at 2 in.

7.3 Compute the development length required for the top no. 9 bars of a cantilever beam that extend into the column support if the bars are

a. Straight

b. Have a 90° hook at the end

c. Have a 180° hook at the end

The bars are confined with no. 3 stirrups spaced at 5 in. and have a clear cover of 2.0 in. Use $f'_c = 4$ ksi and $f_y = 60$ ksi. (Clear spacing = 2.5 in.)

7.4 Repeat Problem 7.3 when no. 7 bars are used.

7.5 Repeat Problem 7.3 when $f'_c = 3$ ksi and $f_y = 40$ ksi.

7.6 Repeat Problem 7.3 when no. 10 bars are used.

7.7 Calculate the lap-splice length for no. 9 tension bottom bars with clear spacing of 2.0 in. and clear cover of 2.0 in. for the following cases:

a. When 50% of the reinforcement is spliced and (A_s provided)/(A_s required) = 2

b. When 75% of the reinforcement is spliced and (A_s provided)/(A_s required) = 1.5

c. When all bars are spliced at one location and (A_s provided)/(A_s required) = 2

d. When all bars are spliced at one location and (A_s provided)/(A_s required) = 1.3. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

7.8 Repeat Problem 7.7 using $f'_c = 3$ ksi and $f_y = 60$ ksi.

7.9 Calculate the lap splice length for no. 9 bars in compression when $f'_c = 5$ ksi and $f_y = 60$ ksi.

7.10 Repeat Problem 7.9 when no. 11 bars are used.

7.11 Repeat Problem 7.9 when $f_y = 80$ ksi.

7.12 Repeat Problem 7.9 when $f'_c = 4$ ksi and $f_y = 60$ ksi.

7.13 A continuous beam has the typical steel reinforcement details shown in Fig. 7.16. The sections at midspan and at the face of the support of a typical interior span are also shown. Check the development lengths of the reinforcing bars at all critical sections. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

7.14 Design the beam shown in Fig. 7.17 using ρ_{max}. Draw the moment—resistance diagram and indicate where the reinforcing bars can be terminated. The beam carries a uniform dead load, including self-weight of 1.5 K/ft, and a live load of 2.2 K/ft. Use $f'_c = 4$ ksi, $f_y = 60$ ksi, and $b = 12$ in.

7.15 Design the beam shown in Fig. 7.18 using a steel ratio $\rho = 1/2\ \rho_b$. Draw the moment—resistance diagram and indicate the cutoff points. Use $f'_c = 3$ ksi, $f_y = 60$ ksi, and $b = 12$ in.

7.16 Design the section at support B of the beam shown in Fig. 7.19, ρ_{max}. Adopting the same dimensions of the section at B for the entire beam ABC, determine the reinforcement required for part AB and draw the moment-resistance diagram for the beam ABC. Use $f'_c = 4$ ksi, $f_y = 60$ ksi, and $b = 12$ in.

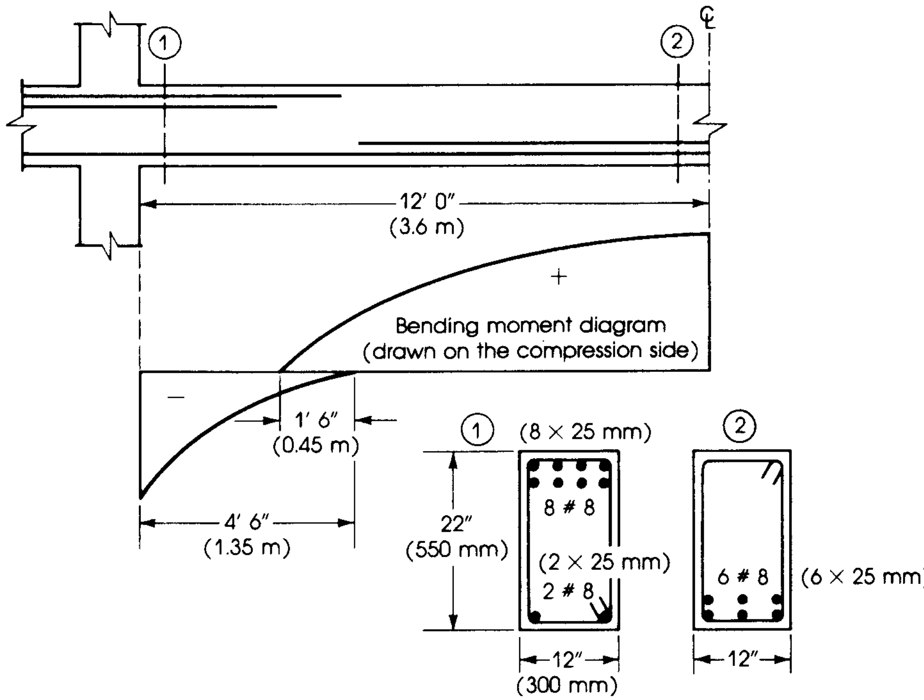


Figure 7.16 Problem 7.13.

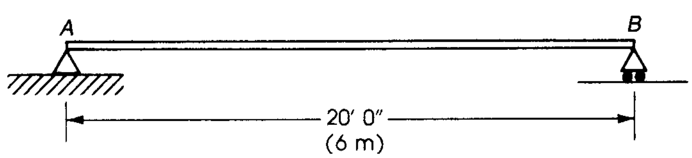


Figure 7.17 Problem 7.14: Dead load = 1.5 K/ft (22.5 kN/m), live load = 2.2 K/ft (33 kN/m).

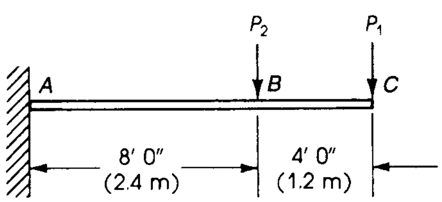


Figure 7.18 Problem 7.15: Dead load = 2 K/ft (30 kN/m), live load (concentrated loads only) is P_1 = 10 K (45 kN), P_2 = 16 K (72 kN).

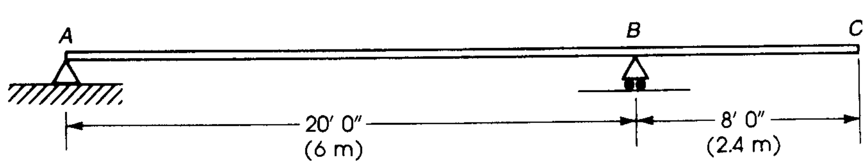


Figure 7.19 Problem 7.16: Dead load = 6 K/ft (90 kN/m), live load = 4 K/ft (60 kN/m).

CHAPTER 8

SHEAR AND DIAGONAL TENSION

Office building, Chicago, Illinois.

8.1 INTRODUCTION

When a simple beam is loaded as shown in Fig. 8.1, bending moments and shear forces develop along the beam. To carry the loads safely, the beam must be designed for both types of forces. Flexural design is considered first to establish the dimensions of the beam section and the main reinforcement needed, as explained in the previous chapters.

The beam is then designed for shear. If shear reinforcement is not provided, shear failure may occur. *Shear failure* is characterized by small deflections and lack of ductility, giving little or no warning before failure. On the other hand, flexural failure is characterized by a gradual increase in deflection and cracking, thus giving warning before total failure. This is due to the ACI Code limitation on flexural reinforcement. The design for shear must ensure that shear failure does not occur before flexural failure.

8.2 SHEAR STRESSES IN CONCRETE BEAMS

The general formula for the shear stress in a homogeneous beam is

$$v = \frac{VQ}{Ib} \tag{8.1}$$

where

V = total shear at the section considered

Q = statical moment about the neutral axis of that portion of cross-section lying between a line through the point in question parallel to the neutral axis and nearest face, upper or lower, of the beam

I = moment of inertia of cross-section about the neutral axis

b = width of beam at the given point

The distribution of bending and shear stresses according to elastic theory for a homogeneous rectangular beam is as shown in Fig. 8.2. The bending stresses are calculated from the flexural

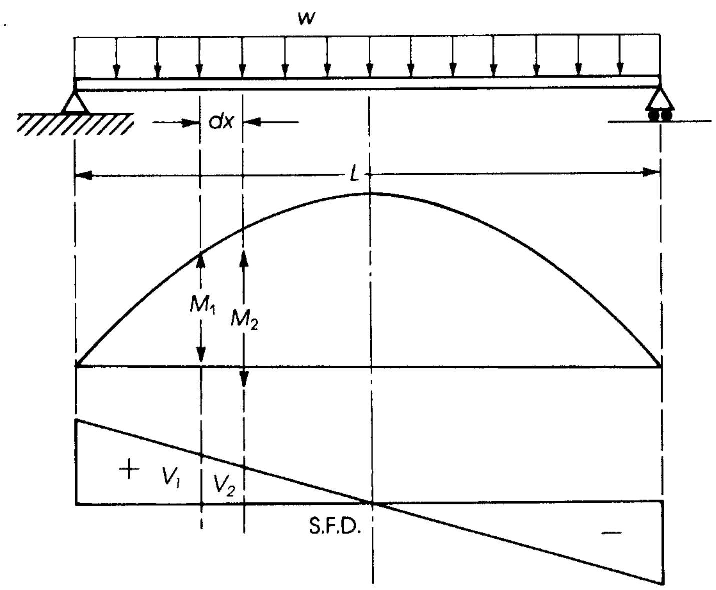


Figure 8.1 Bending moment and shearing force diagrams for a simple beam.

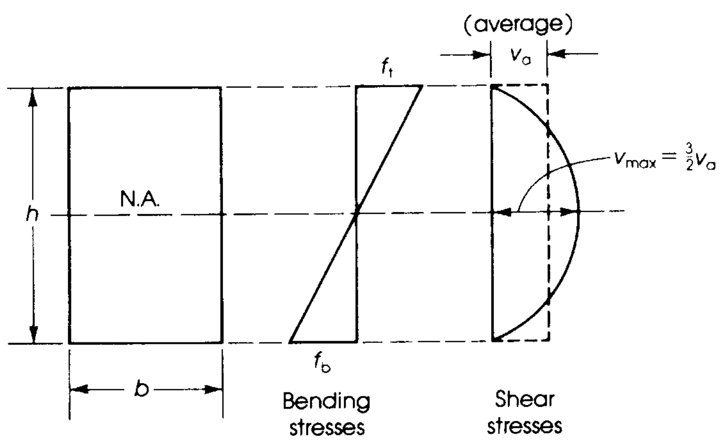


Figure 8.2 Bending and shear stresses in a homogeneous beam, according to elastic theory.

formula $f = Mc/I$, whereas the shear stress at any point is calculated by the shear formula of Eq. 8.1. The maximum shear stress is at the neutral axis and is equal to $1.5v_a$ (average shear), where $v_a = V/bh$. The shear stress curve is parabolic.

For a singly reinforced concrete beam, the distribution of shear stress above the neutral axis is a parabolic curve. Below the neutral axis, the maximum shear stress is maintained down to the level of the tension steel, because there is no change in the tensile force down to this point and the concrete in tension is neglected. The shear stress below the tension steel is 0 (Fig. 8.3). For doubly reinforced and T-sections, the distribution of shear stresses is as shown in Fig. 8.3.

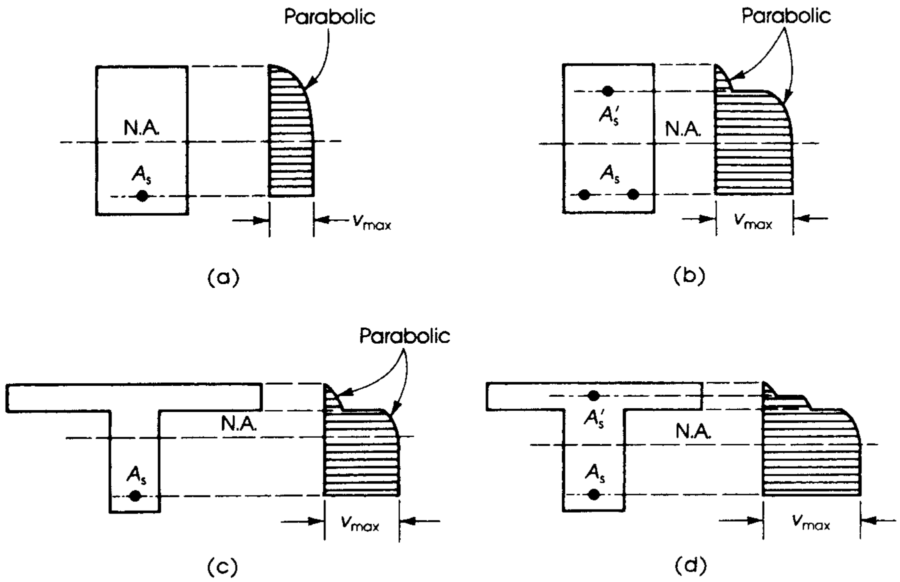


Figure 8.3 Distribution of shear stresses in reinforced concrete beams: (*a*) singly reinforced, (*b*) doubly reinforced, (*c*) T-section, (*d*) T-section with compression steel.

It can be observed that almost all the shear force is resisted by the web, whereas the flange resists a very small percentage; in most practical problems, the shear capacity of the flange is neglected.

Referring to Fig. 8.1 and taking any portion of the beam dx, the bending moments at both ends of the element, M_1 and M_2 are not equal. Because $M_2 > M_1$ and to maintain the equilibrium of the beam portion dx, the compression force C_2 must be greater than C_1 (Fig. 8.4). Consequently, a shear stress v develops along any horizontal section $a\text{-}a_1$ or $b\text{-}b_1$ (Fig. 8.4*a*). The normal and shear stresses on a small element at levels $a\text{-}a_1$ and $b\text{-}b_1$ are shown in Fig. 8.4*b*. Notice that the normal stress at the level of the neutral axis $b\text{-}b_1$ is 0, whereas the shear stress is maximum. The horizontal shear stress is equal to the vertical shear stress, as shown in Fig. 8.4*b*. When the normal stress f is 0 or low, a case of pure shear may occur. In this case, the maximum tensile stress f_t, acts at 45° (Fig. 8.4*c*).

The tensile stresses are equivalent to the principal stresses, as shown in Fig. 8.4*d*. Such principal stresses are traditionally called *diagonal tension stresses*. When the diagonal tension stresses reach the tensile strength of concrete, a diagonal crack develops. This brief analysis explains the concept of diagonal tension and diagonal cracking. The actual behavior is more complex, and it is affected by other factors, as explained later. For the combined action of shear and normal stresses at any point in a beam, the maximum and minimum diagonal tension (principal stresses) f_p are given by the equation

$$f_p = \frac{f}{2} \pm \sqrt{\left(\frac{f}{2}\right)^2 + v^2} \tag{8.2}$$

where

f = intensity of normal stress due to bending
v = shear stress

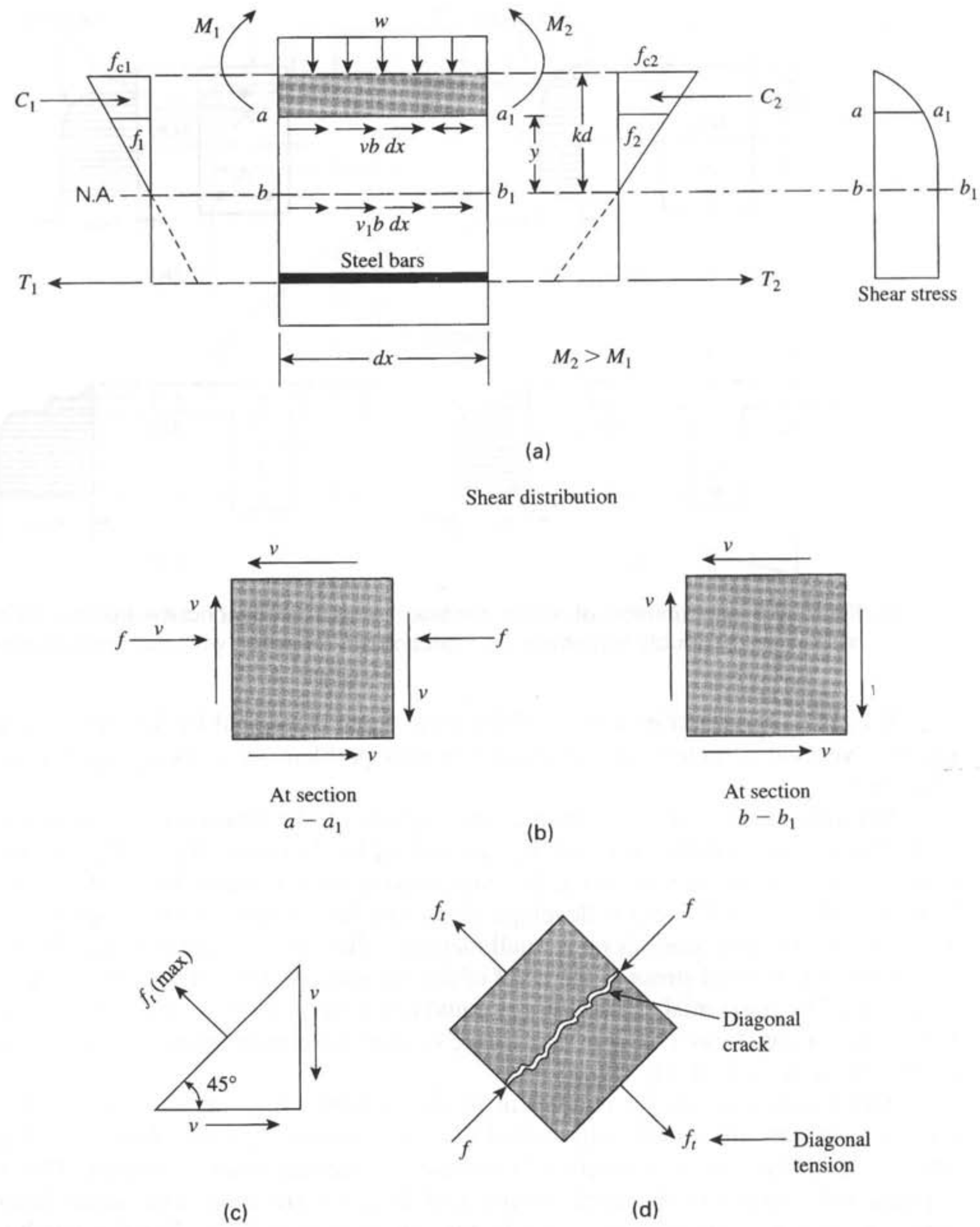


Figure 8.4 (*a*) Forces and stresses along the depth of the section, (*b*) normal and shear stresses, (*c*) pure shear, and (*d*) diagonal tension.

The shear failure in a concrete beam is most likely to occur where shear forces are maximum, generally near the supports of the member. The first evidence of impending failure is the formation of diagonal cracks.

8.3 BEHAVIOR OF BEAMS WITHOUT SHEAR REINFORCEMENT

Concrete is weak in tension, and the beam will collapse if proper reinforcement is not provided. The tensile stresses develop in beams due to axial tension, bending, shear, torsion, or a combination of these forces. The location of cracks in the concrete beam depends on the direction of principal stresses. For the combined action of normal stresses and shear stresses, maximum diagonal tension may occur at about a distance d from the face of the support.

The behavior of reinforced concrete beams with and without shear reinforcement tested under increasing load was discussed in Section 3.3. In the tested beams, vertical flexural cracks developed at the section of maximum bending moment when the tensile stresses in concrete exceeded the modulus of rupture of concrete, or $f_r = 7.5\lambda\sqrt{f'_c}$. Inclined cracks in the web developed at a later stage at a location very close to the support.

An inclined crack occurring in a beam that was previously uncracked is generally referred to as a *web-shear crack*. If the inclined crack starts at the top of an existing flexural crack and propagates into the beam, the crack is referred to as *flexural-shear crack* (Fig. 8.5). Web-shear cracks occur in beams with thin webs in regions with high shear and low moment. They are relatively uncommon cracks and may occur near the inflection points of continuous beams or adjacent to the supports of simple beams.

Flexural-shear cracks are the most common type found in reinforced concrete beams. A flexural crack extends vertically into the beam; then the inclined crack forms, starting from the top of the beam when shear stresses develop in that region. In regions of high shear stresses, beams must be reinforced by stirrups or by bent bars to produce ductile beams that do not rupture at a failure. To avoid a shear failure before a bending failure, a greater factor of safety must be provided against a shear failure. The ACI Code specifies a capacity reduction factor, ϕ, of 0.75 for shear.

Shear resistance in reinforced concrete members is developed by a combination of the following mechanisms [2] (Fig. 8.5):

- Shear resistance of the uncracked concrete, V_z [3]
- Interface shear transfer, V_a, due to aggregate interlock tangentially along the rough surfaces of the crack [3]
- Arch action [4]
- Dowel action, V_d, due to the resistance of the longitudinal bars to the transverse shearing force [5]

In addition to these forces, shear reinforcement increases the shear resistance V_s, by which depends on the diameter and spacings of stirrups used in the concrete member. If shear reinforcement is not provided in a rectangular beam, the proportions of the shear resisted by the various mechanisms are 20% to 40% by V_z 35% to 50% by V_a and 15% to 25% by V_d [6].

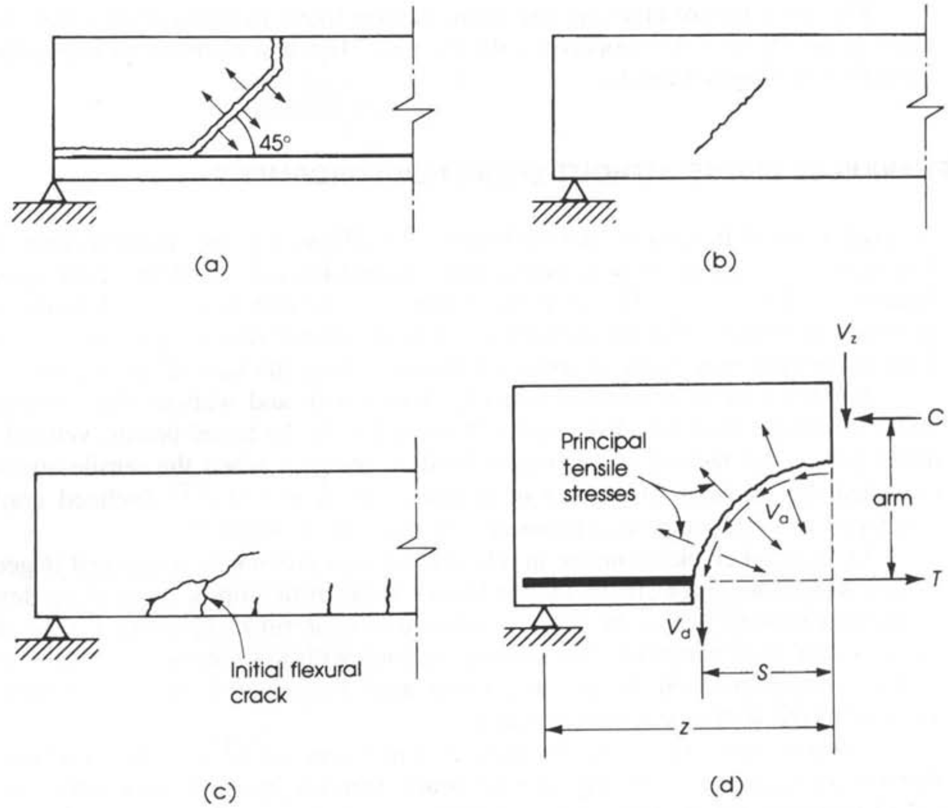


Figure 8.5 Shear failure: (*a*) general form, (*b*) web-shear crack, (*c*) flexural-shear crack, (*d*) analysis of forces involved in shear. V_a is interface shear, V_z is shear resistance, and V_d is dowel force.

8.4 MOMENT EFFECT ON SHEAR STRENGTH

In simply supported beams under uniformly distributed load, the midspan section is subjected to a large bending moment and zero or small shear, whereas sections near the ends are subjected to large shear and small bending moments (Fig. 8.1). The shear and moment values are both high near the intermediate supports of a continuous beam. At a location of large shear force and small bending moment, there will be little flexural cracking, and an average stress v is equal to V/bd. The diagonal tensile stresses are inclined at about 45° (Fig. 8.4*c*). Diagonal cracks can be expected when the diagonal tensile stress in the vicinity of the neutral axis reaches or exceeds the tensile strength of concrete. In general, the factored shear strength varies between $3.5\sqrt{f_c'}$ and $5\sqrt{f_c'}$. After completing a large number of beam tests on shear and diagonal tension [1], it was found that in regions with large shear and small moment, diagonal tension cracks were formed at an average shear force of

$$V_c = 3.5\sqrt{f_c'}b_w d \tag{8.3}$$

where b_w is the width of the web in a T-section or the width of a rectangular section and d is the effective depth of the beam.

In locations where shear forces and bending moments are high, flexural cracks are formed first. At a later stage, some cracks bend in a diagonal direction when the diagonal tension stress at the upper end of such cracks exceeds the tensile strength of concrete. Given the presence of large moments on a beam, for which adequate reinforcement is provided, the nominal shear force at which diagonal tension cracks develop is given by

$$V_c = 1.9\lambda\sqrt{f_c'}b_w d \tag{8.4}$$

This value is a little more than half the value in Eq. 8.3 when bending moment is very small. This means that large bending moments reduce the value of shear stress for which cracking occurs. The following equation has been suggested to predict the nominal shear stress at which a diagonal crack is expected [1]:

$$v_c = \frac{V}{b_w d} = \left(1.9\lambda\sqrt{f_c'} + 2500\rho\frac{Vd}{M}\right) \le 3.5\lambda\sqrt{f_c'} \tag{8.5}$$

1. ACI Code, Section 11.2.2.1, adopted this equation for the nominal shear force to be resisted by concrete for members subjected to shear and flexure only by:

$$V_c = \left[1.9\lambda\sqrt{f_c'} + 2500\rho_w\frac{V_u d}{M_u}\right] b_w d \le 3.5\lambda\sqrt{f_c'}b_w d \tag{8.6}$$

where $\rho_w = A_s/b_w$, d and b_w are the web width in a T-section or the width of a rectangular section, and V_u and M_u are the factored shearing force and bending moment occurring simultaneously on the considered section.

The value of $V_u d/M_u$ must not exceed 1.0 in Eq. 8.6. If M_u is large in Eq. 8.6, the second term becomes small and v_c approaches $1.9\lambda\sqrt{f_c'}$. If M_u is small, the second term becomes large and the upper limit of $3.5\lambda\sqrt{f_c'}$ controls. As an alternative to Eq. 8.6, the ACI Code, Section 11.2.1.1, permits evaluating the shear strength of concrete as follows:

$$V_c = 2\lambda\sqrt{f_c'}b_w d \tag{8.7}$$

$$V_c = 0.17\lambda\sqrt{f_c'}b_w d \quad \text{(SI)}$$

2. For members subjected to significant axial compression force N_u,

$$V_c = \left(1.9\lambda\sqrt{f_c'} + 2500\rho_w\frac{V_u d}{M_m}\right) b_w d \tag{8.8}$$

$$M_m = M_u - N_u\left(\frac{4h - d}{8}\right)$$

where

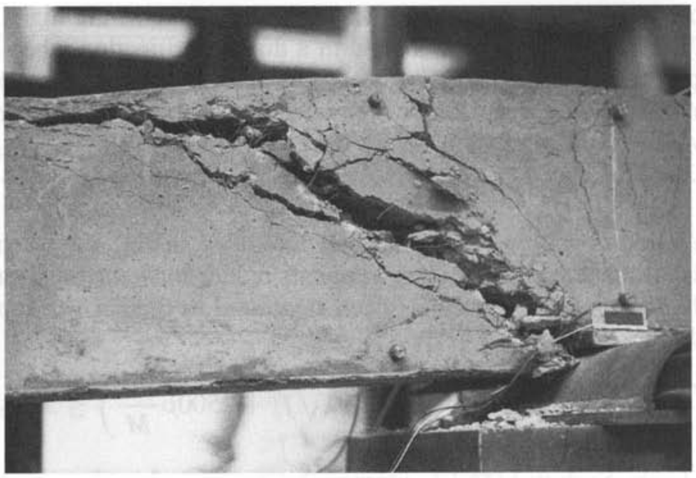
$$\rho_w = \frac{A_s}{b_w d}$$

h = overall depth

$V_u d/M_u$ may be greater than 1.0, but V_c must not exceed

$$V_c = 3.5\lambda\sqrt{f_c'}b_w d\sqrt{1 + \frac{N_u}{500A_g}} \tag{8.9}$$

where A_g is the gross area in.^{2}



Shear failure near a middle support.

Alternatively, V_c may be computed by

$$V_c = b_w d \left(2 + 0.001\frac{N_u}{A_g}\right)\lambda\sqrt{f_c'} \qquad (8.10)$$

3. In the case of members subjected to significant axial tensile force N_u,

$$V_c = b_w d \left(2 + 0.004\frac{N_u}{A_g}\right)\lambda\sqrt{f_c'} \qquad (8.11)$$

where N_u is to be taken as negative for tension and N_u/A_g is in psi. If V_c is negative, V_c should be taken equal to zero.

8.5 BEAMS WITH SHEAR REINFORCEMENT

Different types of shear reinforcement may be used:

1. Stirrups, which can be placed either perpendicular to the longitudinal reinforcement or inclined, usually making a 45° angle and welded to the main longitudinal reinforcement. Vertical stirrups, using no. 3 or no. 4 U-shaped bars, are the most commonly used shear reinforcement in beams (Fig. 8.6*a*).
2. Bent bars, which are part of the longitudinal reinforcement, bent up (where they are no longer needed) at an angle of 30° to 60°, usually at 45°.
3. Combinations of stirrups and bent bars.
4. Welded wire fabric with wires perpendicular to the axis of the member.
5. Spirals, circular ties, or hoops in circular sections, as columns.

The shear strength of a reinforced concrete beam is increased by the use of shear reinforcement. Prior to the formation of diagonal tension cracks, shear reinforcement contributes very little to the shear resistance. After diagonal cracks have developed, shear reinforcement augments the shear resistance of a beam, and a redistribution of internal forces occurs at the cracked section.

When the amount of shear reinforcement provided is small, failure due to yielding of web steel may be expected, but if the amount of shear reinforcement is too high, a shear-compression failure may be expected, which should be avoided.

Concrete, stirrups, and bent bars act together to resist transverse shear. The concrete, by virtue of its high compressive strength, acts as the diagonal compression member of a lattice girder system, where the stirrups act as vertical tension members. The diagonal compression force is such that its vertical component is equal to the tension force in the stirrup. Bent-up reinforcement acts also as tension members in a truss, as shown in Fig. 8.6.

In general, the contribution of shear reinforcement to the shear strength of a reinforced concrete beam can be described as follows [2]:

1. It resists part of the shear, V_s.
2. It increases the magnitude of the interface shear, V_a (Fig. 8.5), by resisting the growth of the inclined crack.

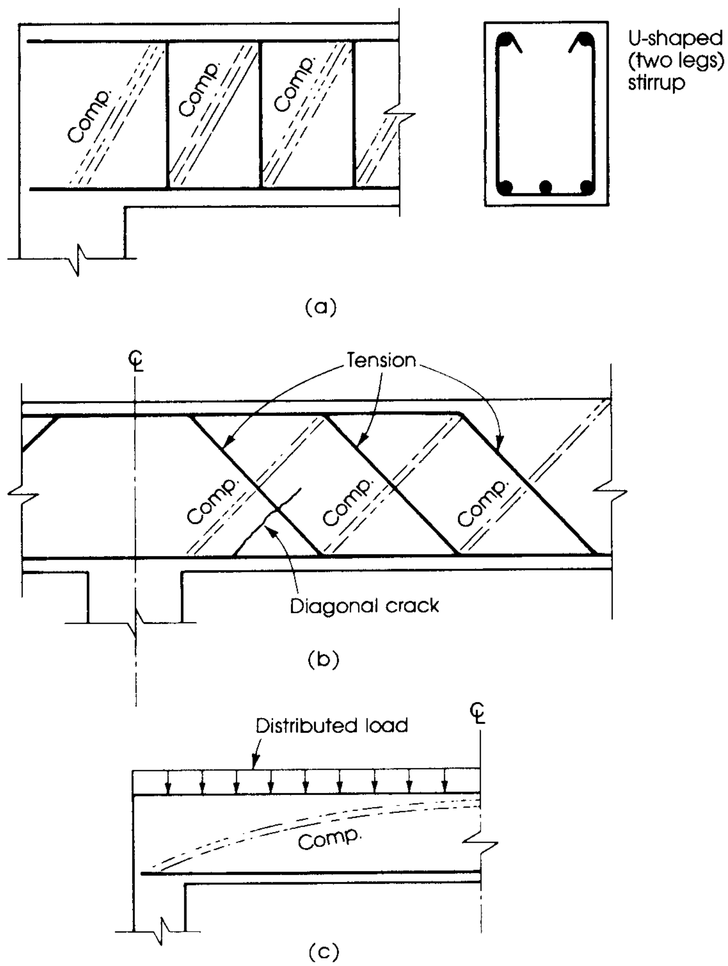


Figure 8.6 Truss action of web reinforcement with (*a*) stirrups, (*b*) bent bars, and (*c*) tension steel.

3. It increases the dowel force, V_d (Fig. 8.5), in the longitudinal bars.
4. The confining action of the stirrups on the compression concrete may increase its strength.
5. The confining action of stirrups on the concrete increases the rotation capacity of plastic hinges that develop in indeterminate structures at ultimate load and increases the length over which yielding takes place [7].

The total nominal shear strength of beams with shear reinforcement V_n is due partly to the shear strength attributed to the concrete V_c and partly to the shear strength contributed by the shear reinforcement V_s:

$$V_n = V_c + V_s \tag{8.12}$$

The shear force V_u produced by factored loads must be less than or equal to the total nominal shear strength V_n or

$$V_u \le \phi V_n = \phi(V_c + V_s) \tag{8.13}$$

where $V_u = 1.2V_D + 1.6V_L$ and $\phi = 0.75$.

An expression for V_s may be developed from the truss analogy (Fig. 8.7). For a 45° crack and a series of inclined stirrups or bent bars, the vertical shear force V_s resisted by shear reinforcement is equal to the sum of the vertical components of the tensile forces developed in the inclined bars. Therefore,

$$V_s = nA_v f_{yt} \sin\alpha \tag{8.14}$$

where A_v is the area of shear reinforcement with a spacing s, and f_{yt} is the yield strength of shear reinforcement, ns is defined as the distance aa_1a_2:

$$d = a_1a_4 = aa_1 \tan 45^\circ \qquad \text{(from triangle } aa_1a_4\text{)}$$

$$d = a_1a_4 = a_1a_2 \tan\alpha \qquad \text{(from triangle } a_1a_2a_4\text{)}$$

$$\begin{aligned} ns &= aa_1 + a_1a_2 \\ &= d(\cot 45^\circ + \cot\alpha) = d(1 + \cot\alpha) \end{aligned}$$

$$n = \frac{d}{S}(1 + \cot\alpha)$$

Substituting this value in Eq. 8.14 gives

$$V_s = \frac{A_v f_{yt} d}{S} \sin\alpha(1 + \cot\alpha) = \frac{A_v f_{yt} d}{S}(\sin\alpha + \cos\alpha) \tag{8.15}$$

For the case of vertical stirrups, $\alpha = 90^\circ$ and

$$V_s = \frac{A_v f_{yt} d}{S} \quad \text{or } s = \frac{A_v f_{yt} d}{V_s} \tag{8.16}$$

In the case of T-sections, b is replaced by the width of web b_w in all shear equations. When $\alpha = 45^\circ$, Eq. 8.15 becomes

$$V_s = 1.4\left(\frac{A_v f_{yt} d}{S}\right) \text{ or } s = \frac{1.4\ A_v f_{yt} d}{V_s} \tag{8.17}$$

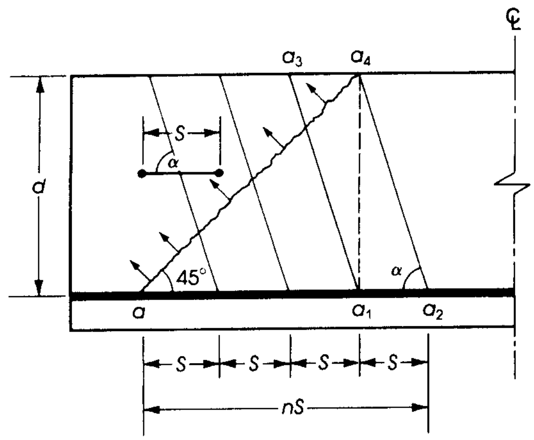


Figure 8.7 Factors in inclined shear reinforcement.

For a single bent bar or group of parallel bars in one position, the shearing force resisted by steel is

$$V_s = A_v f_{yt} \sin\alpha \text{ or } A_v = \frac{V_s}{f_{yt}\sin\alpha} \tag{8.18}$$

For $\alpha = 45^\circ$,

$$A_v = 1.4\left(\frac{V_s}{f_{yt}}\right) \tag{8.19}$$

For circular sections, mainly in columns, V_s shall be computed from Eq. 8.16 using $d = 0.8 \times$ diameter and $A_v =$ two times the area of the bar in a circular tie, hoop, or spiral.

8.6 ACI CODE SHEAR DESIGN REQUIREMENTS

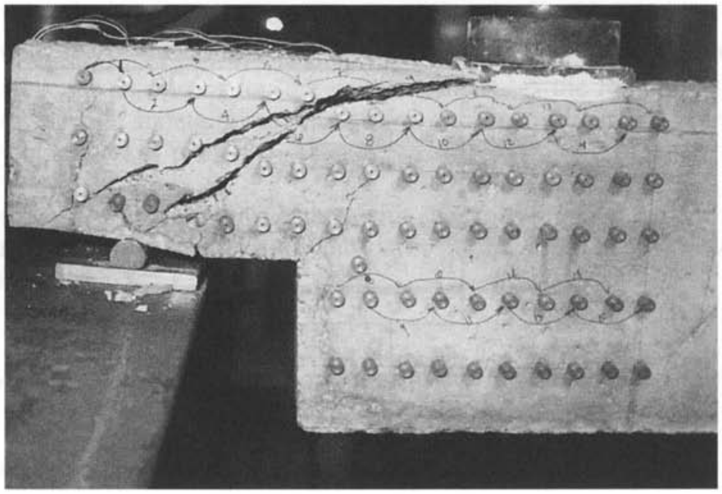
8.6.1 Critical Section for Nominal Shear Strength Calculation

The ACI Code, Section 11.1.3, permits taking the critical section for nominal shear strength calculation at a distance d from the face of the support. This recommendation is based on the fact that the first inclined crack is likely to form within the shear span of the beam at some distance d away from the support. The distance d is also based on experimental work and appeared in the testing of the beams discussed in Chapter 3. This critical section is permitted on the condition that the support reaction introduces compression into the end region, loads are applied at or near the top of the member, and no concentrated load occurs between the face of the support and the location of the critical section.

The Code also specifies that shear reinforcement must be provided between the face of the support and the distance d, using the same reinforcement adopted for the critical section.

8.6.2 Minimum Area of Shear Reinforcement

The presence of shear reinforcement in a concrete beam restrains the growth of inclined cracking. Moreover, ductility is increased, and a warning of failure is provided. If shear reinforcement is



Shear failure in dapped-end beam.

not provided, brittle failure will occur without warning. Accordingly, a minimum area of shear reinforcement is specified by the Code. The ACI Code, Section 11.4.6 requires all stirrups to have a minimum shear reinforcement area, A_v, equal to

$$A_v = 0.75\sqrt{f_c'}\left(\frac{b_w s}{f_{yt}}\right) \geq \frac{50 b_w s}{f_{yt}} \tag{8.20}$$

where b_w is the width of the web and s is the spacing of the stirrups. The minimum amount of shear reinforcement is required whenever V_u exceeds $\phi V_c/2$, except in

- Slabs and footings
- Concrete floor joist construction
- Beams where the total depth does not exceed 10 in., 2.5 times the flange thickness for T-shaped flanged sections, or one-half the web width, whichever is greatest.

If $0.75\sqrt{f_c'} = 50$ then $f_c' = 4444$ psi. This means that when $f_c' < 4500$ psi, the minimum $A_v = 50 b_w s/f_{yt}$ controls and when $f_c' \geq 4500$ psi, then the minimum $A_v = 0.75\sqrt{f_c'}(b_w s/f_{yt})$ controls. This increase in the minimum area of shear reinforcement for $f_c' \geq 4500$ psi is to prevent sudden shear failure when inclined cracking occurs.

It is a common practice to increase the depth of a slab, footing, or shallow beam to increase its shear capacity. Stirrups may not be effective in shallow members, because their compression zones have relatively small depths and may not satisfy the anchorage requirements of stirrups. For beams that are not shallow, reinforcement is not required when V_u is less than $\phi V_c/2$.

The minimum shear reinforcement area can be achieved by using no. 3 stirrups placed at maximum spacing, S_{max}. If $f_y = 60$ ksi and U-shaped (two legs) no. 3 stirrups are used, then Eq. 8.20 becomes

$$S_{max} = \frac{A_v f_{yt}}{(0.75\sqrt{f_c'})b_w} \leq \frac{A_v f_{yt}}{50 b_w} \tag{8.21}$$

$$\begin{aligned}
&\text{For } f'_c < 4500 \text{ psi}, && S_{\max}(\text{in.}) = 0.22(60{,}000)/50b_w = 264/b_w.\\
&\text{For } f'_c = 4500 \text{ psi}, && S_{\max}(\text{in.}) = 262/b_w.\\
&\text{For } f'_c = 5000 \text{ psi}, && S_{\max}(\text{in.}) = 249/b_w.\\
&\text{For } f'_c = 6000 \text{ psi}, && S_{\max}(\text{in.}) = 227/b_w.
\end{aligned} \tag{8.22}$$

If U-shaped no. 4 stirrups are used, then for $f'_c < 4500$ psi,

$$\begin{aligned}
&S_{\max}\ (\text{in.}) = \frac{0.4(60{,}000)}{50b_w} = \frac{480}{b_w}\\
&\text{For } f'_c = 4500 \text{ psi}, \quad S_{\max}(\text{in.}) = 476/b_w.\\
&\text{For } f'_c = 5000 \text{ psi}, \quad S_{\max}(\text{in.}) = 453/b_w.\\
&\text{For } f'_c = 6000 \text{ psi}, \quad S_{\max}(\text{in.}) = 413/b_w.
\end{aligned} \tag{8.23}$$

Note that $S_{\max}$ shall not exceed 24 in., nor $d/2$.

Table 8.1 gives $S_{\max}$ based on Eqs. 8.22 and 8.23. Final spacings should be rounded to the lower inch. For example, $S = 20.3$ in. becomes 20 in.

8.6.3 Maximum Shear Carried by Web Reinforcement V_s

To prevent a shear-compression failure, where the concrete may crush due to high shear and compressive stresses in the critical region on top of a diagonal crack, the ACI Code, Section 11.4.7.9, requires that V_s shall not exceed $(8\sqrt{f'_c})b_w d$. If V_s exceeds this value, the section should be increased. Based on this limitation,

If $f'_c = 3$ ksi, then $V_s \le 0.438 b_w d$ (kips) or $V_s/b_w d \le 438$ psi.

If $f'_c = 4$ ksi, then $V_s \le 0.506 b_w d$ (kips) or $V_s/b_w d \le 506$ psi.

If $f'_c = 5$ ksi, then $V_s \le 0.565 b_w d$ (kips) or $V_s/b_w d \le 565$ psi.

8.6.4 Maximum Spacing of Stirrups

To ensure that a diagonal crack will always be intersected by at least one stirrup, the ACI Code, Section 11.4.5, requires that the spacings between stirrups shall not exceed $d/2$, nor 24 in., provided that $V_s \le (4\sqrt{f'_c})b_w d$. This is based on the assumption that a diagonal crack develops at 45° and extends a horizontal distance of about d. In regions of high shear, where V_s exceeds $(4\sqrt{f'_c})b_w d$, the maximum spacing between stirrups must not exceed $d/4$. This limitation is necessary to ensure that the diagonal crack will be intersected by at least three stirrups. When V_s exceeds the maximum value of $8\sqrt{f'_c}b_w d$, this limitation of maximum stirrup spacing does not apply, and the dimensions of the concrete cross-section should be increased.

Table 8.1 Values of $S_{\max} = (A_v f_y/50b_w) = 24$ in. when $f_{yt} = 60$ ksi and $f'_c < 4500$ psi

b_w (in.)	10	11	12	13	14	15	16	18	20	22	24	b_w
$S_{\max}$ (in.) no. 3 stirrups	24	24	22	20.3	18.9	17.6	16.5	14.7	13.2	12	11	$\frac{264}{b_w}$
S_{max} (in.) no. 4 stirrups	24	24	24	24	24	24	24	24	24	21.8	20	$\frac{480}{b_w}$

A second limitation for the maximum spacing of stirrups may also be obtained from the condition of minimum area of shear reinforcement. A minimum A_v is obtained when the spacing s is maximum (Eq. 8.21).

A third limitation for maximum spacing is 24 in. $V_s \leq (4\sqrt{f_c'})b_w d$ and 12 in. when V_s is greater than $(4\sqrt{f_c'})b_w d$ but less than or equal to $(8\sqrt{f_c'})b_w d$. The least value of all maximum spacings must be adopted. The ACI Code maximum spacing requirement ensures closely spaced stirrups that hold the longitudinal tension steel in place within the beam, thereby increasing their dowel capacity, V_d (Fig. 8.5).

8.6.5 Yield Strength of Shear Reinforcement

The ACI Code, Section 11.4.2, requires that the design yield strength of shear reinforcement shall not exceed 60 ksi (420 MPa). The reason behind this decision is to limit the crack width caused by the diagonal tension and to ensure that the sides of the crack remain in close contact to improve the interface shear transfer, V_a (Fig. 8.5). For welded deformed wire fabric, the design yield strength shall not exceed 80 ksi (560 MPa).

8.6.6 Anchorage of Stirrups

The ACI Code, Section 12.13.1, requires that shear reinforcement be carried as close as possible to the compression and tension extreme fibers, within the Code requirements for concrete cover, because near ultimate load the flexural tension cracks penetrate deep into the beam. Also, for stirrups to achieve their full yield strength, they must be well anchored. Near ultimate load, the stress in a stirrup reaches its yield stress at the point where a diagonal crack intercepts that stirrup. The ACI Code requirements for stirrup anchorage, Section 12.13, are as follows:

1. Each bend in the continuous portion of a simple U-stirrup or multiple U-stirrup shall enclose a longitudinal bar (ACI Code, Section 12.13.3). See Fig. 8.8*a*.
2. The code allows the use of a standard hook of 90°, 135°, or 180° around longitudinal bars for no. 5 bars or D31 wire stirrups. If no. 6, 7, or 8 stirrups with $f_{yt} > 40$ ksi are used, the Code (Section 12.13.2) requires a standard hook plus an embedment length of $0.014 d_b f_{yt}/(\lambda\sqrt{f_c'})$ between midheight of the member and the outside of the hook. If the bars are bent at 90°, extensions shall not be less than $12d_b$. For no. 5 bars or smaller stirrups, the extension is $6d_b$ (ACI Code, Section 7.1.3). See Fig. 8.8*b*.
3. If spliced double U-stirrups are used to form closed stirrups, the lap length shall not be less than $1.3l_d$ (ACI Code, Section 12.13.5). See Fig. 8.8*c*.
4. Welded wire fabric is used for shear reinforcement in the precast industry. Anchorage details are given in the ACI Code, Section 12.13.2.3, and in its commentary.
5. Closed stirrups are required for beams subjected to torsion or stress reversals (ACI Code, Section 7.11).
6. Beams at the perimeter of the structure should contain closed stirrups to maintain the structural integrity of the member (ACI Code, Section 7.13.2.2).

8.6.7 Stirrups Adjacent to the Support

The ACI Code, Section 11.1.3, specifies that shear reinforcement provided between the face of the support and the critical section at a distance d from it may be designed for the same shear V_u at the critical section. It is common practice to place the first stirrup at a distance $S/2$ from the face of the support, where s is the spacing calculated by Eq. 8.16 for V_u at the critical section.

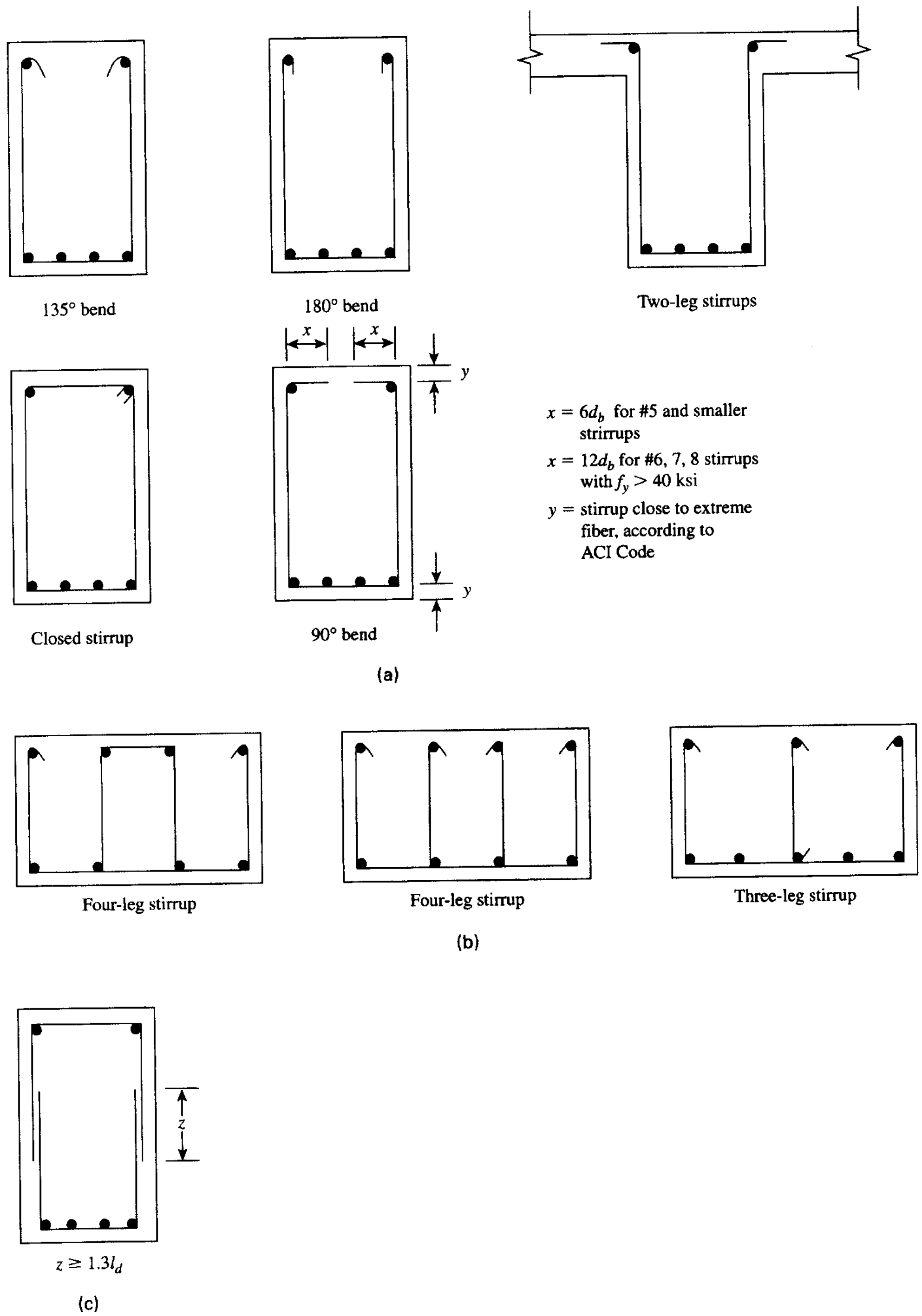


Figure 8.8 Stirrup types: (*a*) U-stirrups enclosing longitudinal bars, anchorage lengths, and closed stirrups, (*b*) multileg stirrups, and (*c*) spliced stirrups.

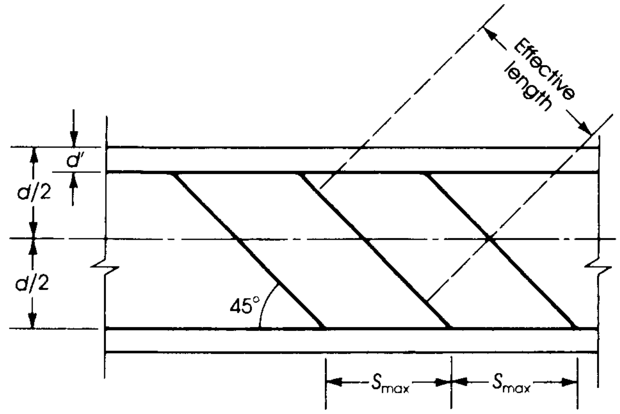


Figure 8.9 Effective length and spacing of bent bars.

8.6.8 Effective Length of Bent Bars

Only the center three-fourths of the inclined portion of any longitudinal bar shall be considered effective for shear reinforcement. This means that the maximum spacing of bent bars is $0.75(d - d')$. From Fig. 8.9, the effective length of the bent bar is $0.75(d - d')/(\sin 45°) = 0.75(1.414)(d - d') = 1.06(d - d')$. The maximum spacing S is equal to the horizontal projection of the effective length of the bent bar. Thus $S_{\max} = 1.06(d - d')\cos 45°$, or $S_{\max} = 0.707[1.06(d - d')] = 0.75(d - d')$.

8.7 DESIGN OF VERTICAL STIRRUPS

Stirrups are needed when $V_u > \frac{1}{2}\phi V_c$. Minimum stirrups are used when V_u is greater than $\frac{1}{2}\phi V_c$ but less than ϕV_c. This is achieved by using no. 3 stirrups placed at maximum spacing. When V_u is greater than ϕV_c stirrups must be provided. The spacing of stirrups may be less than the maximum spacing and can be calculated using Eq. 8.16: $S = A_v f_{yt} d / V_s$.

The stirrups that are commonly used in concrete sections are made of two-leg no. 3 or no. 4 U-stirrups with $f_{yt} = 60$ ksi. If no. 3 stirrups are used, then Eq. 8.16 becomes

$$\frac{S}{d} = \frac{A_v f_y}{V_s} = \frac{0.22(60)}{V_s} = \frac{13.2}{V_s} \tag{8.24}$$

If no. 4 stirrups are used, then

$$\frac{S}{d} = \frac{A_v f_y}{V_s} = \frac{0.4(60)}{V_s} = \frac{24}{V_s} \tag{8.25}$$

The ratio of stirrup spacings relative to the effective depth of the beam, d, depends on V_s. The values of S/d for different values of V_s when $f_y = 60$ ksi are given in Tables 8.2 and 8.3 for no. 3 and no. 4 U-stirrups, respectively. The same values are plotted in Figs. 8.10 and 8.11. The following observations can be made:

1. If no. 3 stirrups are used, $S = d/2$ when $V_s \leq 26.4$ K. When V_s increases, S/d decreases in a nonlinear curve to reach 0.132 at $V_s = 100$ K. If the minimum spacing is limited to

Table 8.2 S/d Ratio for Different Values of V_s (f_{yt} = 60 ksi, $S/d = 13.2/V_s$), No. 3 Stirrups

V_s (K)	26.4	30	40	50	52.8	60	70	80	90	100	125
S/d	0.5	0.44	0.33	0.264	0.25	0.22	0.19	0.165	0.15	0.132	0.106

Table 8.3 S/d Ratio for Different Values of V_s(f_{yt} = 60 ksi, $S/d = 24/V_s$), No. 4 Stirrups

V_s(K)	48	50	60	70	80	90	96	100	110	120	150	175
S/d	0.50	0.48	0.40	0.34	0.3	0.27	0.25	0.24	0.22	0.20	0.16	0.137

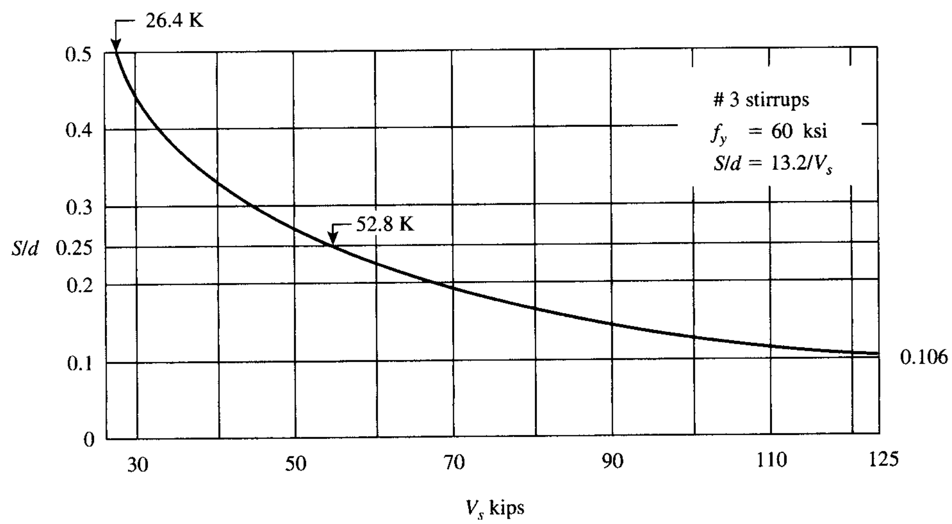


Figure 8.10 V_s versus S/d for no. 3 stirrups and f_{yt} = 60 ksi.

3 in., then d must be equal to or greater than 22.7 in. to maintain that 3-in. spacing. When V_s is equal to or greater than 52.8 K, then $S \leq d/4$.

2. If no. 4 U-stirrups are used, $S = d/2$ when $V_s \leq 48$ K. When V_s increases, S/d decreases to reach 0.16 at $V_s = 150$ K. If the minimum spacing is limited to 3 in., then $d \geq 18.75$ in. to maintain the 3-in. spacing. When V_s is equal to or greater than 96 K, then $S \leq d/4$.
3. If grade 40 U-stirrups are used (f_{yt} = 40 ksi), multiply the S/d values by $\frac{2}{3}$ or, in general, $f_{yt}/60$.

8.8 DESIGN SUMMARY

The design procedure for shear using vertical stirrups according to the ACI Code can be summarized as follows:

1. Calculate the factored shearing force, V_u, from the applied forces acting on the structural member. The critical design shear value is at a section located at a distance d from the face of the support.

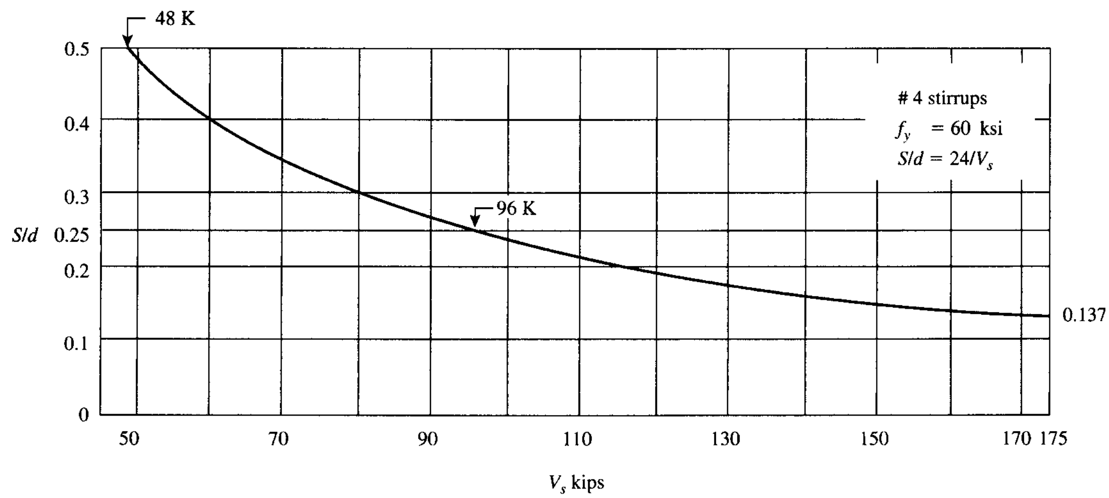


Figure 8.11 V_s versus S/d for no. 4 stirrups and f_{yt} = 60 ksi.

2. Calculate $\phi V_c = \phi 2\lambda\sqrt{f_c'}b_w d$, or

$$\phi V_c = \phi\left[1.9\lambda\sqrt{f_c'} + 2500\rho_w\frac{V_u d}{M_u}\right]b_w d \leq \phi 3.5\lambda\sqrt{f_c'}b_w d$$

   Then calculate $\frac{1}{2}\phi V_c$
3. **a.** If $V_u < \frac{1}{2}\phi V_c$, no shear reinforcement is needed.
   b. If $\frac{1}{2}\phi V_c < V_u \leq \phi V_c$, minimum shear reinforcement is required. Use no. 3 U-stirrups spaced at maximum spacings, as explained in step 7.
   c. If $V_u > \phi V_c$, shear reinforcement must be provided according to steps 4 through 8.
4. If $V_u > \phi V_c$, calculate the shear to be carried by shear reinforcement:

$$V_u = \phi V_c + \phi V_s \text{ or } V_s = \frac{V_u - \phi V_c}{\phi}$$

5. Calculate $V_{c_1} = (4\sqrt{f_c'})b_w d$ and $V_{c_2} = (8\sqrt{f_c'})b_w d = 2V_{c_1}$. Compare the calculated V_s with the maximum permissible value of $V_{c_2} = (8\sqrt{f_c'})b_w d$. If V_s is less than V_{c2}, proceed in the design; if not, increase the dimensions of the section.
6. Calculate the stirrup spacings based on the calculated $S_1 = A_v f_{yt} d/V_s$ or use Figs. 8.10 and 8.11 or Tables 8.2 and 8.3.
7. Determine the maximum spacing allowed by the ACI Code. The maximum spacing is the least of S_2 and S_3:
   a. $S_2 = d/2 \leq 24$ in. if $V_s <= V_{c_1} = (4\sqrt{f_c'})b_w d$.
   $S_2 = d/4 \leq 12$ in. if $V_{c_1} < V_s \leq V_{c_2}$.
   b. $S_3 = A_v f_{yt}/50b_w \geq A_v f_{yt}/(0.75\sqrt{f_c'}b_w)$
   S_{max} is the smaller of S_2 and S_3. Values of S_3 are shown in Table 8.1.
8. If S_1 calculated in step 6 is less than S_{max} (the smaller of S_2 and S_3), then use S_1 to the nearest smaller $\frac{1}{2}$ in. If $S_1 > S_{max}$, then use S_{max} as the adopted S.

9. The ACI Code did not specify a minimum spacing. Under normal conditions, a practical minimum S may be assumed to be equal to 3 in. for $d \leq 20$ in. and 4 in. for deeper beams. If S is considered small, either increase the stirrup bar number or use multiple-leg stirrups (Fig. 8.8).

10. For circular sections, the area used to compute V_c = diameter times the effective depth d, where $d = 0.8$ the diameter, ACI Code, Section 11.2.3.

Example 8.1

A simply supported beam has a rectangular section $b = 12$ in., $d = 21.5$ in., and $h = 24$ in. and is reinforced with four no. 8 bars. Check if the section is adequate for each of the following factored shear forces. If it is not adequate, design the necessary shear reinforcement in the form of U-stirrups. Use $f_c' = 4$ ksi and $f_{yt} = 60$ ksi. Assume normal-weight concrete.

a. $V_u = 12$ K **(b)** $V_u = 24$ K **(c)** $V_u = 54$ K **(d)** $V_u = 77$ K **(e)** $V_u = 128$ K

Solution

In general, $b_w = b = 12$ in., $d = 21.5$ in., and

$$\phi V_c = \phi(2\lambda\sqrt{f_c'})bd = 0.75(2)(1)(\sqrt{4000})(12)(21.5) = 24.5 \text{ K}$$

$$\tfrac{1}{2}\phi V_c = 12.25 \text{ K}$$

$$V_{c_1} = (4\sqrt{f_c'})bd = (4\sqrt{4000})(12)(21.5)/1000 = 65.3 \text{ K}$$

$$V_{c_2} = (8\sqrt{f_c'})bd = 130.6 \text{ K}$$

a. $V_u = 12 \text{ K} < \frac{1}{2}\phi V_c = 12.25$ K, section is adequate, and shear reinforcement is not required.

b. $V_u = 24 \text{ K} > \frac{1}{2}\phi V_c$, but it is less than $\phi V_c = 24.5$ K. Therefore, $V_s = 0$ and minimum shear reinforcement is required. Choose no. 3 U-stirrup (two legs) at maximum spacing. $A_y = 2(0.11) = 0.22$ in^2. Maximum spacing is the least of

$S_2 = d/2 = 21.5/2 = 10.75$ in., say, 10.5 in. (controls).

$S_3 = A_v f_{yt}/50b_w = 0.22(60{,}000)/50(12) = 22$ in. (or use Table 8.1)

$S_4 = 24$ in. Use no. 3 U-stirrups spaced at 10.5 in.

c. $V_u = 54 \text{ K} > \phi V_c$. Shear reinforcement is needed. Calculation may be organized in steps:
Calculate $V_s = (V_u - \phi V_c)/\phi = (54 - 24.5)/0.75 = 39.3$ K.
Check if $V_s \leq V_{c_1} = (4\sqrt{f_c'})b_w d = 65.3$ K. Because $V_s < 65.3$ K, then $S_{max} = d/2$, and the $d/4$ condition does not apply.
Choose no. 3 U-stirrups and calculate the required spacings based on V_s.

$$S_1 = \frac{A_v f_{yt} d}{V_s} = \frac{0.22(60)(21.5)}{39.3} = 7.26 \text{ in.} \qquad \text{say, 7 in.}$$

Calculate maximum spacings: $S_2 = 10.5$ in., $S_3 = 22$ in., and $S_4 = 24$ in. and maximum $S = 10.5$ in. (calculated in (b)).
Because $S = 7 \text{ in.} < S_{max} = 10.5$ in., then use no. 3 U-stirrups spaced at 7 in.

d. $V_u = 77 \text{ K} > \phi V_c$, so stirrups must be provided.
Calculate $V_s = (V_u - \phi V_c)/\phi = (77 - 24.5)/0.75 = 70$ K.
Check if $V_s \leq V_{c_1} = 4\sqrt{f_c'}b_w d = 65.3$ K. Because $V_s > 65.3$ K, then $S_{max} = d/4 = 12$ in. must be used.
Check if $V_s \leq V_{c_2} = 8\sqrt{f_c'}b_w d = 130.6$ K. Because $V_{c_1} < V_s < V_{c_2}$, then stirrups can be used without increasing the section.

Choose no. 3 U-stirrups and calculate S_1 based on V_s:

$$S_1 = \frac{A_v f_{yt} d}{V_s} = \frac{0.22(60)(21.5)}{70} = 4.1 \text{ in.,} \qquad \text{say, 4 in.}$$

Calculate maximum spacings: $S_2 = d/4 = 21.5/4 = 5.3$ in., say, 5.0 in.; $S_3 = 22$ in.; and $S_4 = 12$ in. Hence $S_{max} = 5$-in. controls.
Because $S = 4$ in. $< S_{max} = 5$ in., then use no. 3 stirrups spaced at 4 in.

e. $V_u = 128$ K $> \phi V_c$, so shear reinforcement is required.
Calculate $V_s = (V_u - \phi V_c)/\phi = (128 - 24.5)/0.75 = 138$ K.
Because $V_s > V_{c_2} = 130.2$ K, the section is not adequate. Increase one or both dimensions of the beam section.

Notes : Table 8.2 and Fig. 8.10 can be used to calculate the spacing S for (c) and (d).

1. For (c), $V_s = 39.3$ K, from Fig. 8.10 (or Table 8.2 for no. 3 U-stirrups), $S/d = 0.34$ and $S_1 = 7.3$ in., which is less than $d/2 = 10.5$ in. Note that S_{max} based on V_s is $d/2$ and not $d/4$. Also, from Table 8.1, $S_3 = A_v f_{yt}/50 b_w = 22$ in.
2. For (d), $V_s = 70$ K, $S/d = 0.19$ and $S_1 = 4.1$ in. $V_s = 70$ is greater than 52.8 K, and $S_{max} = d/4$ is required.

Example 8.2

A 17-ft-span simply supported beam has a clear span of 16 ft and carries uniformly distributed dead and live loads of 4.5 K/ft and 3.75 K/ft, respectively. The dimensions of the beam section and steel reinforcement are shown in Fig. 8.12. Check the section for shear and design the necessary shear reinforcement. Given $f'_c = 3$ ksi normal-weight concrete and $f_{yt} = 60$ ksi.

Solution

Given b_w (web) = 14 in., $d = 22.5$ in.

1. Calculate factored shear from external loading:

   factored uniform load = 1.2(4.5) + 1.6(3.75) = 11.4K/ft

   $$V_u(\text{at face of support}) = \frac{11.4(16)}{2} = 91.2\text{K}$$

   Design V_u (at d distance from the face of support) = 91.2 − 22.5(11.4)/12 = 69.83 K.
2. Calculate ϕV_c:

   $$\phi V_c = \phi(2\lambda\sqrt{f'_c})b_w d = \frac{0.75(2)(1)(\sqrt{3000})(14)(22.5)}{1000} = 25.88 \text{ K}$$

   $$\tfrac{1}{2}\phi V_c = 12.94 \text{ K}$$

   Calculate $V_{c_1} = (4\sqrt{f'_c})b_w d = (4\sqrt{3000})(14)(22.5)/1000 = 69$ K. Calculate $V_{c_2} = (8\sqrt{f'_c})b_w d = 138$ K.
3. Design $V_u = 69.83$ K $> \phi V_c = 25.88$ K; therefore, shear reinforcement must be provided. The distance x' at which no shear reinforcement is needed (at $\frac{1}{2}\phi V_c$) is

   $$x' = \left(\frac{91.2 - 12.94}{91.2}\right)(8) = 6.86 \text{ ft} = 82 \text{ in.}$$

   (from the triangles of shear diagram, Fig. 8.12).

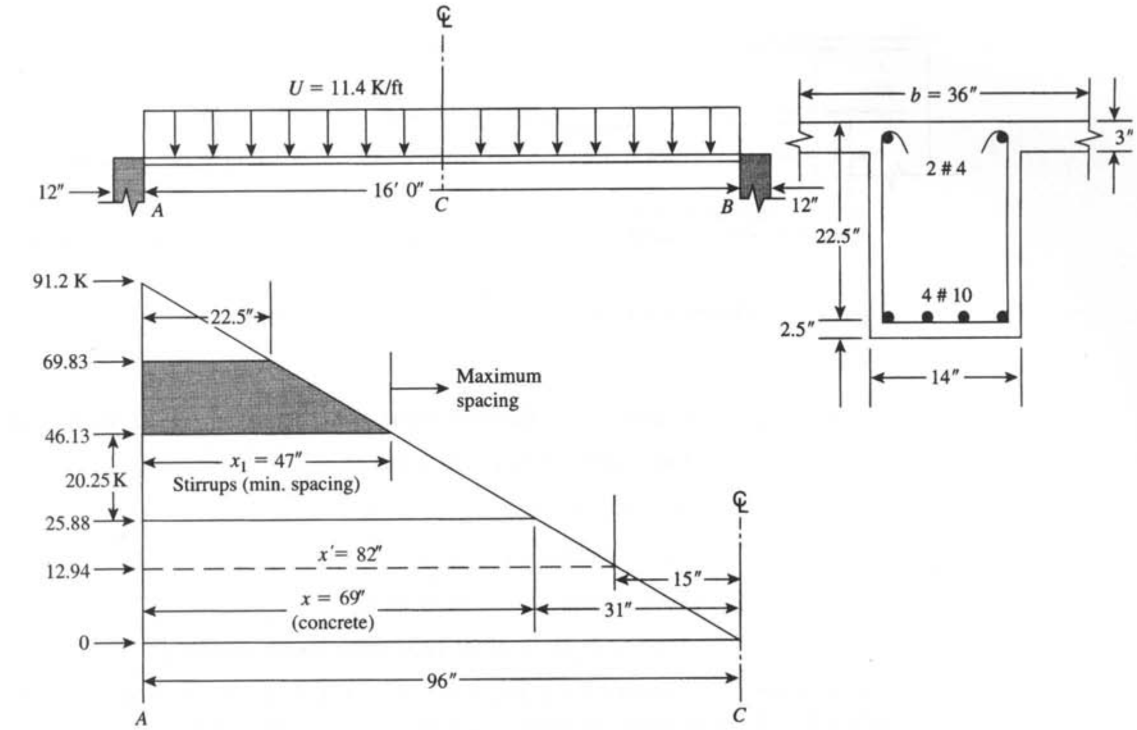


Figure 8.12 Example 8.2.

4. Calculate $V_s = (V_u - \phi V_c)/\phi = (69.83 - 25.88)/0.75 = 58.6$ K. Because V_s is less than $V_{c_1} = (4\sqrt{f_c'})b_w d$, then $S_{\max} = d/2$ must be considered (or refer to Fig. 8.10 or Table 8.2: $V_s < 52.8$ K).
5. Design of stirrups: Choose no. 3 U-stirrups, $A_v = 2(0.11) = 0.22\,\text{in.}^2$ Calculate S_1 based on $V_s = 58.6$ K, $S_1 = A_v f_y d/V_s = 13.2\ d/V_s = 5.07$ in., say, 5 in. (or get $s/d = 0.225$ from Table 8.2 or Fig. 8.10).
6. Calculate maximum spacings: $S_2 = d/2 = 22.5/2 = 11.25$ in., say, 11.0 in.; $S_3 = A_v f_{yt}/50b_w = 0.22(60{,}000)/50(14) = 18.9$ in. (or use Table 8.1); $S_4 = 24$ in.; $S_{\max} = 11$ in. controls.
7. Because $S_1 = 5\,\text{in.} < S_{\max} = 11$ in., use no. 3 U-stirrups spaced at 5 in.
8. Calculate V_s for maximum spacings of 11 in.:

$$V_s = \frac{A_v f_{yt} d}{S} = \frac{0.22(60)(22.5)}{11} = 27\text{ K}$$

$$\phi V_s = 20.25\text{ K}$$

$$\phi V_c + \phi V_s = 25.88 + 20.25 = 46.13\text{ K}$$

The distance x_1 at which $S = 11$ can be used is

$$\left(\frac{91.2 - 46.13}{91.2}\right)(96) = 47\text{ in.}$$

Because x_1 is relatively small, use $S = 5$ in. for a distance greater than or equal to 47 and then use $S = 11$ for the rest of the beam. *Note*: If x_1 is long, then an intermediate spacing between 5 in. and 11 in. may be added.

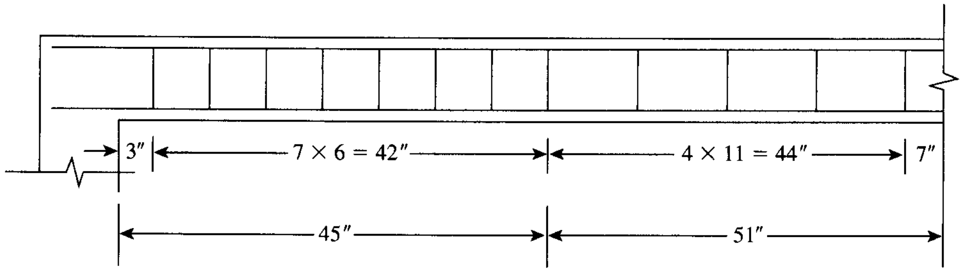


Figure 8.13 Example 8.2: distribution of stirrups.

9. Distribute stirrups as follows: Place the first stirrup at $S/2$ from the face of the support.

First stirrup at $S/2 = 5/2 = 2$ in.

Nine stirrups at $S = 5 = 45$ in.

Total $= 45 + 2$ in. $= 47$ in.

Four stirrups at $S = 11 = 44$ in.

Total $= 91$ in. > 82 in. (minimum length required)

The total number of stirrups for the beam is $2(1 + 9 + 4) = 28$. Distribution of stirrups is shown in Fig. 8.13, whereas calculated shear forces are shown in Fig. 8.12.

10. Place two no. 4 bars at the top of beam section to act as stirrup hangers.

8.9 SHEAR FORCE DUE TO LIVE LOADS

In Example 8.2, it was assumed that the dead and live loads are uniformly distributed along the full span, producing zero shear at midspan. Actually, the dead load does exist along the full span, but the live load may be applied to the full span or part of the span, as needed to develop the maximum shear at midspan or at any specific section. Figure 8.14*a* shows a simply supported beam with a uniform load acting on the full span. The shear force varies linearly along the beam, with maximum shear acting at support A.

In the case of live load, $W_2 = 1.6W_L$, the maximum shear force acts at support A when W_2 is applied on the full span, Fig. 8.14*a*. The maximum shear at midspan develops if the live load is placed on half the beam, BC (Fig. 8.14*b*), producing V_u at midspan equal to $W_2L/8$. Consequently, the design shear force is produced by adding the maximum shear force due to live load (placed at different lengths of the span) to the dead load shear force (Fig. 8.14*c*) to give the shear distribution shown in Fig. 8.14*d*. It is a common practice to consider the maximum shear at support A to be $W_uL/2 = (1.2W_D + 1.6W_L)L/2$, whereas V_u at midspan is $W_2L/8 = (1.6W_L)L/8$ with a straight-line variation along AC and CB, as shown in Fig. 8.14*d*. The design for shear in this case will follow the same procedure explained in Example 8.2. If the approach is applied to the beam in Example 8.2, then V_u (at A) $= 91.2$ K and V_u (at midspan) $= (1.6 \times 3)(16/8) = 10$ K.

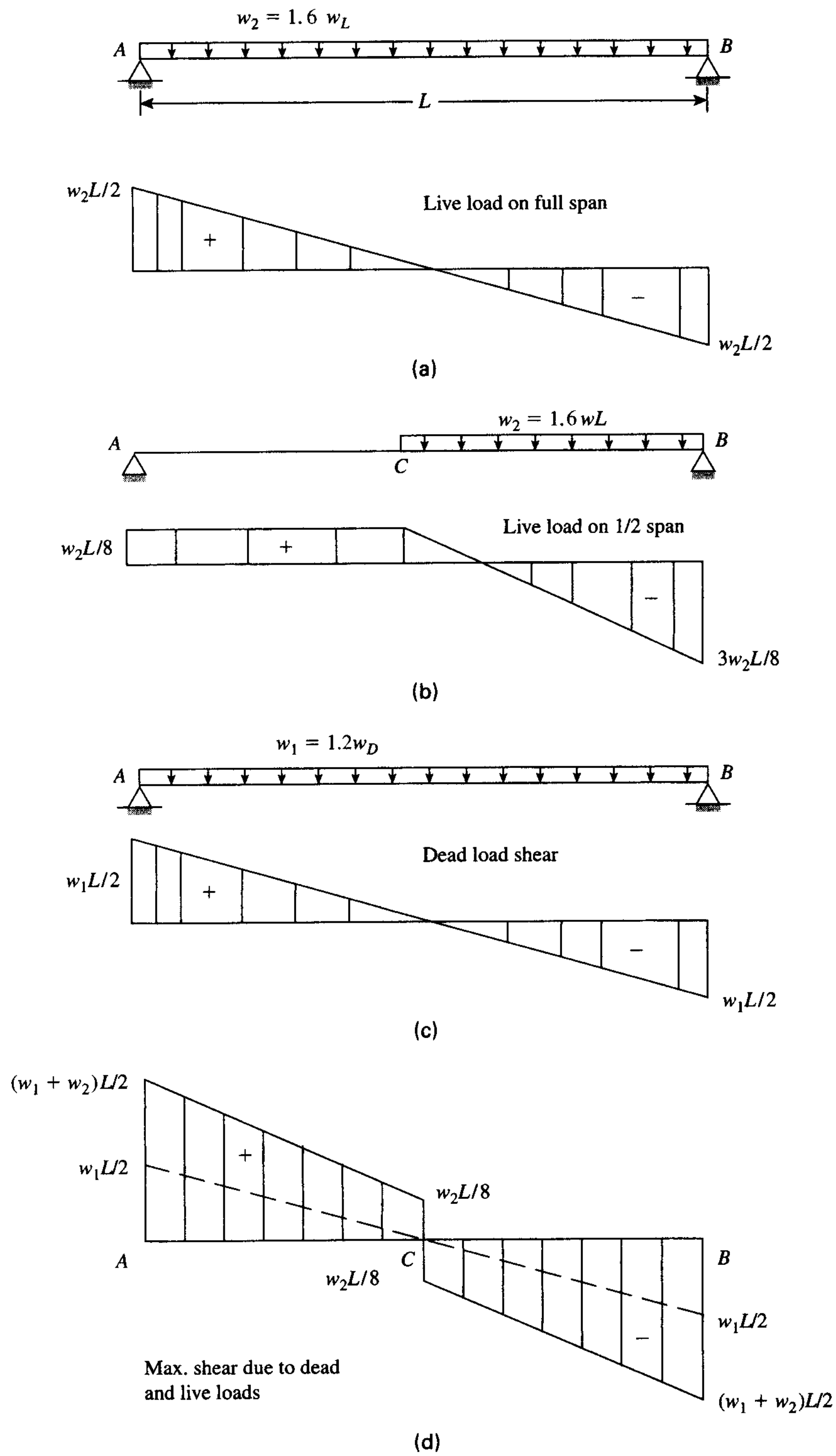


Figure 8.14 Effect of live load application on part of the span.

Example 8.3

A 10-ft-span cantilever beam has a rectangular section and carries uniform and concentrated factored loads (self-weight is included), as shown in Fig. 8.15. Using $f_c' = 4$ ksi normal-weight concrete and $f_y = 60$ ksi, design the shear reinforcement required for the entire length of the beam according to the ACI Code.

Solution

1. Calculate the shear force along the beam due to external loads.

$$V_u \text{ (at support)} = 5.5(10) + 20 + 8 = 83 \text{ K}$$

$$V_{ud} \text{ (at } d \text{ distance)} = 83 - 5.5\left(\frac{20.5}{12}\right) = 73.6 \text{ K}$$

$$V_u \text{ (at 4 ft left)} = 83 - 4(5.5) = 61 \text{ K}$$

$$V_u \text{ (at 4 ft right)} = 61 - 20 = 41 \text{ K}$$

$$V_u \text{ (at free end)} = 8 \text{ K}$$

The shear diagram is shown in Fig. 8.15.

2. Calculate ϕV_c:

$$\phi V_c = 2\lambda\sqrt{f_c'}bd = 2(0.75)(1)\sqrt{4000}(12)(20.5) = 23.34 \text{ K}$$

$$\frac{1}{2}\phi V_c = 11.67 \text{ K}$$

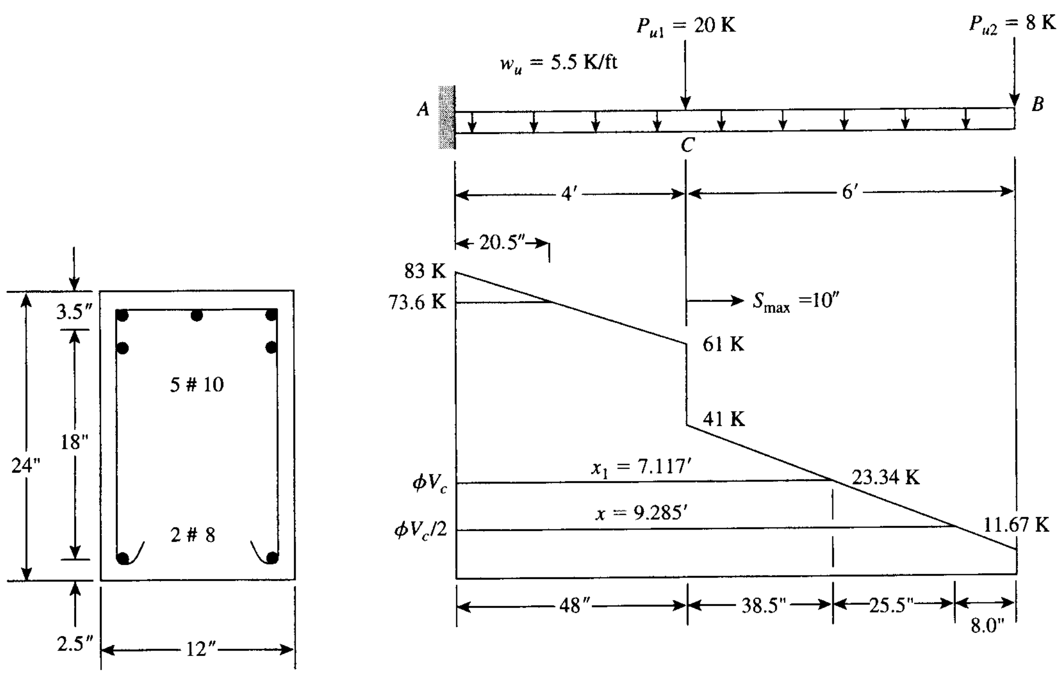


Figure 8.15 Example 8.3.

Because $V_{ud} > \phi V_c$ shear reinforcement is required. Calculate

$$V_{c_1} = 4\sqrt{f_c'}bd = 4\sqrt{4000}(12)(20.5) = 62.2 \text{ K}$$

$$V_{c_2} = 8\sqrt{f_c'}bd = 2V_1 = 124.4 \text{ K}$$

The distance x at which no shear reinforcement is needed (at $\frac{1}{2}\phi V_c = 11.67$ K), measured from support A:

$$x = 4 + \left(\frac{41 - 11.67}{41 - 8}\right)6 = 9.33 \text{ ft} = 112 \text{ in.}$$

(8.0 in. from free end). Similarly, x_1 for ϕV_c is 7.21 ft from A (33.5 in. from the free end).

3. Part AC: Design shear $V_u = V_{ud} = 73.6$ K. Calculate $V_s = (V_u - \phi V_c)/\phi = (73.6 - 23.34)/0.75 = 67$ K. Because $V_{c_1} < V_s < V_{c_2}$, $S_{max} \leq d/4$ must be considered (or check Fig. 8.10).
4. Design stirrups: Choose no. 3 U-stirrups, $A_v = 0.22 \text{ in.}^2$ Calculate S_1 (based on V_s):

$$S_1 = \frac{A_v f_{yt} d}{V_s} = \frac{13.2d}{V_s} = \frac{13.2(20.5)}{67} = 4.0 \text{ in.}$$

Use 4.0 in. (or get $s/d = 0.22$ from Fig. 8.10).

5. Calculate maximum spacings: $S_2 = d/4 = 20.5/4 = 5.12$ in., so use 5.0 in.

$$S_3 = \frac{A_v f_{yt}}{50 b_w} = 22 \text{ in.} \quad \text{(from Table 8.1 for } b = 12 \text{ in.)}$$

$$S_4 = 12 \text{ in.}$$

Then $S_{max} = 5.0$ in.

6. Because $S = 4$ in. $< S_{max} = 5.1$ in., use no. 3 stirrups spaced at 4 in.
7. At C, design shear $V_u = 61 \text{ K} > \phi V_c$. Then $V_s = (61 - 23.34)/0.75 = 50.2$ K, $S_1 = A_v f_{yt} d/V_s = 5.4$ in.

$$V_s = 50.2 \text{ K} < V_{c_1} = 62.2 \text{ K} \quad S_2 = \frac{d}{2} = \frac{20.5}{2} = 10.25 \text{ in.} \quad \text{(or 10 in.)}$$

$S_1 = 5.4$ in. $< S_2$; then $S_1 = 5.4$ or 5.0 in. controls.

8. Because spacings of 5.5 in. and 4.0 in. are close, use no. 3 U-stirrups spaced at 4 in. for part AC.
9. Part BC:
   a.

$$V_u = 41 \text{ K} > \phi V_c$$

$$V_s = (V_u - \phi V_c)/\phi = (41 - 23.34)/0.75 = 23.55 \text{ K} < V_{c_1} = 62.2 \text{ K}$$

   b. $S_1 = A_v f_{yt} d/V_s = (13.2)(20.5)/23.55 = 11.5$ in.
   c. $S_2 = d/2 = 20.5/2 = 10.25$ in. (or less than $S_3 = 22$ in. or $S_4 = 24$ in.). Let $S_{max} = 10$ in. Choose no. 3 stirrups spaced at 10 in. for part BC.
10. Distribution of stirrups measured from support A: Place the first stirrup at

$$\frac{S}{2} = \frac{4}{2} = 2 \text{ in.}$$

$$12 \times 4 \text{ in.} = \underline{48 \text{ in.}}$$

$$50 \text{ in.}$$

$$6 \times 10 \text{ in.} + 1 \times 8 \text{ in.} = \underline{68 \text{ in.}}$$

$$\text{Total} \quad 118 \text{ in.}$$

Distance left to the free end is 7 in., which is less than 8.0 in., where no stirrups are needed. Distribution of stirrups is shown in Fig. 8.16. Total number of stirrups is 20.

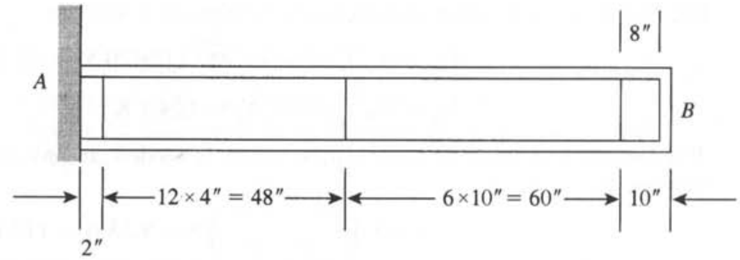


Figure 8.16 Example 8.3: distribution of stirrups.

8.10 SHEAR STRESSES IN MEMBERS OF VARIABLE DEPTH

The shear stress, v, is a function of the effective depth, d; therefore, shear stresses vary along a reinforced concrete beam with variable depth [10]. In such a beam (Fig. 8.17), consider a small element dx. The compression force C at any section is equal to the moment divided by its arm, or $C = M/y$. The first derivative of C is

$$dC = \frac{y\,dM - M\,dy}{y^2}$$

If C_1 is greater than C_2, then $C_1 - C_2 = dC = vb\,dx$

$$vbdx = \frac{y\,dM - M\,dy}{y^2} = \frac{dM}{y} - \frac{M}{y^2}dy$$

$$v = \frac{1}{yb}\left(\frac{dM}{dx}\right) - \frac{M}{by^2}\left(\frac{dy}{dx}\right)$$

Because $y = jd$, dM/dx is equal to the shearing force V and $d(jd)/dx$ is the slope,

$$v = \frac{V}{bjd} - \frac{M}{b(jd)^2}\left[\frac{d}{dx}(jd)\right] \quad \text{and} \quad v = \frac{V}{bjd} \pm \frac{M}{b(jd)^2}(\tan\alpha) \tag{8.26}$$

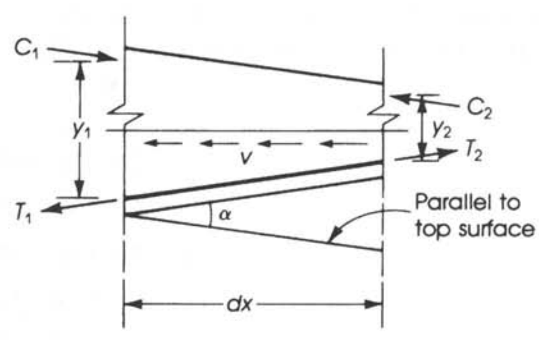


Figure 8.17 Shear stress in a beam with variable depth.

where V and M are the external shear and moment, respectively, and α is the slope angle of one face of the beam relative to the other face. The plus sign is used when the beam depth decreases as the moment increases, whereas the minus sign is used when the depth increases as the moment increases. This formula is used for small slopes, where the angle α is less than or equal to 30°.

A simple form of Eq. 8.26 can be formed by eliminating the j value:

$$v = \frac{V}{bd} \pm \frac{M}{bd^2} (\tan \alpha) \tag{8.27}$$

For the strength design method, the following equation may be used:

$$v_u = \frac{V_u}{\phi bd} \pm \frac{M_u}{\phi bd^2} (\tan \alpha) \tag{8.28}$$

For the shearing force,

$$\phi V_n = V_u \pm \frac{M_u}{d} (\tan \alpha) \tag{8.29}$$

Figure 8.18 shows a cantilever beam with a concentrated load P at the free end. The moment and the depth d increase toward the support. In this case a negative sign is used in Eqs. 8.27, 8.28, and 8.29. Similarly, a negative sign is used for section t in the simply supported beam shown, and a positive sign is used for section Z, where moment increases as the depth decreases.

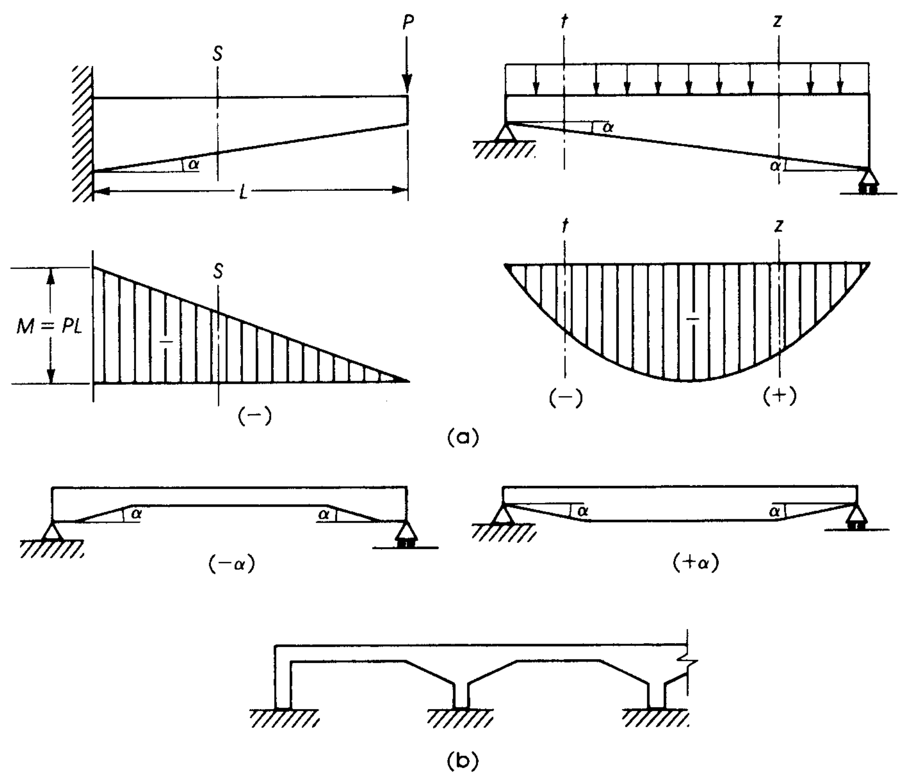


Figure 8.18 Beams with variable depth: (*a*) moment diagrams and (*b*) typical forms.

In many cases, the variation in the depth of beams occurs on parts of the beams near their supports (Fig. 8.18).

Tests [11] on beams with variable depth indicate that beams with greater depth at the support fail mainly by shear compression. Beams with smaller depth at the support fail generally by an instability type of failure, caused by the propagation of the major crack in the beam upward and then horizontally to the beam's top section. Tests also indicate that for beams with variable depth (Fig. 8.18) with an inclination α of about 10° and subjected to shear and flexure, the concrete shear strength, V_{cv}, may be computed by

$$V_{cv} = V_c(1 + \tan\alpha) \tag{8.30}$$

where

V_{cv} = shear strength of beam with variable depth

V_c = ACI Code Eq. 11.5

$= \left(1.9\lambda\sqrt{f'_c} + 2500\rho_w \frac{V_u d_s}{M_u}\right) b_w d_s \leq 3.5\lambda\sqrt{f'_c} b_w d_s$

α = angle defining the orientation of reinforcement, considered positive for beams of small depth at the support and negative for beams with greater depth at the support (Fig. 8.18)

d_s = effective depth of the beam at the support

The simplified ACI Code, Eq. 11.3,can also be used to compute V_c:

$$V_c = (2\lambda\sqrt{f'_c})b_w d_s \tag{8.31}$$

Example 8.4

Design the cantilever beam shown in Fig. 8.19 under the factored loads applied if the total depth at the free end is 12 in., and it increases toward the support. Use a steel percentage $\rho = 1.5\%$, $f'_c = 4$ ksi normal-weight concrete, $f_y = 60$ ksi, and $b = 10$ in.

Solution

1. M_u (support) $= (2.5/2)(8)^2(12) + (14)(8)(12) = 2304$ K·in.
2. For $\rho = 1.5\%$, $R_u = 703$ psi (from Table 4.1).

$$d = \sqrt{\frac{M}{R_u b}} = \sqrt{\frac{2304}{0.703 \times 10}} = 18.1 \text{ in.}$$

$A_s = 0.015 \times 10 \times 18.1 = 2.72 \text{ in.}^2$ (use three no. 9 bars); let actual $d = 19.5$ in., $h = 22$ in.

3. Design for shear: Maximum shear at the support is $14 + 20 = 34$ K. Because the beam section is variable, moment effect shall be considered; because the beam depth increases as the moment increases, a minus sign is used in Eq.8.28.

$$v_u = \frac{V_u}{\phi bd} - \frac{M_u}{\phi bd^2}(\tan\alpha)$$

To find $\tan\alpha$, let d at the free end be 9.5 in., and d at the support be 19.5 in.:

$$\tan\alpha = \frac{19.5 - 9.5}{8 \times 12} = 0.1042$$

$$v_u \text{ (at the support)} = \frac{34{,}000}{(0.75 \times 10 \times 19.5)} - \frac{2304 \times 1000 \times 0.1042}{[0.75 \times 10 \times (19.5)^2]}$$

$$= 148 \text{ psi}$$

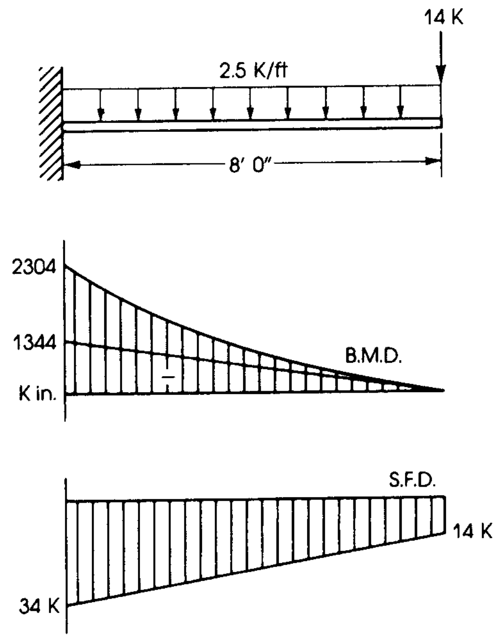


Figure 8.19 Example 8.4 with bending moment diagram (*middle*) and shear force diagram (*bottom*).

4. Shear stress at the free end is $V_u/\phi bd(M_u = 0)$.

$$v_u = \frac{14{,}000}{0.75 \times 10 \times 9.5} = 196 \text{ psi}$$

5. At a distance $d = 18$ in. from the face of the support, the effective depth is 17.6 in. (from geometry),

$$V_u = 34 - 2.5 \times \frac{18}{12} = 30.25 \text{ K}$$

$$M_u \text{ (at 18 in. from support)} = 14 \times 78 + \frac{2.5}{12} \times \frac{(78)^2}{2}$$

$$= 1726 \text{ K·in.}$$

$$v_u = \frac{30.25}{0.75 \times 10 \times 18} - \frac{1726 \times 1000 \times 0.1042}{0.75 \times 10 \times (18)^2}$$

$$= 150 \text{ psi}$$

6. At midspan (48 in. from the support),

$$d = 14.5 \text{ in.}$$

$$V_u = 14 + 10 = 24 \text{ K}$$

$$M_u = 14 \times 48 + \frac{2.5}{12} \times \frac{(48)^2}{2} = 912 \text{ K·in.}$$

$$v_u = \frac{24{,}000}{0.75 \times 10 \times 14.5} - \frac{912 \times 1000 \times 0.1042}{0.75 \times 10 \times (14.5)^2} = 160 \text{ psi}$$

Similarly, at 6 ft from the support (2 ft from the free end),

$$d = 12 \text{ in.} \quad V_u = 19 \text{ K} \quad M_u = 396 \text{ K·in.}$$

$$v_u = 173 \text{ psi}$$

At 1 ft from the free end,

$$d = 10.75 \text{ in.} \quad V_u = 16.5 \text{ K} \quad M_u = 183 \text{ K·in.}$$

$$v_u = 182 \text{ psi}$$

These values are shown in Fig. 8.20.

7. Shear stress resisted by concrete is

$$2\lambda\sqrt{f'_c} = (2)(1)\sqrt{4000} = 126.5 \text{ psi}$$

Minimum shear stress to be resisted by shear reinforcement

$$v_{us} = 196 - 126.6 = 69.5 \text{ psi}$$

(V_u and consequently v_{us} have already been increased by the ratio $1/\phi$ in Eq. 8.28).

8. Choose no. 3 stirrups with two legs.

$$A_v = 2 \times 0.11 = 0.22 \text{ in.}^2$$

$$S \text{ (required)} = \frac{A_v f_{yt}}{v_s b_w} = \frac{0.22 \times 60{,}000}{69.5 \times 10} = 19 \text{ in.}$$

$$S_{max}\left(\text{for } \frac{d}{2}\right) = 9.5 \text{ in. to } 4.5 \text{ in. at the free end}$$

$$S_{max} \text{ (for minimum} A_v) = \frac{A_v f_{yt}}{50 b_w} = \frac{0.22 \times 60{,}000}{50 \times 10} = 26.4 \text{ in.}$$

9. Check for maximum spacing ($d/4$): $v_{us} \leq 4\sqrt{f'_c}$.

$$4\sqrt{f'_c} = (4)\sqrt{4000} = 253 > 69.5 \text{ in.}$$

10. Distribution of stirrups (distances from the free end):

1 stirrup at 2 in. = 42 in.

10 stirrups at 4.5 in. = 45 in.

3 stirrups at 7 in. = 21 in.

3 stirrups at 8 in. = 24 in.

Total = 92 in.

There is 4 in. left to the face of the support.

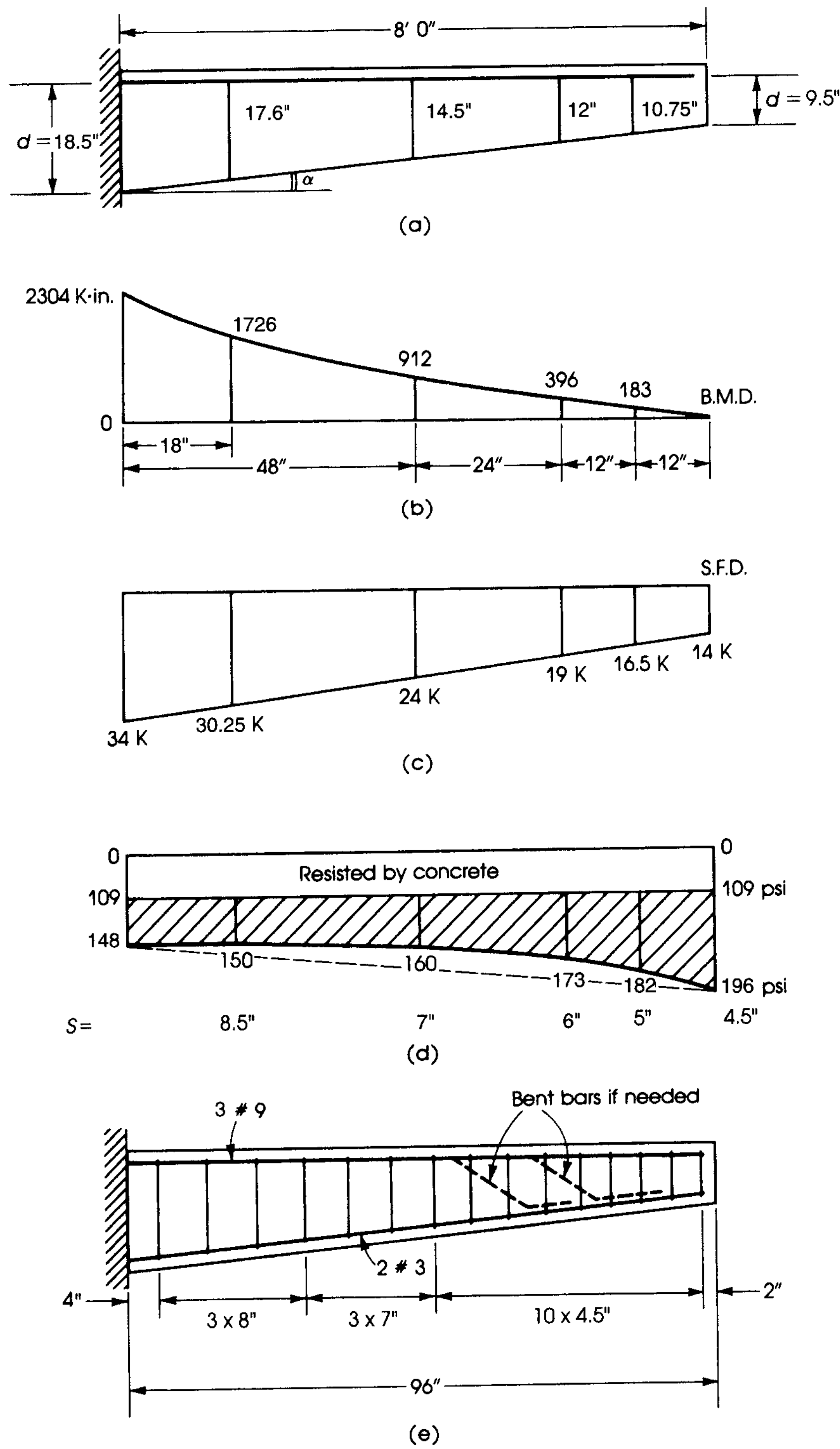


Figure 8.20 Example 8.4: web reinforcement for a beam of variable depth.

8.11 DEEP FLEXURAL MEMBERS

Flexural members should be designed as deep beams if the ratio of the clear span, l_n (measured from face to face of the supports; Fig. 8.21), to the overall depth, h, is less than 4 (ACI Code, Section 11.8). The members should be loaded on one face and supported on the opposite face so that compression struts can develop between the loads and supports (Fig. 8.22). If the loads are applied through the bottom or sides of the deep beam, shear design equations for ordinary beams given earlier should be used. Examples of deep beams are short-span beams supporting heavy loads, vertical walls under gravity loads, shear walls, and floor slabs subjected to horizontal loads.

The definition of deep flexural members is also presented in the ACI Code, Section 10.7.1. It indicates that flexural members where the ratio of the clear span, l_n, to the overall depth, h (Fig. 8.21), is less than 4 and regions loaded with concentrated loads within twice the member depth from the face of the support are considered deep flexural members. Such beams should be designed taking into account nonlinear distribuion of stress and lateral buckling (Fig. 8.22*a*).

Figure 8.22*a* shows the elastic stress distribution at the midspan section of a deep beam, and Fig. 8.22*b* shows the principal trajectories in top-loaded deep beams. Solid lines indicate tensile stresses, whereas dashed lines indicate compressive stress distribution. Under heavy loads, inclined vertical cracks develop in the concrete in a direction perpendicular to the principal tensile stresses and almost parallel to the dashed trajectories (Fig. 8.22*c*). Hence, both horizontal and vertical reinforcement is needed to resist principal stresses. Moreover, tensile flexural reinforcement is needed within about the bottom one-fifth of the beam along the tensile stress trajectories (Fig. 8.22*b*). In general, the analysis of deep beams is complex and can be performed using truss models or more accurately using a finite-element approach or similar methods. A simplified provision for the shear design of deep beams can be presented in steps as follows:

1. Critical section: If the critical section for shear design in deep beams supporting top vertical loads is located at a distance X from the face of the support, then the distance X can be determined as follows (Fig. 8.23):
   a. For deep beams supporting uniformly distributed loads, $X = 0.15 l_n$, where l_n = clear span.
   b. For concentrated loads, $X_1 = 0.5a_1$ (left support) or $X_2 = 0.5a_2$ (right support) (Fig. 8.23), where a_1 and a_2 equal the shear span near each support. The shear span is the distance between the concentrated load and the face of the support.
   In all cases, the distances X, X_1, and X_2 must not exceed the effective depth, d.

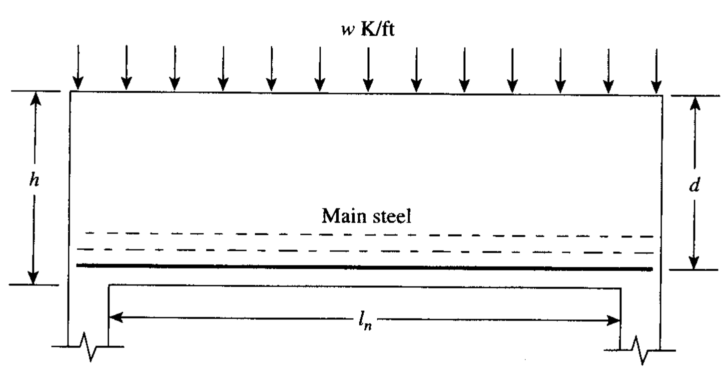


Figure 8.21 Single-span deep beam ($l_n/d < 5$).

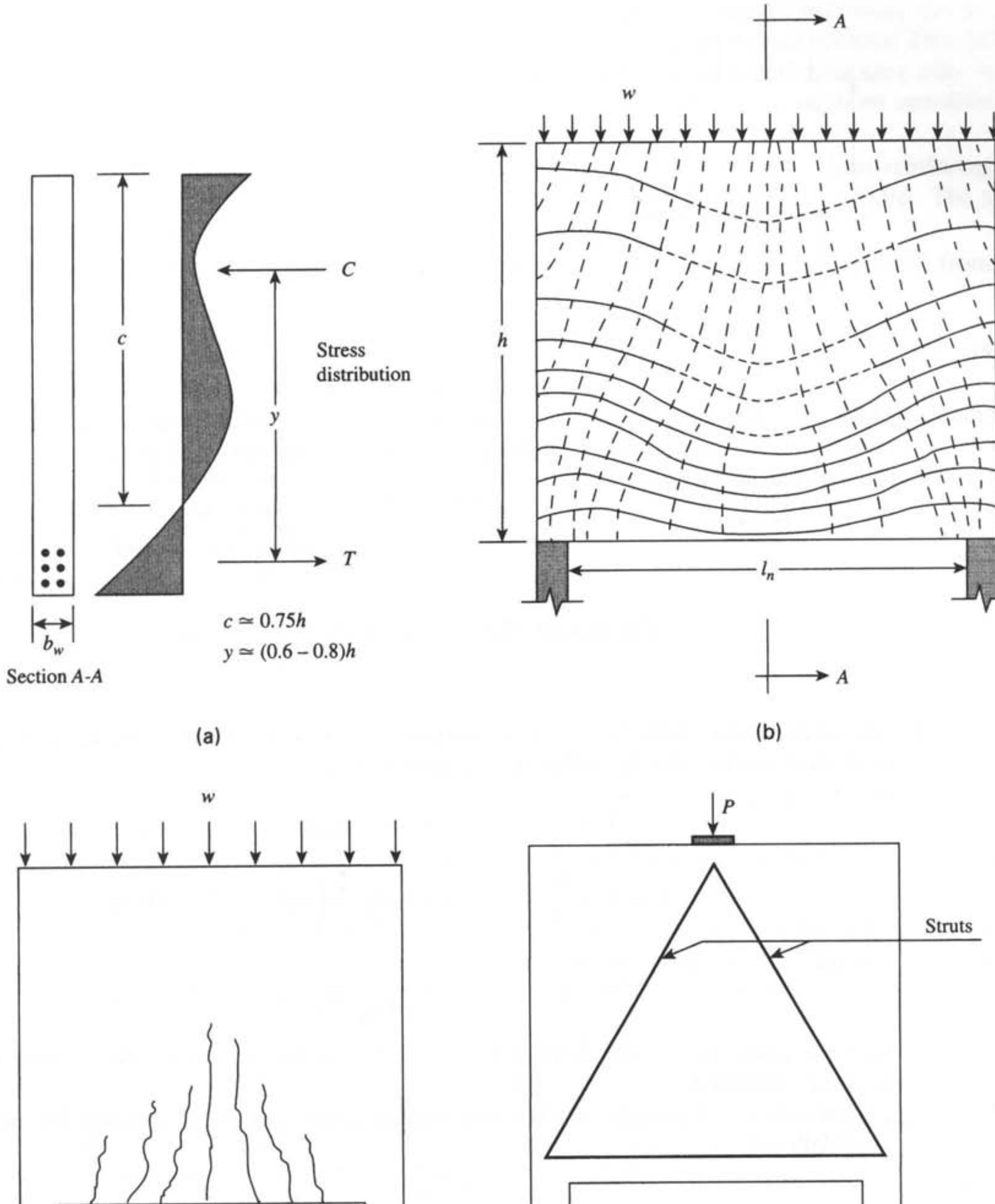


Figure 8.22 Stress distribution and cracking: (*a*) elastic stress distribution, (*b*) stress trajectories (tension, solid lines, and compression, dashed lines), (*c*) cracks pattern, and (*d*) truss model for a concentrated load applied at the wall upper surface.

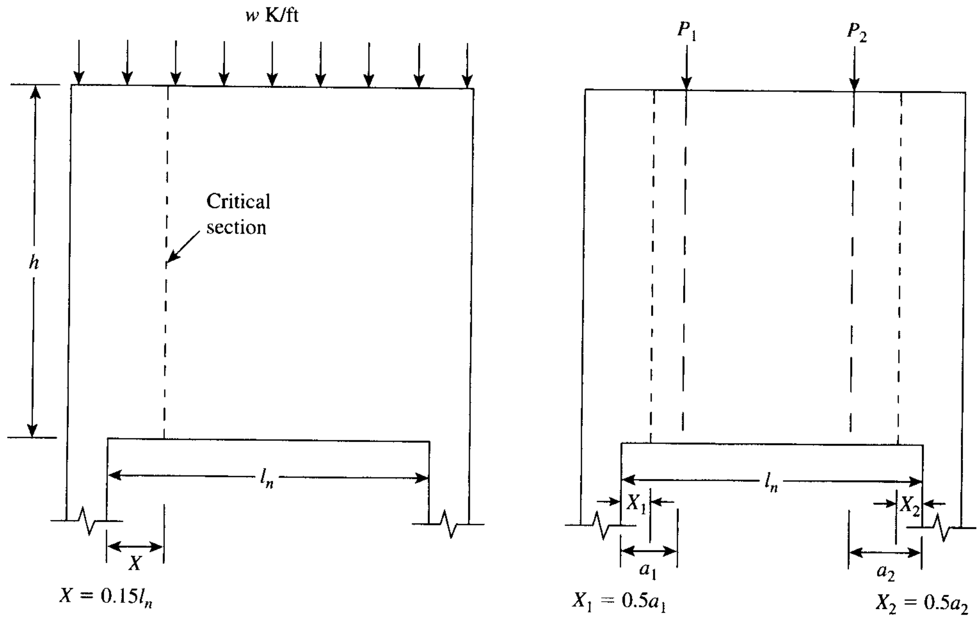


Figure 8.23 Critical sections for shear design.

2. Maximum shear strength ϕV_n: The maximum shear strength, ϕV_n, for deep flexural members shall not exceed the following values ($\phi = 0.75$):

$$\text{For } \frac{l_n}{d} < 2, \quad \phi V_n = \phi 8\sqrt{f_c'}b_w d \tag{8.32a}$$

$$\text{For } 2 \le \frac{l_n}{d} \le 5, \quad \phi V_n = \phi\frac{2}{3}\left(10 + \frac{l_n}{d}\right)\sqrt{f_c'}b_w d \tag{8.32b}$$

or let

$$\phi V_n = \phi 10\sqrt{f_c'}b_w d \tag{8.33}$$

for both cases, ACI Code, Section 11.7.3. If V_u exceeds ϕV_n, then the section dimensions must be increased.

3. **a.** Concrete shear strength, V_c: The nominal shear strength, V_c, of concrete can be estimated as follows:

$$V_c = 2\lambda\sqrt{f_c'}b_w d \tag{8.34}$$

This V_c is similar to the concrete shear strength for regular beams, as in the previous sections of this chapter.

b. Alternatively, another expression may be used that takes into account the effect of the factored moment and shear at the critical section:

$$V_c = \left(3.5 - \frac{2.5M_u}{V_u d}\right)\left(1.9\lambda\sqrt{f_c'} + \frac{2500\rho_w V_u d}{M_u}\right)b_w d \tag{8.35}$$

but V_c should not exceed $6\sqrt{f_c'}b_w d$.

The value of $(3.5 - 2.5\ M_u/V_u d)$ may not be greater than 2.5 and must not be less than 1.0. The values of M_u and V_u are taken at the critical design section. This higher shear strength of Eq. 8.35 is used with the idea that minor unsightly cracking may occur in the deep beam and can be tolerated. Cracks may start to develop at about one-third the factored load.

4. Shear reinforcement: When the factored shear force, V_u, exceeds ϕV_c, shear reinforcement must be provided, considering that $V_u = \phi(V_c + V_s)$, or $V_s = (V_u - \phi V_c)/\phi$. The steps are as follows:
   a. Determine V_s: The force resisted by shear reinforcement V_s is determined from the following expression:

$$V_s = \left[\frac{A_v}{S_v}\left(\frac{1 + l_n/d}{12}\right) + \frac{A_{\text{vh}}}{S_h}\left(\frac{11 - l_n/d}{12}\right)\right] f_y d \tag{8.36}$$

   where A_v = total area of vertical shear reinforcement spaced at S_v and perpendicular to the main flexural tensile reinforcement on both faces of the beam and A_{vh} = total area of horizontal shear reinforcement spaced at S_h parallel to the main flexural tensile reinforcement on both faces of the beam.
   b. Spacing of shear reinforcement is

$$\text{Maximum vertical spacings} \quad S_v \le \frac{d}{5} \le 12 \text{ in.}$$

$$\text{Maximum horizontal spacings} \quad S_h \le \frac{d}{5} \le 12 \text{ in.}$$

   c. Minimum shear reinforcement: The area of vertical shear reinforcement is $A_v = 0.0025 b_w S_v$. The area of horizontal shear reinforcement is $A_{\text{vh}} = 0.0015\ b_w S_h$.
   d. The shear reinforcement required at the critical section should be extended throughout the length and depth of the deep beam.
   e. For continuous deep beams, the same shear reinforcement may be used in all spans if the spans are almost equal with similar loading.
5. Flexural reinforcement of deep beams: The flexural behavior of deep beams is complex and requires nonlinear analysis of stresses and strains along the depth of the beam. For a preliminary design, the following simplified approach may be used:

$$\phi M_n = \phi A_s f_y y$$

   where y = moment arm = $(d - a/2)$. Because the value of $(d - a/2)$ is not easy to calculate, the moment arm y may be taken approximately equal to $0.6h$ for $l_n/h = 1.0$ and equal to $0.8h$ for $l_n/h = 2.0$. Linear interpolation may be used to estimate y when l_n/h varies between 1.0 and 2.0. Therefore,

$$A_s = \frac{M_u}{\phi y f_y} \tag{8.37}$$

   The value of A_s may not be less than the minimum flexural reinforcement required for regular beams given next, assuming $d = 0.9h$:

$$\text{Minimum } A_s = \left(\frac{3\sqrt{f_c'}}{f_y}\right) b_w d \ge \left(\frac{200}{f_y}\right) b_w d \tag{8.38}$$

   The second term controls when $f_c' < 4500$ psi. Note that, and f_y are in psi.

The flexural tension reinforcement shoul d be placed within $h/4$ to $h/5$ of the beam and should be adequately spaced along the bottom tension zone. Tension bars should be well anchored to the supports.

For more accurate analysis and design and for continuous deep beams, a rigorous non-linear approach should be used to determine the proper amount and distribution of the tension reinforcement.

Example 8.5

A simply supported deep beam has a span = 14 ft, a clear span of $l_n = 12$ ft, a total height of $h = 8$ ft, and width of $b = 16$ in. The deep beam supports a uniform service dead load of 41 K/ft (including self-weight) and a live load of 22 K/ft on top of the beam. Design the beam for moment and shear using $f_c' = 4$ ksi normal-weight concrete, and $f_y = 60$ ksi. Refer to Fig. 8.24.

Solution

1. Design for moment:

$$W_u = 1.2W_D + 1.6W_L = 1.2(41) + 1.6(22) = 84.4 \text{ K/ft}$$

$$M_u = \frac{W_u L^2}{8} = \frac{84.4(14)^2}{8} = 2067.8 \text{ K·ft}$$

$$\frac{l_n}{h} = \frac{12}{8} = 1.5$$

Determine the moment arm, y. For $l_n/h = 1.0$, $y = 0.6d$, and for $l_n/h = 2.0$, $y = 0.8d$; hence for $l_n/h = 1.5$, $y = 0.7d$ (by interpolation) $= 0.7(8 \times 12) = 67.2$ in.

$$A_s = \frac{M_u}{\phi y f_y} = \frac{2067.8 \times 12}{0.9(67.2)(60)} = 6.84 \text{ in.}^2$$

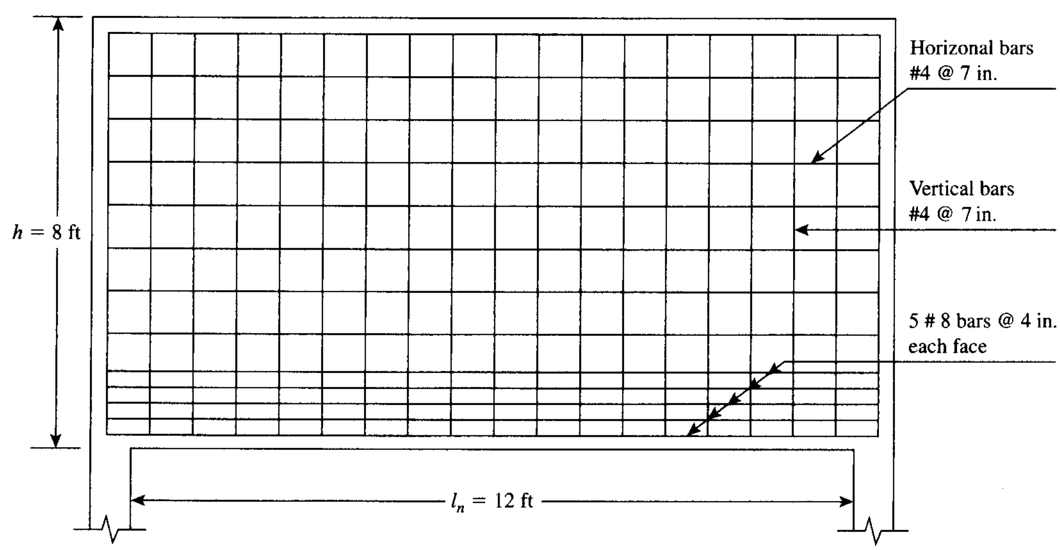


Figure 8.24 Example 8.5.

Assume $d = 0.9h = 0.9(8 \times 12) = 86.4$ in. Because $f_c' < 4500$ psi,

$$A_s \text{ (minimum)} = \left(\frac{200}{f_y}\right) b_w d = \frac{200(16)(86.4)}{60{,}000} = 4.6 \text{ in.}^2$$

Therefore, $A_s = 6.84$ in.^{2} controls. Choose 10 no. 8 bars (7.85 in.^{2}), five on each face, distributed within $h/5 = 8(12)/5 = 19.2$ in. of the tension zone of the beam. Spacing of bars $= 19.2/5 = 3.84$ in., or 4 in. Bars should be well anchored into the supports.

2. Design for shear:

a. Calculate V_u and M_u at the distance $x = 0.15l_n = d$ from the face of the support.

$$0.15l_n = 0.15(12 \times 12) = 21.6 \text{ in.} = 1.8 \text{ ft} < d = 86.4 \text{ in.}$$

$$\text{Design } V_u = 84.4(12/2) - 84.4(1.8) = 354.5 \text{ K}$$

$$M_u = 84.4(6)(1.8) - \frac{84.4(1.8)^2}{2} = 774.8 \text{ K·ft}$$

$$\frac{M_u}{V_u d} = \frac{774.8(12)}{354.5(86.4)} = 0.304$$

b. Calculate V_c:

$$3.5 - 2.5\frac{M_u}{V_u d} = 3.5 - 2.5(0.304) = 2.74 > 2.5$$

So, use 2.5 In this case, determine $M_u/V_u d$ to be used to calculate V_c: $2.5 = 3.5 - 2.5\, M_u/(V_u d)$, and $M_u/(V_u d) = 0.4$.

$$\frac{V_u d}{M_u} = \frac{1}{0.4} = 2.5$$

$$\rho_w = \frac{A_s}{b_w d} = \frac{7.85}{16 \times 86.4} = 0.00496$$

$$V_c = 2.5\left(1.9\lambda\sqrt{f_c'} + \frac{2500\rho_w V_u d}{M_u}\right) b_w d$$

$$= 2.5[(1.9)(1)\sqrt{4000} + (2500)(0.00496)(2.5)](16)(86.4)$$

$$= 522.4 \text{ K}$$

$$V_c \le 6\sqrt{f_c'} b_w d = 6\sqrt{4000}(16)(86.4) = 524.6 \text{ K}$$

Hence, $V_c = 522.4$ controls and $\phi V_c = 392$ K.

c. Calculate $V_s = (V_u - \phi V_c)/\phi$. Because $\phi V_c = 392\text{ K} > V_u = 354.5$ K, then $V_s = 0$, and only minimum shear reinforcement is required.

d. Calculate shear reinforcement: Assume no. 4 bars placed on both faces in the horizontal and vertical directions; then $A_v = A_{vh} = 2(0.2) = 0.4$ in.^{2} Maximum allowable spacing of vertical bars is $S_v = d/5 = 18$ in. $S_v = 86.4/5 = 17.3$ in. > 12 in. use $S_u = 12$ in. Maximum allowable spacing of horizontal bars is $S_h = d/5 = 18$ in. $S_h = 86.4/5 = 17.3$ in > 12 in.; use $S_h = 12$ in. Minimum A_v (vertical) $= 0.0025 b_w S_v = 0.0025(16)(72) = 0.48$ in.^{2} > 0.4 in.^{2} Minimum A_{vh} (horizontal) $= 0.0015 b_w S_h = 0.0015(16)(12) = 0.288$ in.^{2} < 0.4 in.^{2} Reduce spacing to $S_v = 0.4/(0.0025 \times 16) = 10$ in. Therefore, use no. 4 vertical bars spaced at 10 in., and use no. 4 horizontal bars spaced at 12 in.

3. If $V_c = 2\lambda\sqrt{f_c'} b_w d$ is used, then $V_c = (2)(1)\sqrt{4{,}000}(16)(86.4) = 174.9$ K and $\phi V_c = 0.75 V_c = 131$ K, which is less than $V_u = 354.5$ K. Hence, shear reinforcement is required.

$$V_s = \frac{V_u - \phi V_c}{\phi} = \frac{354.1 - 131}{0.75} = 297.5 \text{ K}$$

Assuming no. 4 bars placed on both faces in the vertical and horizontal directions, then $A_v = A_{vh} = 2(0.2) = 0.4\,\text{in.}^2$ Assuming that the spacings of bars in both directions are equal, $S_v = S_h = S$, and $l_n/d = 12 \times 12/86.4 = 1.67$, then

$$V_s = \left[\frac{A_v}{S_v}\left(\frac{1+l_n/d}{12}\right) + \frac{A_{vh}}{S_h}\left(\frac{11-l_n/d}{12}\right)\right] f_y d$$
$$297.5 = \left[\frac{0.4}{S}\left(\frac{1+1.67}{12}\right) + \frac{0.4}{S}\left(\frac{11-1.67}{12}\right)\right](60)(86.4) \tag{8.36}$$

$S = 7$ in., which is less than the maximums $S_v = 17.3$ in. and $S_h = 18$ in. Use $S = 7$ in. for both vertical and horizontal spacing.

$$\text{Minimum } A_v \text{ (vertical)} = 0.0025(16)(7) = 0.28\ \text{in.}^2 < 0.4\ \text{in.}^2$$

$$\text{Minimum } A_{vh} \text{ (horizontal)} = 0.0015(16)(7) = 0.168\ \text{in.}^2 < 0.4\ \text{in.}^2$$

Then use no. 4 bars spaced at 7 in. on both faces in the horizontal and vertical directions. A welded wire fabric mesh may be adopted to replace the preceding bar arrangements. It can be seen that this solution is more conservative than that given in step 2. Reinforcement details are shown in Fig. 8.24 on page 281.

Example 8.6: Strut and Tie Deep Beam

A simply supported deep beam has a clear span = 12 ft, a total height = 6 ft, and a width = 18 in. The beam supports an 18-in. square column at midspan carrying a dead load = 300 K, and a live load = 240 K. Design the beam using the strut and tie model, using $f'_c = 4$ ksi and $f_y = 60$ ksi. (Refer to Fig. 8.25).

Solution

1. Calculate the factored loads (Fig. 8.25):

$$\text{Weight of the beam} = 15 \times 6 \times 1.5 \times 0.150 = 20\text{K}$$

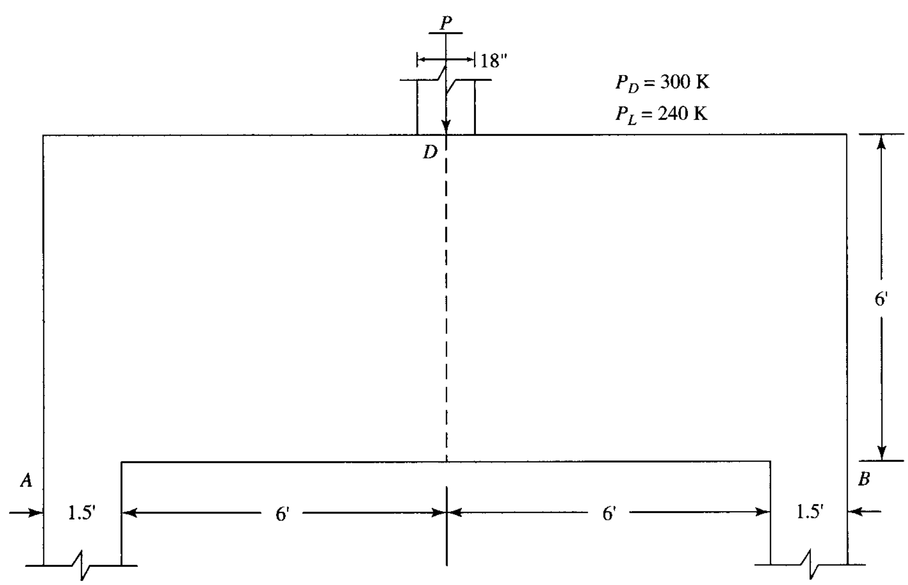


Figure 8.25 Example 8.6

Since the weight of the beam is small relative to the concentrated loads at midspan, add it to the concentrated load at midspan.

$$P_u = 1.2D + 1.6L = 1.2(300 + 20) + 1.6(240) = 768 \text{ K}$$

$$R_A = R_B = 768/2 = 384 \text{ K}$$

2. Check if the beam is deep according to to the ACI Code, Section 11.8: Clear span, $l_n = 12$ ft, $h = 6$ ft, and $l_n/h = 2 < 4$, a deep beam.
3. Calculate the maximum shear strength of the beam cross-section: Let V_u at $A = R_A = 384$ K, and assume $d = 0.9h = 0.9 \times 72 = 64$ in.

$$V_n = 10\sqrt{f_c'}b_w d = 10 \times \sqrt{4000}(18 \times 64) = 728.6 \text{ K}$$

$$\phi V_n = 0.75(728.6) = 546 \text{ K} > V_u \qquad \text{(o.k.)}$$

4. Select a truss model.
A triangular truss model is chosen. Assume that the nodes act at the centerline of the supports and at 6.0 in from the lower or upper edge of the beam (Fig. 8.26). The strut and tie model consists of a tie AB and two struts AD and BD. Also, the reactions at A and B and the load P_u at D represent vertical struts.

$$\text{Length of the diagonal strut } AD = \sqrt{(60)^2 + (80)^2} = 100.8 \text{ in.}$$

Let the angle between the strut and the tie $= \theta$, $\tan\theta = 60/81 = 0.7407$, and $\theta = 36.5$ degrees >26 degrees, which is o.k.

5. Calculate the forces in the truss members: The compression force in strut $AD = F_{AD} = F_{BD}$ $= 384$ (100.8/60) $= 645$ K. The tension force in the tie $AB = F_{AB} = 384$ (100.8/81) $= 478$ K.
6. Calculate the effective strength, f_{cu}. Assume that confining reinforcement is provided to resist the splitting forces. Struts AD and BD represent the bottle-shape compression members, and therefore, $\beta_s = 0.75$.

$$f_{ce} = 0.85\beta_s f_c' = 0.85 \times 0.75 \times 4 = 2.55 \text{ ksi}$$

The vertical struts at A, B, and D have uniform sections, and therefore $\beta_s = 1.0$.

$$f_{ce} = 0.85 \times 1.0 \times 4 = 3.4 \text{ ksi}$$

The nodal zone D has a C-C-C force and therefore, $\beta_s = 1.0$. The effective strength at nodal zone D is:

$$f_{ce} = 0.85 \times 1.0 \times 4 = 3.4 \text{ ksi}$$

Since the struts AD and BD connect to the other nodes, $f_{ce} = 2.55$ ksi controls to all nodal zones.

7. Design of nodal zones:
   a. Design of nodal zone at A : Assume that the faces of the nodal zone have the same stress of 2.55 ksi and the faces are perpendicular to their respective forces.

$$\phi F_n \geq F_u \quad \text{or} \quad \phi f_{ce} A_{cs} \geq F_u$$

   where $\phi = 0.75$ for struts, ties, and nodes. The length of the horizontal face ab (Fig. 8.27a) is equal to $F_u/(\phi f_{ce} b) = 384/(0.75 \times 2.55 \times 18) = 11.2$ in. From geometry, the length $ac =$ 11.2 (478/384) $= 13.94$ in., say 14 in. Similarly, the length $bc = 11.2$ (645/384) $= 18.8$ in. The center of the nodal zone is located at 14/2 = 7 in. from the bottom of the beam, which is close to 6.0 in., assumed earlier.

   b. Design of nodal zone at D (Fig. 8.27b): The length of the horizontal face $de = 768/(0.75 \times 2.55 \times 18) = 22.3$ in. The length of $df = ef = 22.3(645/768) = 18.7$ in. The length of $fg = 15.0$ in., and the center of the nodal zone is located at 15/3 = 5.0 in from the top, which is close to the assumed 6.0 in.

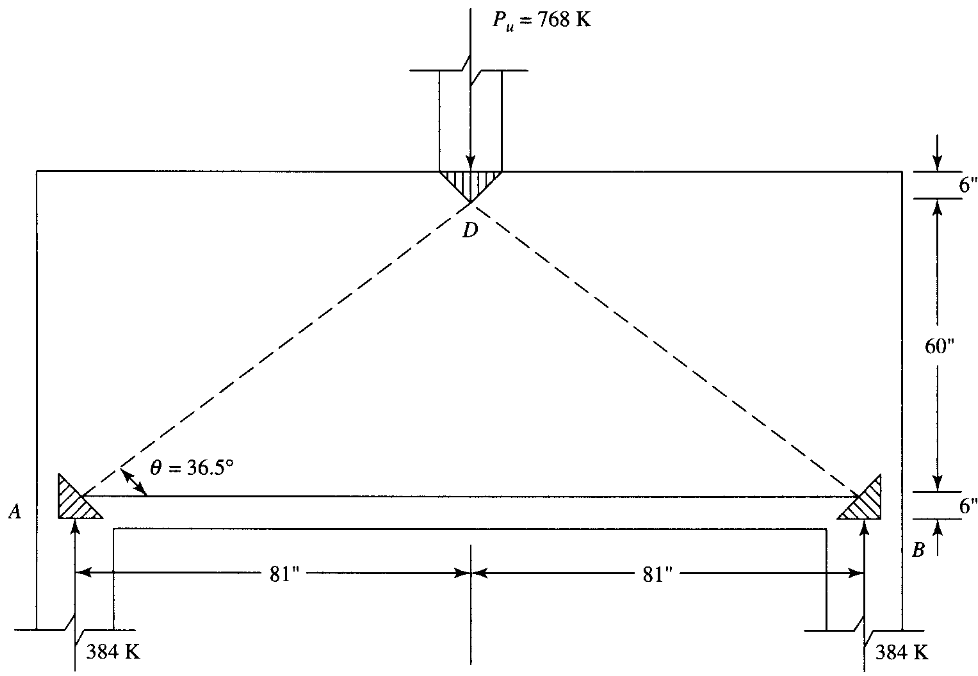


Figure 8.26 Example 8.6: idealized beam.

8. Design of vertical and horizontal reinforcement:

a. Vertical bars: The angle between the vertical bars and strut = 53.5°, from Fig. 8.27*a*. Use No. 5 bars spaced at 12 in., two branches, A_s = 2 (0.31) = 0.62 in2. Sin 53.5° = 0.804.

$$(A_{\text{si}}/b_s s) \sin \gamma_i = (0.62/18 \times 12)(0.804) = 0.0023$$

b. Horizontal bars: The angle between the horizontal bars and strut = 36.5°, (Fig. 8.27*a*). Use No. 5 bars spaced at 12 in., two branches, $A_s = 0.62\,\text{in.}^2$. Sin 36.5° = 0.595.

$$(A_{\text{si}}/b_s s) \sin \gamma_i = (0.62/18 \times 12)(0.595) = 0.0017$$

c. Total $(A_{si}/b_s s)$ sin γ_i = 0.0023 + 0.0017 = 0.004 > 0.003, which is o.k.

9. Design of the horizontal tie AB :

a. Calculate A_s:

$$F_u = \phi A_s f_y \quad A_s = 478/(0.75 \times 60) = 10.6\ \text{in.}^2$$

Use 12 no. 9 bars, $A_s = 12\,\text{in.}^2$ in three rows as shown in Fig. 8.27*c*.

b. Calculate anchorage length: Anchorage length is measured from the point beyond the extended nodal zone, Fig. 8.28. Tan 36.5 = $7/x$. Then x = 9.5 in. Available anchorage length = 9.5 + 5.6 + 9 − 1.5 in. (cover) = 22.6 in. Development length of no. 9 bars required = 47.5 in. (Table 7.1), which is greater than 22.6 in. Use a standard 90° hook enclosed within the column reinforcement.

$$l_{\text{dh}} = 0.02\psi_e f_y d_b/\lambda\sqrt{f_c'} \tag{7.15}$$

$$\psi_e = \lambda = 1.0 \qquad d_b = 1.128\ \text{in.}$$

$$l_{\text{dh}} = 0.02(1)(60{,}000)(1.128)/\big((1)(\sqrt{4000})\big) = 21.4\ \text{in} < 22.6\ \text{in.}$$

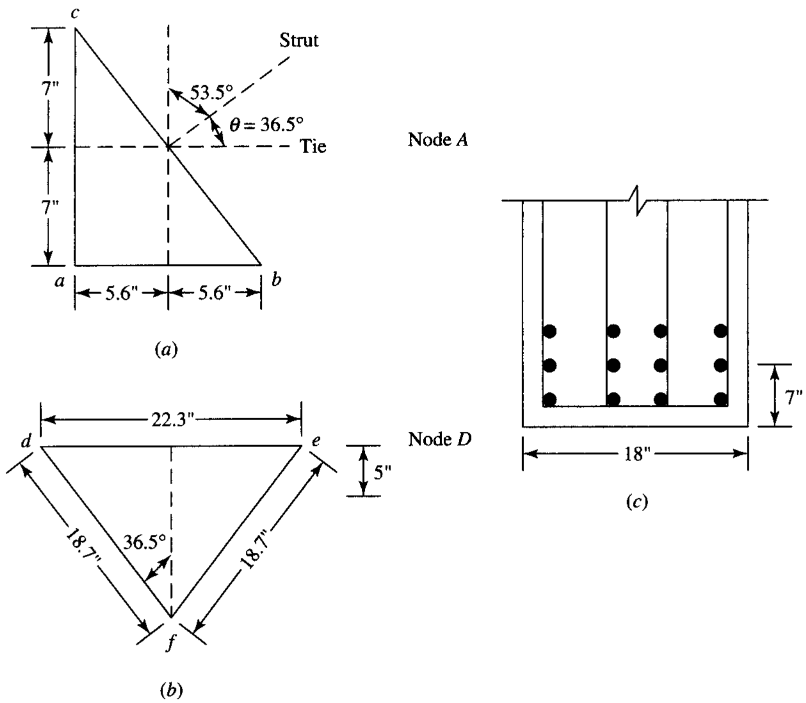


Figure 8.27 Example 8.6: nodal zones, (*a*) at node *A*, (*b*) at node *D*, and (*c*) reinforcement details.

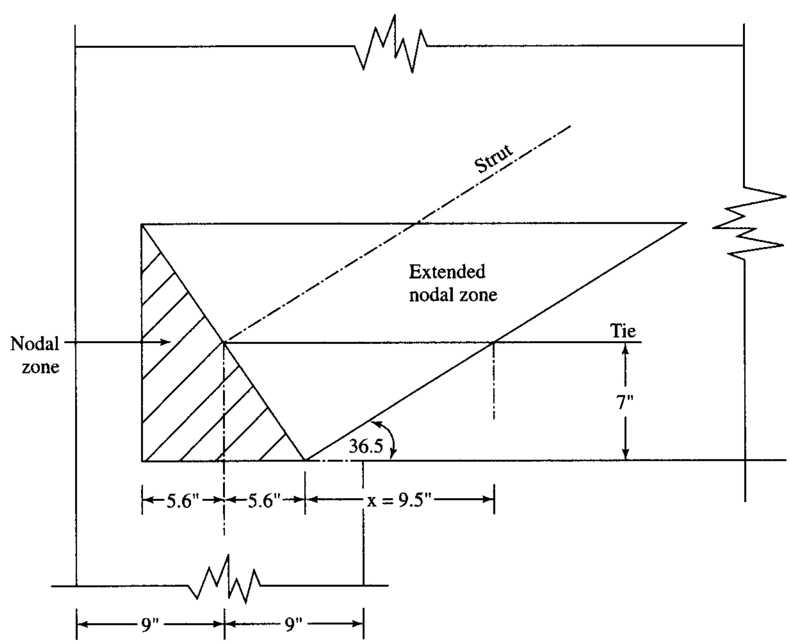


Figure 8.28 Example 8.6: development of tie reinforcement.

Table 8.4 Shear Reinforcement Formulas

U.S. Customary Units	SI Units
V_u = design shear	V_u = design shear
(Maximum design V_u is at a distance d from the face of the support.)	
$V_c = (2.0\lambda\sqrt{f_c'})\, b_w d$	$V_c = (0.17\lambda\sqrt{f_c'})\, b_w d$
$V_c = \left[1.9\lambda\sqrt{f_c'} + \left(2500\rho_w \frac{V_u d}{M_u}\right)\right] b_w d$	$V_c = \left[0.16\lambda\sqrt{f_c'} + \left(17.2\rho_w \frac{V_u d}{M_u}\right)\right] b_w d$
$\rho_w = \frac{A_s}{b_w d} \quad \frac{V_u d}{M_u} \le 1.0$	$\rho_w = \frac{A_s}{b_w d} \quad \frac{V_u d}{M_u} \le 1.0$
$V_c \le (3.5\lambda\sqrt{f_c'})\, b_w d$	$V_c \le (0.29\lambda\sqrt{f_c'})\, b_w d$
$V_u = \phi V_c + \phi V_s$	$V_u = \phi V_c + \phi V_s$
Vertical stirrups	
$\phi V_s = V_u - \phi V_c$	$\phi V_s = V_u - \phi V_c$
$S = \frac{A_v f_{yt} d}{V_s}$	$S = \frac{A_v f_{yt} d}{V_s}$
Minimum $A_v = \frac{50 b_w S}{f_{yt}} \le 0.75\sqrt{f_c'}\left(\frac{b_w S}{f_{yt}}\right)$	Minimum $A_v = \frac{0.35 b_w S}{f_{yt}} \le 0.0062\sqrt{f_c'}\left(\frac{b_w S}{f_y}\right)$
Maximum $S = \frac{A_v f_{yt}}{50 b_w} \ge \frac{A_v f_{yt}}{0.75\sqrt{f_c'} b_w}$	Maximum $S = \frac{A_v f_{yt}}{0.35 b_w} \ge \frac{A_v f_y}{0.062\sqrt{f_c'} b_w}$
For vertical web reinforcement	
Maximum $S = \frac{d}{2} \le 24$in.	Maximum $S = \frac{d}{2} \le 600$ mm
if $V_s \le 4.0\sqrt{f_c'}(b_w d)$	if $V_s \le 0.33\sqrt{f_c'}(b_w d)$
Maximum $S = d/4 = 12$ in.	Maximum $S = d/4 = 300$ mm
if $V_s > 4.0\sqrt{f_c'}(b_w d)$	if $V_s > 0.33\sqrt{f_c'}(b_w d)$
$V_s \le 8\sqrt{f_c'}(b_w d)$	$V_s \le 0.67\sqrt{f_c'}(b_w d)$
Otherwise increase the dimensions of the section.	
Series of bent bars or inclined stirrups	
$A_v = \frac{V_s S}{f_{yt}\, d(\sin\alpha + \cos\alpha)}$	$A_v = \frac{V_s S}{f_{yt}\, d(\sin\alpha + \cos\alpha)}$
For $\alpha = 45°$, $S = \frac{1.4 A_v f_y d}{V_s}$	For $\alpha = 45°$, $S = \frac{1.4 A_v f_y d}{V_s}$
For a single bent bar or group of bars, parallel and bent in one position	
$A_v = \frac{V_s}{f_{yt}\sin\alpha}$	$A_v = \frac{V_s}{f_{yt}\sin\alpha}$
For $\alpha = 45°$, $A_y = \frac{1.4 V_s}{f_{yt}}$	For $\alpha = 45°$, $A_v = \frac{1.4 V_s}{f_{yt}}$
$V_s \le (3\sqrt{f_c'})\, b_w d$	$V_s \le (0.25\sqrt{f_c'})\, b_w d$

8.12 EXAMPLES USING SI UNITS

The general design requirements for shear reinforcement according to the ACI Code are summarized in Table 8.4, which gives the necessary design equations in both U.S. customary and SI units. The following example shows the design of shear reinforcement using SI units.

Example 8.7

A 6-m clear span simply supported beam carries a uniform dead load of 47.5 kN/m and a live load of 25 kN/m (Fig. 8.29). The dimensions of the beam section are $b = 350$ mm, $d = 550$ mm. The beam is reinforced with four bars of 25-mm diameter in one row. It is required to design the necessary shear reinforcement. Given: $f_c' = 28$ MPa and $f_y = 280$ MPa.

Solution

1. Factored load is

$$1.2D + 1.6L = 1.2 \times 47.5 + 1.6 \times 25 = 97 \text{ kN/m}$$

2. Factored shear force at the face of the support is

$$V_u = 97 \times \frac{6}{2} = 291 \text{ kN}$$

3. Maximum design shear at a distance d from the face of the support is

$$V_u \text{ (at distance} d) = 291 - 0.55 \times 97 = 237.65 \text{ kN}$$

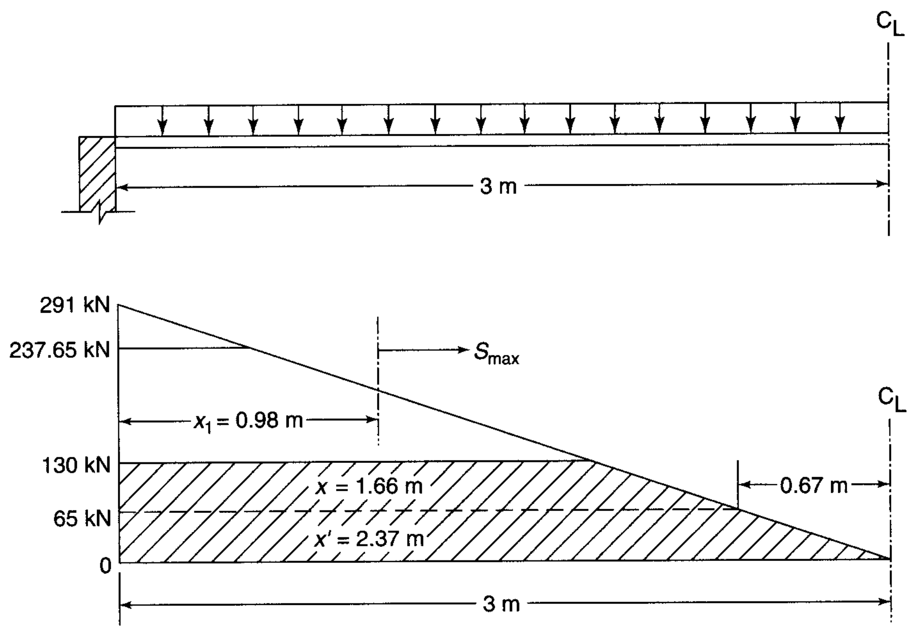


Figure 8.29 Example 8.7.

4. The nominal shear strength provided by concrete is

$$V_c = (0.17\lambda\sqrt{f_c'})bd = (0.17\sqrt{28}) \times 350 \times 550 = 173.2 \text{ kN}$$
$$V_u = \phi V_c + \phi V_s$$
$$\phi V_c = 0.75 \times 173.2 = 130 \text{ kN}$$
$$\frac{1}{2}\phi V_c = 65 \text{ kN}$$
$$\phi V_s = 237.65 - 130 = 107.65 \text{ kN}$$
$$V_s = \frac{107.65}{0.75} = 143.5 \text{ kN}$$

5. Distance from the face of the support at which $\frac{1}{2}\phi V_c = 65$ kN is

$$x' = \frac{(291 - 65)}{291}(3) = 2.33 \text{ m} \quad \text{(from triangles)}$$

6. Design of stirrups:
   a. Choose stirrups 10 mm in diameter with two branches ($A_s = 78.5 \text{ mm}^2$).

$$A_v = 2 \times 78.5 = 157\text{mm}^2$$
$$\text{Spacing } S_1 = \frac{A_y f_{yt} d}{V_s} = \frac{157 \times 280 \times 550}{143.5 \times 10^3} = 168.5 \text{ mm} < 600 \text{ mm}$$

   Thus, use 160 mm. Check maximum spacing of stirrups:

$$\text{Maximum } S_2 = \frac{d}{2} = \frac{550}{2} = 275 \text{ mm}$$
$$S_3 = \frac{A_v f_{yt}}{0.35b} = \frac{157 \times 280}{0.35 \times 350} = 359 \text{ mm}$$

   $S = S_1 = 160$ mm controls.

   b. Check for maximum spacing of $d/4$:

$$\text{If } V_s \leq \left(0.33\sqrt{f_c'}\right)bd, \quad S_{\max} = \frac{d}{2}.$$
$$\text{If } V_s > \left(0.33\sqrt{f_c'}\right)bd, \quad S_{\max} = \frac{d}{4}.$$
$$bd\left(0.33\sqrt{f_c'}\right) = 0.33\sqrt{28} \times 350 \times 550 = 336.1\text{kN}$$

   Actual $V_s = 143.5$ kN < 336.1 kN. Therefore, $S_{\max}$ is limited to $d/2 = 275$ mm.

7. The shear reinforcement, stirrups 10 mm in diameter and spaced at 160 mm, will be needed only for a distance $d = 0.55$ m from the face of the support. Beyond that, the shear stress V_s decreases to 0 at a distance $x = 1.66$ m when $\phi V_c = 130$ kN. It is not practical to provide stirrups at many different spacings. One simplification is to find out the distance from the face of support where maximum spacing can be used, and then only two different spacings may be adopted.

$$\text{Maximum spacing} = \frac{d}{2} = 275 \text{ mm}$$

$$V_s \text{ (for } s_{\max} = 275 \text{ mm)} = \frac{A_v f_{yt} d}{S} = \frac{157 \times 0.280 \times 550}{275} = 87.9 \text{ kN}$$

$$\phi V_s = 87.9 \times 0.75 = 65.94 \text{ kN}$$

The distance from the face of the support where $S_{\max} = 275$ mm can be used (from the triangles):

$$x_1 = \frac{291 - (130 + 65.94)}{291}(3) = 0.98 \text{ m}$$

Then, for 0.98 m from the face of support, use stirrups of 10-mm diameter at 160 mm, and for the rest of the beam, minimum stirrups (with maximum spacings) can be used.

8. Distribution of stirrups:

$$\text{one stirrup at } \frac{S}{2} = \frac{160}{2} = 80 \text{ mm}$$

$$\text{six stirrups at 160 mm} = \underline{960 \text{ mm}}$$

$$\text{Total} = 1040 \text{ mm} = 1.04 \text{ m} > 0.98 \text{ m}$$

$$\text{six stirrups at 270 mm} = \underline{1620 \text{ mm}}$$

$$\text{Total} = 2660 \text{ mm} = 2.66 \text{ m} < 3 \text{ m}$$

The last stirrup is $(3 - 2.66) = 0.34$ m $= 340$ mm from the centerline of the beam, which is adequate. A similar stirrup distribution applies to the other half of the beam, giving a total number of stirrups of 28.

The other examples in this chapter can be worked out in a similar way using SI equations.

SUMMARY

Sections 8.1–8.2

The shear stress in a homogeneous beam is $v = VQ/Ib$. The distribution of the shear stress above the neutral axis in a singly reinforced concrete beam is parabolic. Below the neutral axis, the maximum shear stress is maintained down to the level of the steel bars.

Section 8.3

The development of shear resistance in reinforced concrete members occurs by

- Shear resistance of the uncracked concrete
- Interface shear transfer

- Arch action
- Dowel action

Section 8.4

The shear stress at which a diagonal crack is expected is

$$v_c = \frac{V}{bd} = \left(1.9\lambda\sqrt{f_c'} + 2500\rho_w \frac{V_u d}{M_u}\right) \le 3.5\sqrt{f_c'}$$

The nominal shear strength is

$$V_c = v_c b_w d = 2\lambda\sqrt{f_c'} b_w d$$

Sections 8.5–8.6

1. The common types of shear reinforcement are stirrups (perpendicular or inclined to the main bars), bent bars, or combinations of stirrups and bent bars.

$$V_u = \phi V_n = \phi V_c + \phi V_s \quad \text{and} \quad V_s = \frac{1}{\phi}(V_u - \phi V_c)$$

2. The ACI Code design requirements are summarized in Table 8.4.

Sections 8.7–8.8

Design of vertical stirrups and shear summary is given in these sections.

Sections 8.9–8.10

1. Variation of shear force along the span due to live load may be considered.
2. For members with variable depth,

$$\phi V_n = V_u \pm \frac{M_u(\tan\alpha)}{d} \tag{8.29}$$

Section 8.11

For deep beams, the shear capacity, V_c, may be determined from the following expressions:

$$V_c = 2\lambda\sqrt{f_c'} b_w d \tag{8.35}$$

or

$$V_c = \left(3.5 - \frac{2.5M_u}{V_u d}\right)\left(1.9\lambda\sqrt{f_c'} + \frac{2500\rho_w V_u d}{M_u}\right) b_w d \tag{8.36}$$

The critical section for shear design is at $X = 0.15l_n$ for uniform loads and $X = 0.5a$ for concentrated loads.

Also, refer to Section 5.7 in text.

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PROBLEMS

8.1 Design the necessary shear reinforcement (if needed) in the form of U-stirrups (two legs) for the T-section shown in Fig. 8.30. Use $f'_c = 4$ ksi (28 MPa) and $f_y = 60$ ksi (420 MPa).

a. $V_u = 22$ K (98 kN)

b. $V_u = 56$ K (246 kN)

c. $V_u = 69$ K (306 kN)

8.2 Repeat Problem 8.1 for the section shown in Fig. 8.31.

8.3 Design the necessary shear reinforcement (if needed) in the form of U-stirrups (two legs) for the rectangular section shown in Fig. 8.32 using $f'_c = 3$ ksi (21 MPa) and $f_{yt} = 60$ ksi (420 MPa).

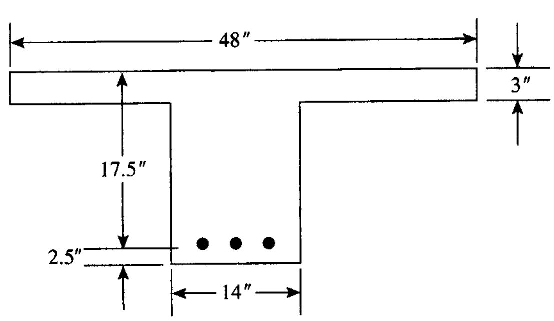


Figure 8.30 Problem 8.1.

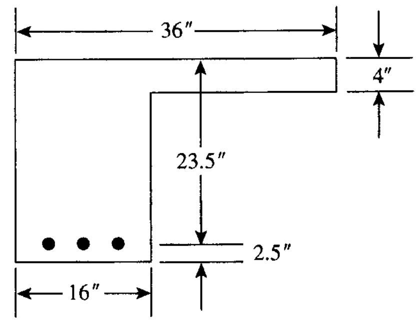


Figure 8.31 Problem 8.2.

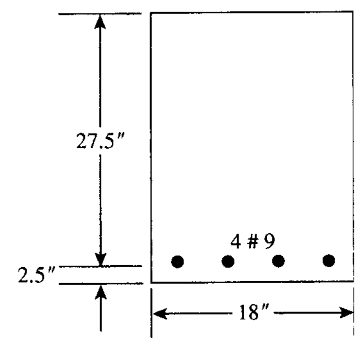


Figure 8.32 Problem 8.3.

- **a.** $V_u = 55\,\text{K}$ (245 kN)
- **b.** $V_u = 110\,\text{K}$ (490 kN)
- **c.** $V_u = 144\,\text{K}$ (640 kN)

8.4 A 16-ft- (4.8-m-)span simply supported beam, Fig. 8.33; has a clear span of 15 ft (4.5 m) and is supported by 12 × 12-in. (300 × 300-mm) columns. The beam carries a factored uniform load of 11.1 K/ft (166 kN/m). The dimensions of the beam section and the flexural steel reinforcement are shown in Fig. 8.33. Design the necessary shear reinforcements using $f_c' = 3$ksi (21 MPa) and $f_{yt} =$ 60 ksi (420 MPa). Show the distribution of stirrups along the beam.

8.5 An 18-ft- (5.4-m-)span simply supported beam carries a uniform dead load of 4 K/ft (60 kN/m) and a live load of 1.5 K/ft (22 kN/m). The beam has a width of $b = 12$ in. (300 mm) and a depth of $d = 24$ in. (600 mm) and is reinforced with six no. 9 bars (6 × 28 mm) in two rows. Check the beam for shear and design the necessary shear reinforcement. Given: $f_c' = 3$ ksi (21 MPa) and $f_{yt} = 50$ ksi (280 MPa).

8.6 Design the necessary shear reinforcement for a 14-ft (4.2-m) simply supported beam that carries a factored uniform load of 10 K/ft (150 kN/m) (including self-weight) and a factored concentrated load at midspan of $P_u = 24$ K (108 kN). The beam has a width of $b = 14$ in. (350 mm) and a depth of $d = 16.5$ (400 mm) and is reinforced with four no. 8 bars (4 × 25 mm). Given: $f_c' = 4$ ksi (28 MPa) and $f_{yt} = 60$ ksi (420 MPa).

8.7 A cantilever beam with 7.4-ft (2.20-m) span carries a uniform dead load of 2.5 K/ft (36 kN/m) (including self-weight) and a concentrated live load of 18 K (80 kN) at a distance of 3 ft (0.9 m) from the face of the support. Design the beam for moment and shear. Given: $f_c' = 3$ ksi (21 MPa), $f_{yt} = 60$ ksi (420 MPa), and $b = 12$ in. (200 mm), and use $\rho = 3/4\rho_{max}$.

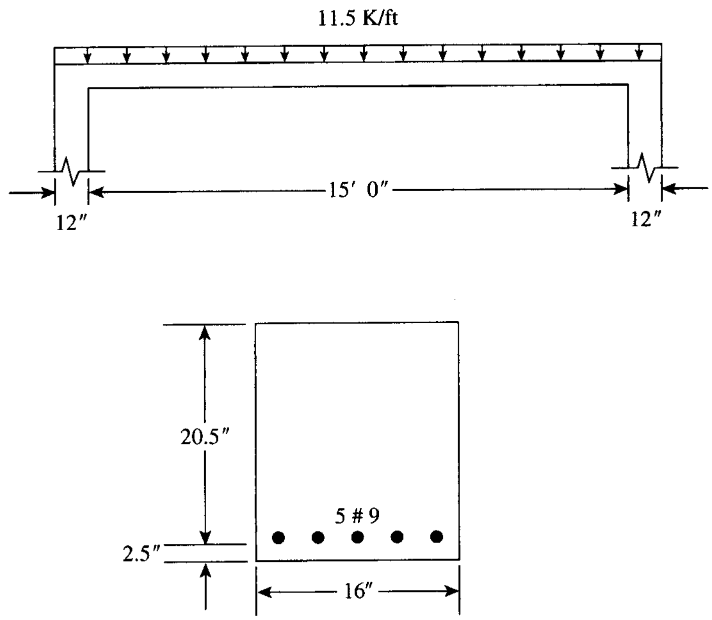


Figure 8.33 Problem 8.4.

8.8 Design the critical sections of an 11-ft-(3.3-m-)span simply supported beam for bending moment and shearing forces using $\rho = 0.016$. Given: $f_c' = 3$ ksi (21 MPa), $f_{yt} = 60$ ksi (420 MPa), and $b = 10$ in. (250 mm). Dead load is 2.75 K/ft (40 kN/m) and live load is 1.375 K/ft (20 kN/m).

8.9 A rectangular beam is to be designed to carry a factored shearing force of 75 kips (335 kN). Determine the minimum beam section if controlled by shear ($V_c = 2\lambda\sqrt{f_c'}bd$) using the minimum shear reinforcement as specified by the ACI Code and no. 3 stirrups. Given: $f_c' = 4$ ksi (28 MPa), $f_{yt} = 40$ ksi (280 MPa), and $b = 16$ in. (400 mm).

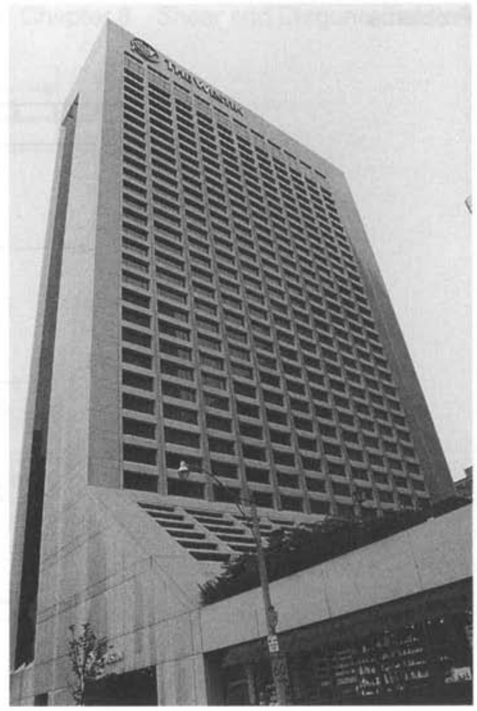
8.10 Redesign Problem 8.5 using $f_{yt} = 60$ ksi.

8.11 Redesign the shear reinforcement of the beam in Problem 8.6 if the uniform factored load of 6 K/ft (90 kN/m) is due to dead load and the concentrated load $P_u = 24$ k (108 kN) is due to a moving live load. Change the position of the live load to cause maximum shear at the support and at midspan.

8.12 Design a cantilever beam that has a span of 9 ft (2.7 m) to carry a factored triangular load that varies from 0 load at the free end to maximum load of 8 K/ft (120 kN/m) at the face of the support. The beam shall have a variable depth, with minimum depth at the free end of 10 in. (250 mm) and increasing linearly toward the support. Use steel percentage $\rho = 0.016$ for flexural design. Given: $f_c' = 4$ ksi (28 MPa), $f_{yt} = 60$ ksi (420 MPa) or flexural reinforcement, $f_{yt} = 40$ ksi (280 MPa) for stirrups, and $b = 11$ in. (275 mm).

CHAPTER 9

ONE-WAY SLABS



The Westin Hotel, Toronto, Canada.

9.1 TYPES OF SLABS

Structural concrete slabs are constructed to provide flat surfaces, usually horizontal, in building floors, roofs, bridges, and other types of structures. The slab may be supported by walls, by reinforced concrete beams usually cast monolithically with the slab, by structural steel beams, by columns, or by the ground. The depth of a slab is usually very small compared to its span. See Fig. 9.1.

Structural concrete slabs in buildings may be classified as follows:

1. *One-way slabs:* If a slab is supported on two opposite sides only, it will bend or deflect in a direction perpendicular to the supported edges. The structural action is one way, and the loads are carried by the slab in the deflected short direction. This type of slab is called a *one-way slab* (Fig. 9.1*a*). If the slab is supported on four sides and the ratio of the long side to the short side is equal to or greater than 2, most of the load (about 95% or more) is carried in the short direction, and one-way action is considered for all practical purposes (Fig. 9.1*b*). If the slab is made of reinforced concrete with no voids, then it is called a *one-way solid slab*. Fig. 9.1*c*, *d*, and *e* shows cross-sections and bar distribution.
2. *One-way joist floor system:* This type of slab is also called a *ribbed slab*. It consists of a floor slab, usually 2 to 4 in. (50 to 100 mm) thick, supported by reinforced concrete ribs (or joists). The ribs are usually tapered and are uniformly spaced at distances that do not exceed 30 in. (750 mm). The ribs are supported on girders that rest on columns. The spaces between the ribs may be formed using removable steel or fiberglass form fillers (pans), which may be used many times (Fig. 9.2). In some ribbed slabs, the spaces between ribs may be filled with permanent fillers to provide a horizontal slab.

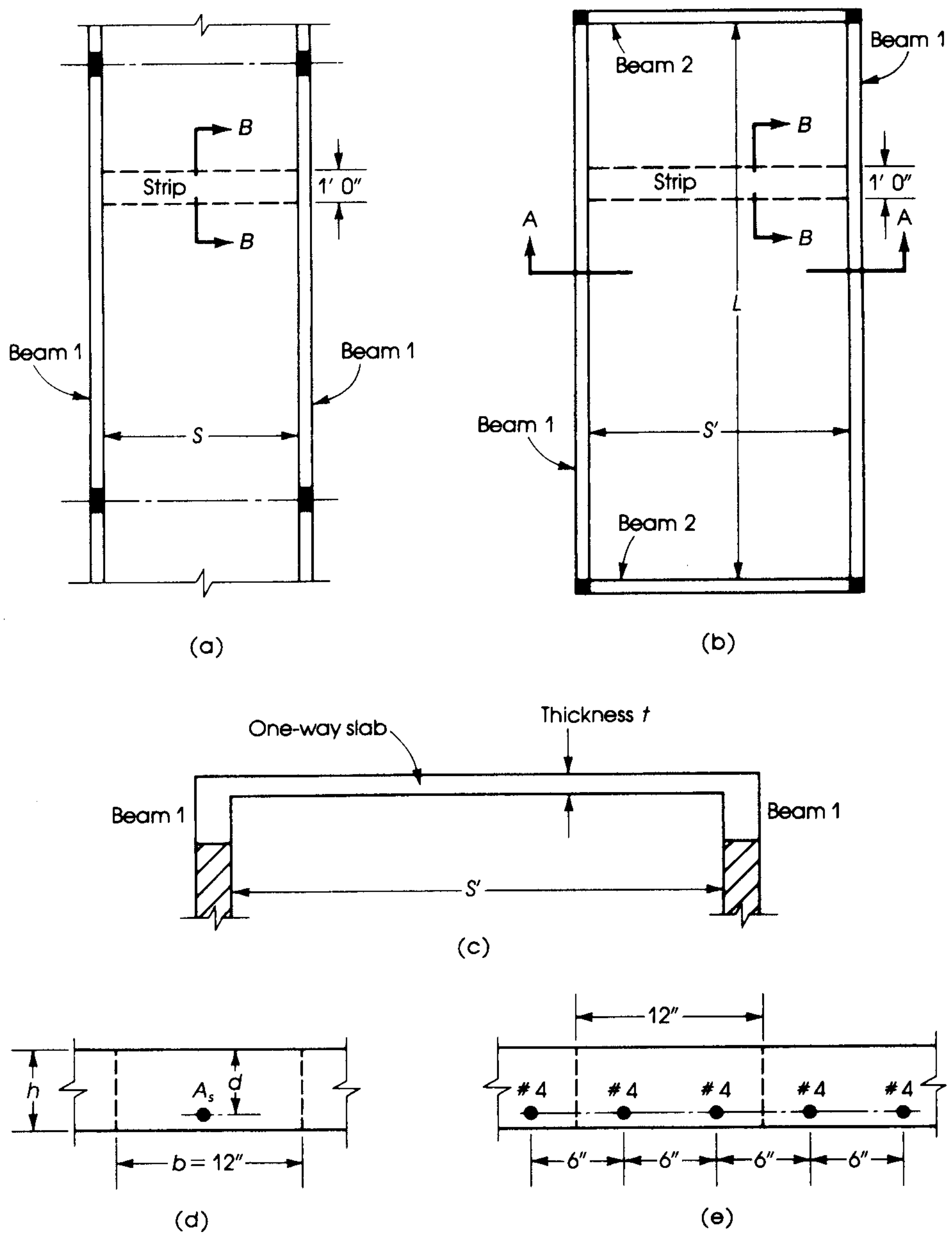


Figure 9.1 One-way slabs.

3. *Two-way floor systems:* When the slab is supported on four sides and the ratio of the long side to the short side is less than 2, the slab will deflect in double curvature in both directions. The floor load is carried in two directions to the four beams surrounding the slab (refer to Chapter 17). Other types of *two-way floor systems* are flat plate floors, flat slabs, and waffle slabs, all explained in Chapter 17. This chapter deals only with one-way floor systems.

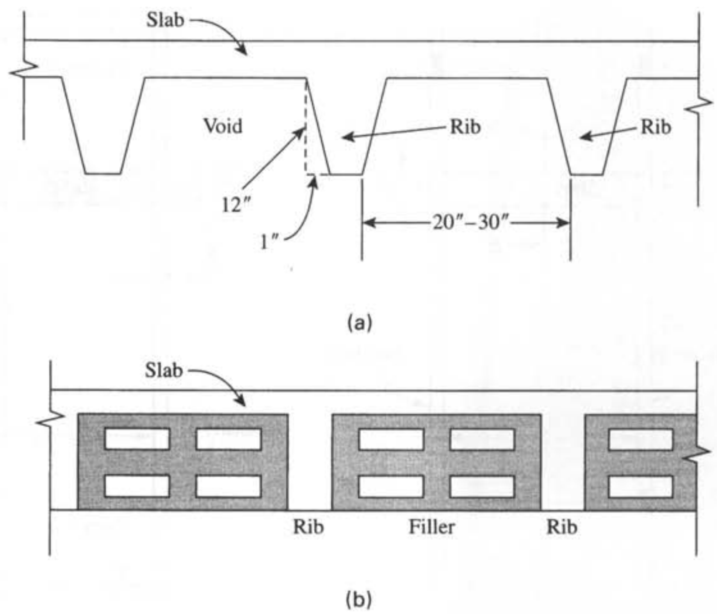


Figure 9.2 Cross-sections of one-way ribbed slab: (*a*) without fillers and (*b*) with fillers.

9.2 DESIGN OF ONE-WAY SOLID SLABS

If the concrete slab is cast in one uniform thickness without any type of voids, it can be referred to as a *solid slab*. In a one-way slab, the ratio of the length of the slab to its width is greater than 2. Nearly all the loading is transferred in the short direction, and the slab may be treated as a beam. A unit strip of slab, usually 1 ft (or 1 m) at right angles to the supporting girders, is considered a rectangular beam. The beam has a unit width with a depth equal to the thickness of the slab and a span length equal to the distance between the supports. A one-way slab thus consists of a series of rectangular beams placed side by side (Fig. 9.1).

If the slab is one span only and rests freely on its supports, the maximum positive moment M for a uniformly distributed load of w psf is $M = (wL^2)/8$, where L is the span length between the supports. If the same slab is built monolithically with the supporting beams or is continuous over several supports, the positive and negative moments are calculated either by structural analysis or by moment coefficients as for continuous beams. The ACI Code, Section 8.3, permits the use of moment and shear coefficients in the case of two or more approximately equal spans (Fig. 9.3). This condition is met when the larger of two adjacent spans does not exceed the shorter span by more than 20%. For uniformly distributed loads, the unit live load shall not exceed three times the unit dead load. When these conditions are not satisfied, structural analysis is required. In structural analysis, the negative bending moments at the centers of the supports are calculated. The value that may be considered in the design is the negative moment at the face of the support. To obtain this value, subtract from the maximum moment value at the center of the support a quantity equal to $Vb/3$, where V is the shearing force calculated from the analysis and b is the width of the support:

$$M_f \text{ (at face of the support)} = M_c \text{ (at centerline of support)} - \frac{Vb}{3} \tag{9.1}$$

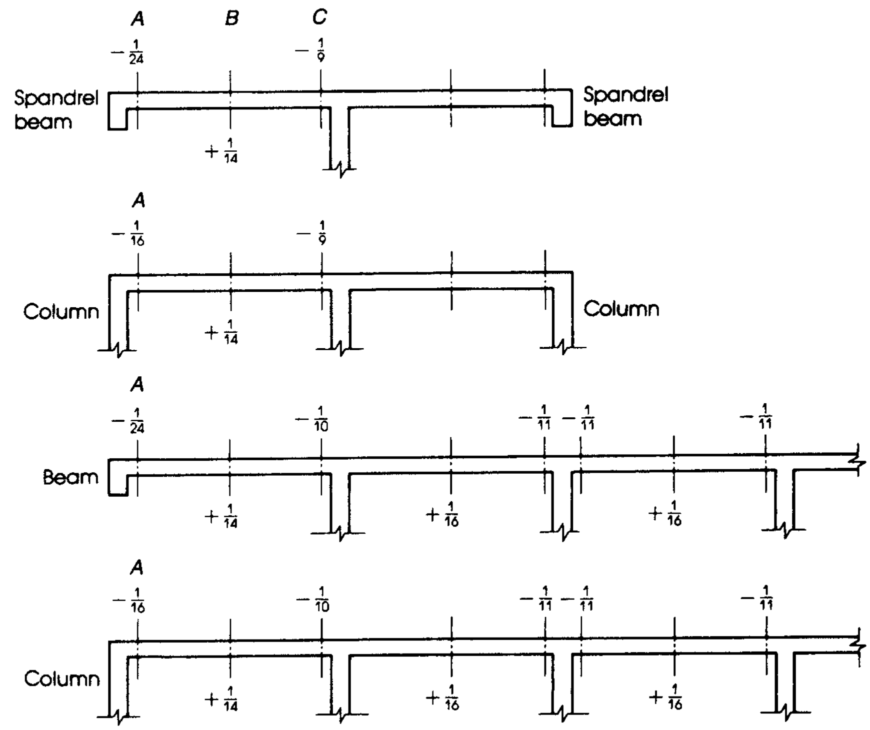


Figure 9.3 Moment coefficients for continuous beams and slabs (ACI Code, Section 8.3).

In addition to moment, diagonal tension and development length of bars should also be checked for proper design.

The conditions under which the moment coefficients for continuous beams and slabs given in Fig. 9.3 should be used can be summarized as follows:

1. Spans are approximately equal: Longer span $\leq$ 1.2 (shorter span).
2. Loads are uniformly distributed.
3. The ratio (live load/dead load) is less than or equal to 3.
4. For slabs with spans less than or equal to 10 ft, negative bending moment at face of all supports is $\left(\frac{1}{12}\right) w_u l_n^2$.
5. For an unrestrained discontinuous end at A, the coefficient is 0 at A and $+\frac{1}{11}$ at B.
6. Shearing force at C is $1.15 w_u l_n/2$ and at the face of all other support is $\frac{1}{2} w_u l_n$.
7. M_u = (coefficient) $(w_u l_n^2)$ and l_n = clear span.

9.3 DESIGN LIMITATIONS ACCORDING TO THE ACI CODE

The following limitations are specified by the ACI Code.

1. A typical imaginary strip 1 ft (or 1 m) wide is assumed.

2. The minimum thickness of one-way slabs using grade 60 steel according to the ACI Code, Table 9.5*a*, for solid slabs and for beams or ribbed one-way slabs should be equal to the following:
   - For simply supported spans: solid slabs, $h = L/20$ (ribbed slabs, $h = L/16$).
   - For one-end continuous spans: solid slabs, $h = L/24$ (ribbed slabs, $h = L/18.5$).
   - For both-end continuous spans: solid slabs, $h = L/28$ (ribbed slabs, $h = L/21$).
   - For cantilever spans: solid slabs, $h = L/10$ (ribbed slabs, $h = L/8$).
   - For f_y other than 60 ksi, these values shall be multiplied by $0.4 + 0.01 f_y$, where f_y is in ksi. This minimum thickness should be used unless computation of deflection indicates a lesser thickness can be used without adverse effects.
3. Deflection is to be checked when the slab supports are attached to construction likely to be damaged by large deflections. Deflection limits are set by the ACI Code, Table 9.5*b*.
4. It is preferable to choose slab depth to the nearest $\frac{1}{2}$ in. (or 10 mm).
5. Shear should be checked, although it does not usually control.
6. Concrete cover in slabs shall not be less than $\frac{3}{4}$ in. (20 mm) at surfaces not exposed to weather or ground. In this case, $d = h - \left(\frac{3}{4}\text{ in.}\right) - \text{(half-bar diameter)}$. Refer to Fig. 9.1*d*.
7. In structural slabs of uniform thickness, the minimum amount of reinforcement in the direction of the span shall not be less than that required for shrinkage and temperature reinforcement (ACI Code, Section 7.12).
8. The principal reinforcement shall be spaced not farther apart than three times the slab thickness nor more than 18 in. (ACI Code, Section 7.6.5).
9. Straight-bar systems may be used in both tops and bottoms of continuous slabs. An alternative bar system of straight and bent (trussed) bars placed alternately may also be used.
10. In addition to main reinforcement, steel bars at right angles to the main must be provided. This additional steel is called *secondary, distribution, shrinkage*, or *temperature reinforcement*.

9.4 TEMPERATURE AND SHRINKAGE REINFORCEMENT

Concrete shrinks as the cement paste hardens, and a certain amount of shrinkage is usually anticipated. If a slab is left to move freely on its supports, it can contract to accommodate the shrinkage. However, slabs and other members are joined rigidly to other parts of the structure, causing a certain degree of restraint at the ends. This results in tension stresses known as *shrinkage stresses*. A decrease in temperature and shrinkage stresses is likely to cause hairline cracks. Reinforcement is placed in the slab to counteract contraction and distribute the cracks uniformly. As the concrete shrinks, the steel bars are subjected to compression.

Reinforcement for shrinkage and temperature stresses normal to the principal reinforcement should be provided in a structural slab in which the principal reinforcement extends in one direction only. The ACI Code, Section 7.12.2, specifies the following minimum steel ratios: For slabs in which grade 40 or 50 deformed bars are used, $\rho = 0.2\%$, and for slabs in which grade 60 deformed bars or welded bars or welded wire fabric are used, $\rho = 0.18\%$. In no case shall such reinforcement be placed farther apart than five times the slab thickness or more than 18 in.

For temperature and shrinkage reinforcement, the whole concrete depth h exposed to shrinkage shall be used to calculate the steel area. For example, if a slab has a total depth of $h = 6$ in.

and $f_y = 60$ ksi, then the area of steel required per 1-ft width of slab is $A_s = 6(12)(0.0018) = 0.129\,\text{in.}^2$. The spacings of the bars, S, can be determined as follows:

$$S = \frac{12A_b}{A_s} \tag{9.2}$$

where A_b = area of the bar chosen and A_s = calculated area of steel.

For example, if no. 3 bars are used ($A_b = 0.11\,\text{in.}^2$), then $S = 12(0.11)/0.129 = 10.33$ in., say, 10 in. If no. 4 bars are chosen ($A_b = 0.2\,\text{in.}^2$), then $S = 12(0.2)/0.129 = 18.6$ in., say, 18 in. Maximum spacing is the smaller of five times slab thickness (30 in.) or 18 in. Then no. 4 bars spaced at 18 in. are adequate (or no. 3 bars at 10 in.). These bars act as secondary reinforcement and are placed normal to the main reinforcement calculated by flexural analysis. Note that areas of bars in slabs are given in Table A.14.

9.5 REINFORCEMENT DETAILS

In continuous one-way slabs, the steel area of the main reinforcement is calculated for all critical sections, at midspans, and at supports. The choice of bar diameter and detailing depends mainly on the steel areas, spacing requirements, and development length. Two bar systems may be adopted.

In the straight-bar system (Fig. 9.4), straight bars are used for top and bottom reinforcement in all spans. The time and cost to produce straight bars is less than that required to produce bent bars; thus, the straight-bar system is widely used in construction.

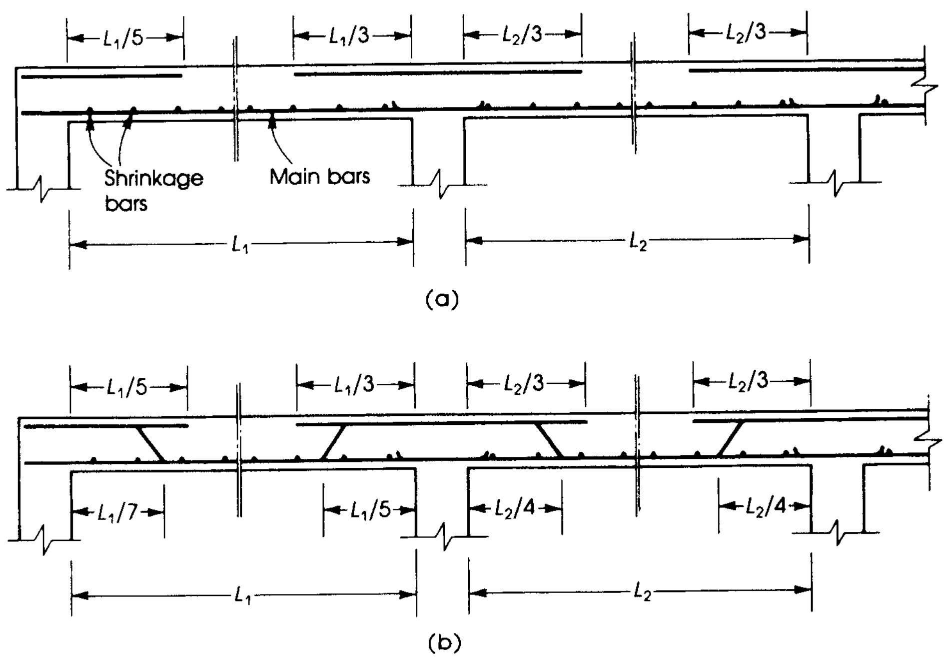


Figure 9.4 Reinforcement details in continuous one-way slabs: (*a*) straight bars and (*b*) bent bars.

In the bent-bar, or trussed, system, straight and bent bars are placed alternately in the floor slab. The location of bent points should be checked for flexural, shear, and development length requirements. For normal loading in buildings, the bar details at the end and interior spans of one-way solid slabs may be adopted as shown in Fig. 9.4.

9.6 DISTRIBUTION OF LOADS FROM ONE-WAY SLABS TO SUPPORTING BEAMS

In one-way floor slab systems, the loads from slabs are transferred to the supporting beams along the long ends of the slabs. The beams transfer their loads in turn to the supporting columns.

From Fig. 9.5 it can be seen that beam B_2 carries loads from two adjacent slabs. Considering a 1-ft length of beam, the load transferred to the beam is equal to the area of a strip 1 ft wide and S feet in length multiplied by the intensity of load on the slab.

This load produces a uniformly distributed load on the beam:

$$U_B = U_S \cdot S$$

The uniform load on the end beam, B_1, is half the load on B_2, because it supports a slab from one side only.

The load on column C_4 is equal to the reactions from two adjacent B_2 beams,

$$\text{Load on column } C_4 = U_B L = U_S L S$$

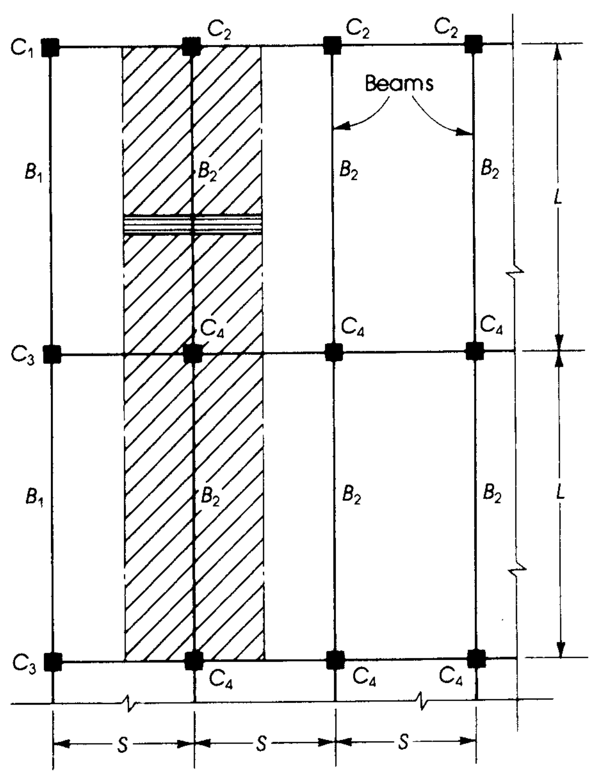


Figure 9.5 Distribution of loads on beams.

The load on column C_3 is one-half the load on column C_4, because it supports loads from slabs on one side only. Similarly, the loads on columns C_2 and C_1 are

$$\text{Load on } C_2 = U_S \frac{L}{2} S = \text{load on } C_3$$

$$\text{Load on } C_1 = U_S \left(\frac{L}{2}\right)\left(\frac{S}{2}\right)$$

From this analysis, it can be seen that each column carries loads from slabs surrounding the column and up to the centerline of adjacent slabs: up to $L/2$ in the long direction and $S/2$ in the short direction.

Distribution of loads from two-way slabs to their supporting beams and columns is discussed in Chapter 17.

Example 9.1

Calculate the design moment strength of a one-way solid slab that has a total depth of $h = 7$ in. and is reinforced with no. 6 bars spaced at $S = 7$ in. Use $f'_c = 3$ ksi and $f_y = 60$ ksi.

Solution

1. Determine the effective depth, d:

$$d = h - \frac{3}{4} \text{ in. (cover)} - \text{half-bar diameter} \quad \text{(See Fig. 9.1}d\text{).}$$

$$d = 7 - \frac{3}{4} - \frac{6}{16} = 5.875 \text{ in.}$$

2. Determine the average A_s provided per 1-ft width (12 in.) of slab. The area of no. 6 bar is $A_b = 0.44\,\text{in.}^2$.

$$A_s = \frac{12A_b}{S} = \frac{12(0.44)}{7} = 0.754 \text{ in.}^2/\text{ft}$$

Areas of bars in slabs are given in Table A.14 in Appendix A.

3. Compare the steel ratio used with ρ_{max} and ρ_{min}. For $f'_c = 3$ ksi and $f_y = 60$ ksi, $\rho_{max} = 0.01356$ and $\rho_{min} = 0.00333$. ρ (used) $= 0.754/(12 \times 5.875) = 0.0107$, which is adequate ($\phi = 0.9$).

4. Calculate $\phi M_n = \phi A_s f_y (d - a/2)$.

$$a = A_s f_y/(0.85 f'_c b) = 0.754(60)/(0.85 \times 3 \times 12) = 1.48 \text{ in.}$$

$$\phi M_n = 0.9(0.754)(60)(5.875 - 1.48/2) = 209 \text{ K·in.} = 17.42 \text{ K·ft}$$

Example 9.2

Determine the allowable uniform live load that can be applied on the slab of the previous example if the slab span is 16 ft between simple supports and carries a uniform dead load (excluding self-weight) of 100 psf.

Solution

1. The design moment strength of the slab is 17.42 K·ft per 1-ft width of slab.

$$M_u = \phi M_n = 17.42 = \frac{W_u L^2}{8} = \frac{W_u (16)^2}{8}$$

The factored uniform load is $W_u = 0.544$ K/ft^2 = 544 psf.

2.

$$W_u = 1.2D + 1.6L$$

$$D = 100 \text{ psf} + \text{self-weight} = 100 + \frac{7}{12}(150) = 187.5 \text{ psf}$$

$$544 = 1.2(187.5) + 1.6L \quad L = 200 \text{ psf}$$

Example 9.3

Design a 12-ft simply supported slab to carry a uniform dead load (excluding self-weight) of 120 psf and a uniform live load of 100 psf. Use $f'_c = 3$ ksi, $f_y = 60$ ksi, $\lambda = 1$, and the ACI Code limitations.

Solution

1. Assume a slab thickness. For $f_y = 60$ ksi, the minimum depth to control deflection is $L/20 = 12(12)/20 = 7$ in. Assume a total depth of $h = 7$ in. and assume $d = 6$ in. (to be checked later).
2. Calculate factored load: weight of slab $= \frac{7}{12}(150) = 87.5$ psf.

$$W_u = 1.2D + 1.6L = 1.2(87.5 + 120) + 1.6(100) = 409 \text{ psf}$$

For a 1-ft width of slab, $M_u = W_u L^2/8$.

$$M_u = \frac{0.409(12)^2}{8} = 7.362 \text{ K·ft}$$

3. Calculate A_s: For $M_u = 7.362$ K·ft, $b = 12$ in., and $d = 6$ in., $R_u = M_u/bd^2 = 7.362(12{,}000)/(12)(6)^2 = 205$ psi. From tables in Appendix A, $\rho = 0.0040 < \rho_{max} = 0.01356$, $\phi = 0.9$.

$$A_s = \rho bd = 0.0040(12)(6) = 0.28 \text{ in.}^2$$

Choosing no. 4 bars ($A_b = 0.2$ in.^{2}), and $S = 12A_b/A_s = 12(0.2)/0.28 = 8.6$ in. Check actual $d = h - \frac{3}{4} - \frac{4}{16} = 6$ in. It is acceptable. Let $S = 8$ in. and $A_s = 0.3$ in^2.

4. Check the moment capacity of the final section.

$$a = \frac{A_s f_y}{(0.85 f'_c b)} = \frac{0.3(60)}{0.85 \times 3 \times 12} = 0.59 \text{ in.}$$

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right) = 0.9(0.3)(60)(6 - 0.59/2) = 92.42 \text{ K·in.} = 7.7 \text{ K·ft} > M_u = 7.362 \text{ K·ft}$$

5. Calculate the secondary (shrinkage) reinforcement normal to the main steel. For $f_y = 60$ ksi,

$$\rho_{min} = 0.0018$$

$$A_{sh} = \rho bh = 0.0018(12)(7) = 0.1512 \text{ in.}^2$$

Choose no. 4 bars, $A_b = 0.2$ in.^{2}, $S = 12A_b/A_s = 12(0.2)/0.1512 = 15.9$ in. Use no. 4 bars spaced at 15 in.

6. Check shear requirements: V_u at a distance d from the support is $0.409\left(\frac{12}{2} - \frac{6}{12}\right) = 2.25$ K.

$$\phi V_c = \phi 2\lambda\sqrt{f'_c}bd = \frac{0.75(2)(1)(\sqrt{3000})(12 \times 6)}{1000} = 5.9 \text{ K}$$

$\frac{1}{2}\phi V_c = 2.95 \text{ K} > V_u$, so the shear is adequate.

7. Final section: $h = 7$ in., main bars = no. 4 spaced at 8 in., and secondary bars = no. 4 spaced at 15 in.

Example 9.4

The cross-section of a continuous one-way solid slab in a building is shown in Fig. 9.6. The slabs are supported by beams that span 12 ft between simple supports. The dead load on the slabs is that due to self-weight plus 77 psf; the live load is 130 psf. Design the continuous slab and draw a detailed section. Given: $f_c' = 3$ ksi and $f_y = 40$ ksi.

Solution

1. The minimum thickness of the first slab is $L/30$, because one end is continuous and the second end is discontinuous. The distance between centers of beams may be considered the span L, here equal to 12 ft. For $f_y = 40$ ksi,

$$\text{Minimum total depth} = \frac{L}{30} = \frac{12 \times 12}{30} = 4.8 \text{ in.}$$

$$\text{Minimum total depth for interior span} = \frac{L}{35} = 4.1 \text{ in.}$$

Assume a uniform thickness of 5 in., which is greater than 4.8 in.; therefore, it is not necessary to check deflection.

2. Calculate loads and moments in a unit strip:

$$\text{Dead load} = \text{weight of slab} + 60 \text{ psf}$$

$$= \left(\frac{5}{12} \times 150\right) + 77 = 139.5 \text{ psf}$$

$$\text{Factored load } (U) = 1.2D + 1.6L = 1.2 \times 139.5 + 1.6 \times 130 = 375.5 \text{ psf}$$

The clear span is 11.0 ft. The required moment in the first span is over the support and equals $UL^2/10$.

$$M_u = \frac{U(11)^2}{10} = (0.3755)\frac{121}{10} = 4.54 \text{ K·ft} = 54.5 \text{ K·in.}$$

3. Assume $\rho = 1.4\%$; then $R_u = 450$ psi $= 0.45$ ksi. This value is less than ρ_{max} of 0.0203 (Table 4.1), and greater than ρ_{min} of 0.005 ($\phi = 0.9$).

$$d = \sqrt{\frac{M_u}{R_u b}} = \sqrt{\frac{54.5}{0.45 \times 12}} = 3.18 \text{ in.}$$

$$A_s = \rho b d = 0.014(12)(3.18) = 0.53 \text{ in.}^2$$

Choosing no. 5 bars,

$$\text{Total depth} = d + \tfrac{1}{2} \text{ bar diameter} + \text{cover} = 3.18 + \tfrac{5}{16} + \tfrac{3}{4} = 4.25 \text{ in.}$$

Use slab thickness of 5 in., as assumed earlier.

$$\text{Actual } d \text{ used} = 5 - \tfrac{3}{4} - \tfrac{5}{16} = 3.9 \text{ in.}$$

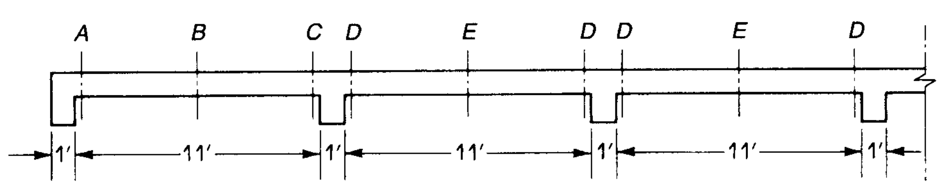


Figure 9.6 Example 9.4.

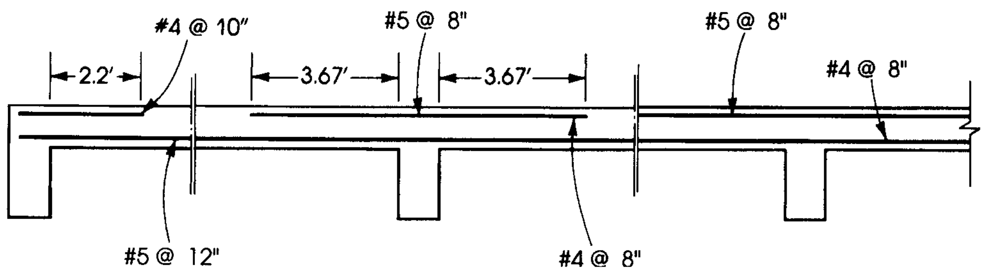


Figure 9.7 Example 9.4: Reinforcement details.

4. Moments and steel reinforcement required at other sections using $d = 3.9$ in. are as follows:

Location	Moment Coefficient	M_u (K·in.)	$R_u = M_u/bd^2$ (psi)	ρ (%)	A_s (in.^{2})	Bars and Spacings
A	$-\frac{1}{24}$	22.7	Small	0.50	0.23	No. 4 at 10 in.
B	$+\frac{1}{14}$	38.9	213	0.65	0.30	No. 5 at 12 in.
C	$-\frac{1}{10}$	54.5	300	0.90	0.44	No. 5 at 8 in.
D	$-\frac{1}{11}$	49.6	271	0.80	0.38	No. 5 at 8 in.
E	$+\frac{1}{16}$	34.1	187	0.55	0.26	No. 4 at 8 in.

The arrangement of bars is shown in Fig. 9.7.

5. Maximum shear occurs at the exterior face of the second support, section C.

$$V_u \text{ (at } C) = 1.15UL_n/2 = \frac{1.15(0.3755)(11)}{2} = 2.375 \text{ K/ft of width}$$

$$\phi V_c = \phi 2\lambda\sqrt{f_c'}bd = \frac{0.75(2)(1)(\sqrt{3000})(12)(3.9)}{1000} = 3.84 \text{ K}$$

This result is acceptable. Note that the provision of minimum area of shear reinforcement when V_u exceeds $\frac{1}{2}\phi V_c$ does not apply to slabs (ACI Code, Section 11.5.5).

Example 9.5

Determine the uniform factored load on an intermediate beam supporting the slabs of Example 9.4. Also calculate the axial load on an interior column; refer to the general plan of Fig. 9.5.

Solution

1. The uniform factored load per foot length on an intermediate beam is equal to the factored uniform load on slab multiplied by S, the short dimension of the slab. Therefore,

$$U \text{ (beam)} = U \text{ (slab)} \times S = 0.3755 \times 12 = 4.5 \text{ K/ft}$$

The weight of the web of the beam shall be added to this value. Span of the beam is 24 ft.

$$\text{Estimated total depth} = \frac{L}{20} \times 0.8 = \left(\frac{24}{20} \times 0.8\right) \times 12 = 11.5 \text{ in.} \quad \text{say, 12 in.}$$

Slab thickness is 5 in. and height of the web is $12 - 5 = 7$ in.

$$\text{Factored weight of beam web} = \left(\frac{7}{12} \times 150\right) \times 1.2 = 105 \text{ lb/ft}$$

$$\text{Total uniform load on beam} = 4.5 + 0.105 = 4.605 \text{ K/ft}$$

2. Axial load on an interior column:

$$P_u = 4.605 \times 24 \text{ ft} = 110.5 \text{ K}$$

9.7 ONE-WAY JOIST FLOOR SYSTEM

A one-way joist floor system consists of hollow slabs with a total depth greater than that of solid slabs. The system is most economical for buildings where superimposed loads are small and spans are relatively large, such as schools, hospitals, and hotels. The concrete in the tension zone is ineffective; therefore, this area is left open between ribs or filled with lightweight material to reduce the self-weight of the slab.

The design procedure and requirements of ribbed slabs follow the same steps as those for rectangular and T-sections explained in Chapter 3. The following points apply to design of one-way ribbed slabs:

1. Ribs are usually tapered and uniformly spaced at about 16 to 30 in. (400 to 750 mm). Voids are usually formed by using pans (molds) 20 in. (500 mm) wide and 6 to 20 in. (150 to 500 mm) deep, depending on the design requirement. The standard increment in depth is 2 in. (50 mm).
2. The ribs shall not be less than 4 in. (100 mm) wide and must have a depth of not more than 3.5 times the width. Clear spacing between ribs shall not exceed 30 in. (750 mm) (ACI Code, Section 8.13).
3. Shear strength, V_c, provided by concrete for the ribs may be taken 10% greater than that for beams. This is mainly due to the interaction between the slab and the closely spaced ribs (ACI Code, Section 8.13.8).
4. The thickness of the slab on top of the ribs is usually 2 to 4 in. (50 to 100 mm) and contains minimum reinforcement (shrinkage reinforcement). This thickness shall not be less than $\frac{1}{12}$ of the clear span between ribs or 1.5 in. (38 mm) (ACI Code, Section 8.13.5.2).
5. The ACI coefficients for calculating moments in continuous slabs can be used for continuous ribbed slab design.
6. There are additional practice limitations, which can be summarized as follows:
   - The minimum width of the rib is one-third of the total depth or 4 in. (100 mm), whichever is greater.
   - Secondary reinforcement in the slab in the transverse directions of ribs should not be less than the shrinkage reinforcement or one-fifth of the area of the main reinforcement in the ribs.
   - Secondary reinforcement parallel to the ribs shall be placed in the slab and spaced at distances not more than half of the spacings between ribs.
   - If the live load on the ribbed slab is less than 3 kN/m^2 (60 psf) and the span of ribs exceeds 5 m (17 ft), a secondary transverse rib should be provided at midspan (its direction

is perpendicular to the direction of main ribs) and reinforced with the same amount of steel as the main ribs. Its top reinforcement shall not be less than half of the main reinforcement in the tension zone. These transverse ribs act as floor stiffeners.

- If the live load exceeds 3 kN/m^2 (60 psf) and the span of ribs varies between 4 and 7 m (13 and 23 ft), one traverse rib must be provided, as indicated before. If the span exceeds 7 m (23 ft), at least two transverse ribs at one-third span must be provided with reinforcement, as explained before.

Example 9.6

Design an interior rib of a concrete joist floor system with the following description: Span of rib = 20 ft (simply supported), dead load (excluding own weight) = 16 psf, live load = 85 psf, $f_c' = 4$ ksi, and $f_y = 60$ ksi.

Solution

1. Design of the slab: Assume a top slab thickness of 2 in. that is fixed to ribs that have a clear spacing of 20 in. No fillers are used. The self-weight of the slab is $\frac{2}{12} \times 150 = 25$ psf.

$$\text{Total D.L.} = 16 + 16 = 41 \text{ psf}$$

$$U = 1.2D + 1.6L = 1.2 \times 41 + 1.6 \times 85 = 185 \text{ psf}$$

$$M_u = \frac{UL^2}{12} \quad \text{(Slab is assumed fixed to ribs.)}$$

$$= \frac{0.185}{12}\left(\frac{20}{12}\right)^2 = 0.043 \text{ K·ft} = 0.514 \text{ K·in.}$$

Considering that the moment in slab will be carried by plain concrete only, the allowable flexural tensile strength is $f_t = 5\sqrt{f_c'}$, with a capacity-reduction factor $\phi = 0.55$, $f_t = 5\sqrt{4000} =$ 316 psi.

$$\text{Flexural tensile strength} = \frac{Mc}{I} = \phi f_t \quad I = \frac{bh^3}{12} = \frac{12(2)^3}{12} = 8 \text{ in.}^4 \quad c = \frac{h}{2} = \frac{2}{2} = 1 \text{ in.}$$

$$M = \phi f_t \frac{I}{c} = 0.55 \times 0.316 \times \frac{8}{1} = 1.39 \text{ K·in.}$$

This value is greater than $M_u = 0.514$ K·in., and the slab is adequate. For shrinkage reinforcement, $A_s = 0.0018 \times 12 \times 2 = 0.043$ in.^{2} Use no. 3 bars spaced at 12 in. laid transverse to the direction of the ribs. Welded wire fabric may be economically used for this low amount of steel reinforcement. Use similar shrinkage reinforcement no. 3 bars spaced at 12 in. laid parallel to the direction of ribs, one bar on top of each rib and one bar in the slab between ribs.

2. Calculate moment in a typical rib:

$$\text{Minimum depth} = \frac{L}{20} = \frac{20 \times 12}{20} = 12 \text{ in.}$$

The total depth of rib and slab is 10 + 2 = 12 in. Assume a rib width of 4 in. at the lower end that tapers to 6 in. at the level of the slab (Fig. 9.8). The average width is 5 in. Note that the increase in the rib width using removable forms has a ratio of about 1 horizontal to 12 vertical.

$$\text{Weight of rib} = \frac{5}{12} \times \frac{10}{12} \times 150 = 52 \text{ lb/ft}$$

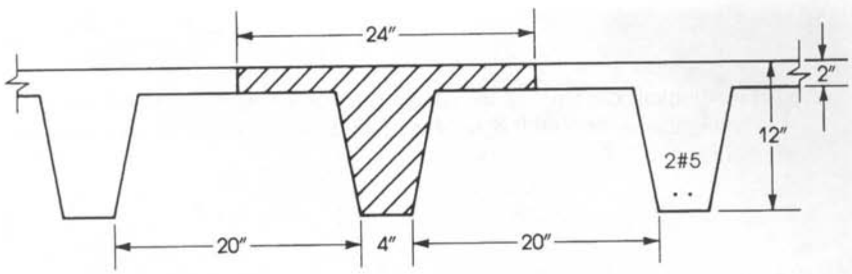
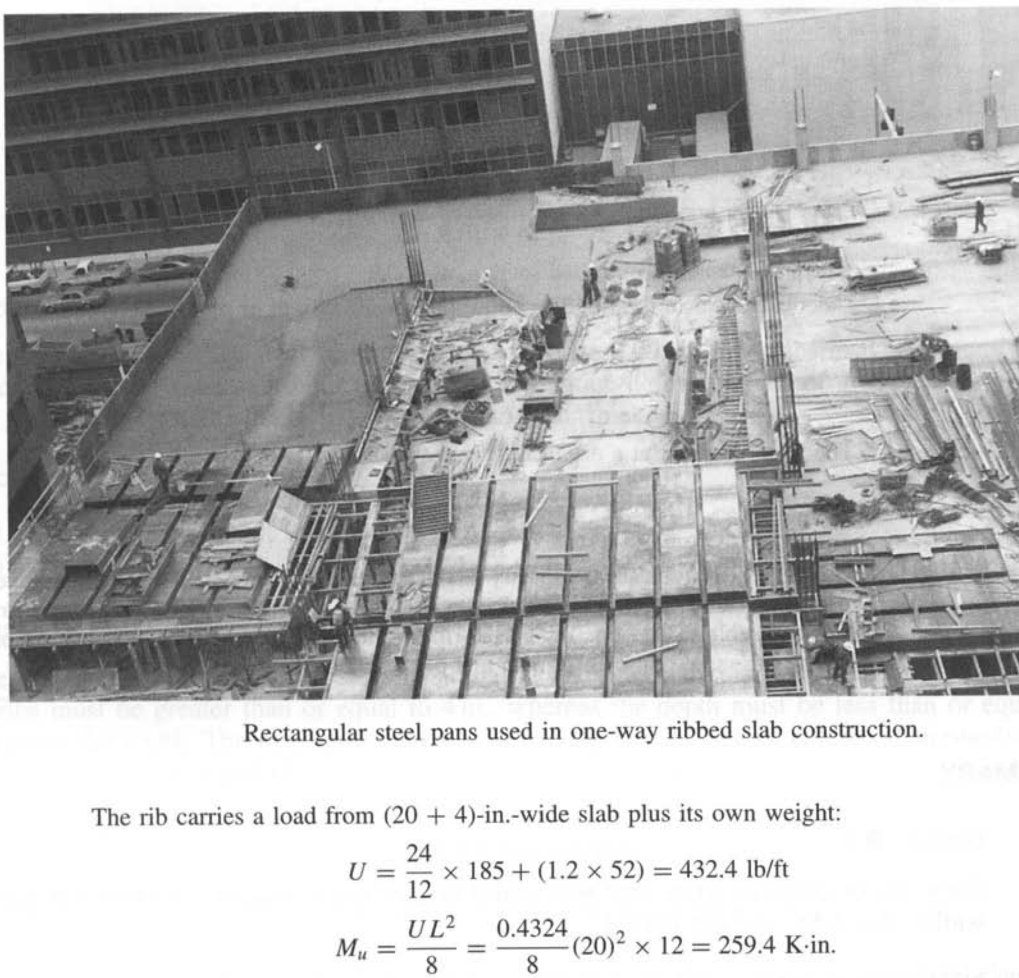


Figure 9.8 Example 9.6.



Rectangular steel pans used in one-way ribbed slab construction.

The rib carries a load from (20 + 4)-in.-wide slab plus its own weight:

$$U = \frac{24}{12} \times 185 + (1.2 \times 52) = 432.4 \text{ lb/ft}$$

$$M_u = \frac{UL^2}{8} = \frac{0.4324}{8}(20)^2 \times 12 = 259.4 \text{ K·in.}$$

3. Design of rib: The total depth is 12 in. Assuming no. 5 bars and concrete cover of $\frac{3}{4}$ in., the effective depth d is $12 - \frac{3}{4} - \frac{5}{16} = 10.9$ in. Check the moment capacity of the flange (assume tension-controlled section, $\phi = 0.9$):

$$\phi M_n \text{ (flange)} = \phi C \left(d - \frac{t}{2}\right), \quad \text{where } C = 0.85 f_c' bt$$

$$M_u = 0.9(0.85 \times 4 \times 24 \times 2)\left(10.9 - \frac{2}{2}\right) = 1454 \text{ K·in.}$$

The moment capacity of the flange is greater than the applied moment; thus, the rib acts as a rectangular section with $b = 24$ in., and the depth of the equivalent compressive block a is less than 2 in.

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right) = \phi A_s f_y \left(d - \frac{A_s f_y}{1.7 f_c' b}\right)$$

$$259.4 = 0.9 A_s \times 60 \left(10.9 - \frac{A_s \times 60}{1.7 \times 4 \times 24}\right) \quad A_s = 0.45 \text{ in.}^2$$

$$a = \frac{A_s f_y}{0.85 \times f_c' b} = 0.33 \text{ in.} < 2 \text{ in.}$$

Use two no. 5 bars per rib ($A_s = 0.65$ in.^{2}).

$$A_{s\,\min} = 0.0033 b_w d = 0.0033(5)(10.9) = 0.18 \text{ in.}^2 < 0.45 \text{ in.}^2$$

Check

$$\rho = \frac{0.45}{24 \times 10.9} = 0.00172 < \rho_{\max} = 0.01806$$

which is a tension-controlled section, $\phi = 0.9$.

4. Calculate shear in the rib: The allowable shear strength of the rib web is

$$\phi V_c = \phi(1.1) \times 2\lambda\sqrt{f_c'} b_w d$$
$$= 0.75 \times 1.1 \times 2(1)\sqrt{4000} \times 5 \times 10.9 = 5687 \text{ lb}$$

The factored shear at a distance d from the support is

$$V_u = 432.4\left(10 - \frac{10.9}{12}\right) = 3931 \text{ lb}$$

This is less than the shear capacity of the rib. Minimum stirrups may be used, and in this case an additional no. 4 bar will be placed within the slab above the rib to hold the stirrups in place. It is advisable to add one transverse rib at midspan perpendicular to the direction of the ribs having the same reinforcement as that of the main ribs to act as a stiffener.

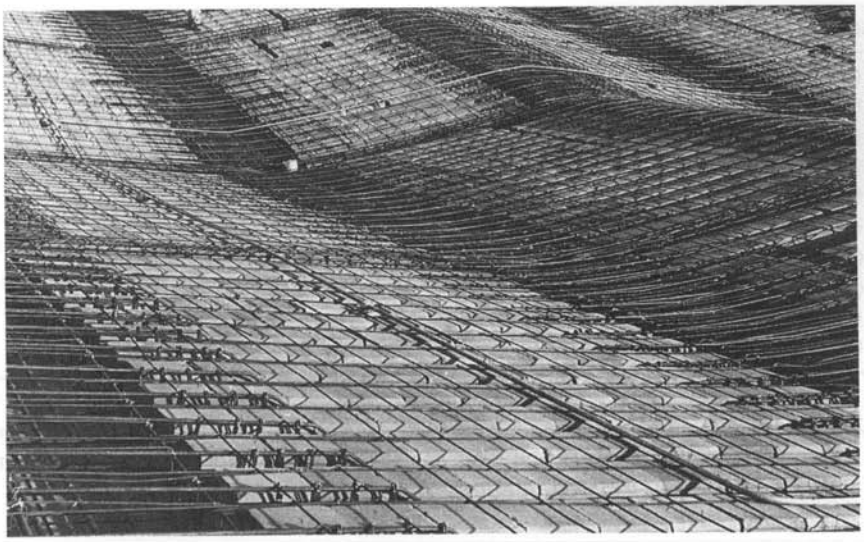
SUMMARY

Section 9.1

Slabs are of different types, one way (solid or joist floor systems) and two way (solid, ribbed, waffle, flat slabs, and flat plates).

Sections 9.2–9.3

1. The ACI Code moment and shear coefficients for continuous one-way slabs are given in Fig. 9.3.
2. The minimum thickness of one-way slabs using grade 60 steel is $L/20$, $L/24$, $L/28$, and $L/10$ for simply supported, one-end continuous, both-end continuous, and cantilever slabs, respectively.



One-way ribbed slab roof. The wide beams have the same total depth as the ribbed slab.

Section 9.4

The minimum shrinkage steel ratios, ρ_{min}, in slabs are 0.002 in. for slabs in which grade 40 or grade 50 bars are used and 0.0018 in. for slabs in which deformed bars of grade 60 are used.

Maximum spacings between bars $\leq$ 5 times rib thickness $\leq$ 18 in.

Sections 9.5–9.6

1. Reinforcement details are shown in Fig. 9.4.
2. Distribution of loads from one-way slabs to the supporting beams is shown in Fig. 9.5.

Section 9.7

The design procedure of ribbed slabs is similar to that of rectangular and T-sections. The width of ribs must be greater than or equal to 4 in., whereas the depth must be less than or equal to 3.5 times the width. The minimum thickness of the top slab is 2 in. or not less than one-twelfth of the clear span between ribs.

REFERENCES

1. Concrete Reinforcing Steel Institute. *CRSI Design Handbook*. Chicago, 2002.
2. Portland Cement Association. *Continuity in Concrete Building Frames*. Chicago, 1959.
3. American Concrete Institute. ACI Code 318-08, Building Code Requirements for Structural Concrete. Detroit, Michigan, 2008.

PROBLEMS

9.1 For each problem, calculate the factored moment capacity of each concrete slab section using $f_y = 60$ ksi.

Number	f_c'(ksi)	h (in.)	Bars and Spacings (in.)	Answer ϕM_n (K·ft)
(a)	3	5	No. 4 at 6	6.35
(b)	3	6	No. 5 at 8	9.29
(c)	3	7	No. 6 at 9	14.06
(d)	3	8	No. 8 at 12	21.01
(e)	4	$5\frac{1}{2}$	No. 5 at 10	6.93
(f)	4	6	No. 7 at 12	11.80
(g)	4	$7\frac{1}{2}$	No. 6 at 6	22.68
(h)	4	8	No. 8 at 12	21.23
(i)	5	5	No. 5 at 10	6.19
(j)	5	6	No. 5 at 8	9.66

9.2 For each slab problem, determine the required steel reinforcement, A_s, and the total depth, if required; then choose adequate bars and their spacings. Use $f_y = 60$ ksi for all problems, $b = 12$ in., and a steel ratio close to the steel ratio $\rho = A_s/bd$ given in some problems.

					One Answer	
Number	f_c'(ksi)	M_u (K·ft)	h (in.)	ρ (%)	h (in.)	Bars
(a)	3	5.4	6	—	6	No. 4 at 9 in.
(b)	3	13.8	$7\frac{1}{2}$	—	$7\frac{1}{2}$	No. 6 at 10 in.
(c)	3	24.4	—	0.85	9	No. 8 at 12 in.
(d)	3	8.1	5	—	5	No. 5 at 7 in.
(e)	4	22.6	—	1.18	$7\frac{1}{2}$	No. 7 at 8 in.
(f)	4	13.9	$8\frac{1}{2}$	—	$8\frac{1}{2}$	No. 6 at 12 in.
(g)	4	13.0	—	1.10	6	No. 6 at 8 in.
(h)	4	11.2	—	0.51	$7\frac{1}{2}$	No. 5 at 9 in.
(i)	5	20.0	9	—	9	No. 7 at 12 in.
(j)	5	10.6	—	0.90	6	No. 6 at 10 in.

9.3 A 16-ft- (4.8-m-)span simply supported slab carries a uniform dead load of 200 psf (10 kN/m^2) (excluding its own weight). The slab has a uniform thickness of 7 in. (175 mm) and is reinforced with no. 6 (20-mm) bars spaced at 5 in. (125 mm). Determine the allowable uniformly distributed load that can be applied on the slab if $f_c' = 4$ ksi (28 MPa) and $f_y = 60$ ksi (420 MPa)

9.4 Design a 10-ft (3-m) cantilever slab to carry a uniform total dead load of 170 psf (8.2 kN/m^2) and a concentrated live load at the free end of 2 K/ft (30 kN/m), when $f_c' = 4$ ksi (28 MPa) and $f_y =$ 60 ksi (420 MPa).

9.5 A 6-in. (150-mm) solid one-way slab carries a uniform dead load of 190 psf (9.2 kN/m^2) (including its own weight) and a live load of 80 psf (3.9 kN/m^2). The slab spans 12 ft (3.6 m) between 10-in.-(250-mm-)wide simple supports. Determine the necessary slab reinforcement using $f_c' = 4$ ksi (28 MPa) and $f_y = 50$ ksi (350 MPa).

9.6 Repeat Problem 9.4 using a variable section with a minimum total depth at the free end of 4 in. (100 mm).

9.7 Design a continuous one-way solid slab supported on beams spaced at 14 ft (4.2 m) on centers. The width of the beams is 12 in. (300 mm), leaving clear slab spans of 13 ft (3.9 m). The slab carries a uniform dead load of 126 psf (6.0 kN/m$_2$) (including self-weight of slab) and a live load of 120 psf (5.8 kN/m$_2$). Use $f_c' = 3$ ksi (21 MPa), $f_y = 40$ ksi (280 MPa), and the ACI coefficients. Show bar arrangements using straight bars for all top and bottom reinforcement.

9.8 Repeat Problem 9.7 using equal clear spans of 10 ft (3 m), $f_c' = 3$ ksi (21 MPa), and $f_y = 60$ ksi (420 MPa).

9.9 Repeat Problem 9.7 using $f_c' = 4$ ksi (28 MPa) and $f_y = 60$ ksi (420 MPa).

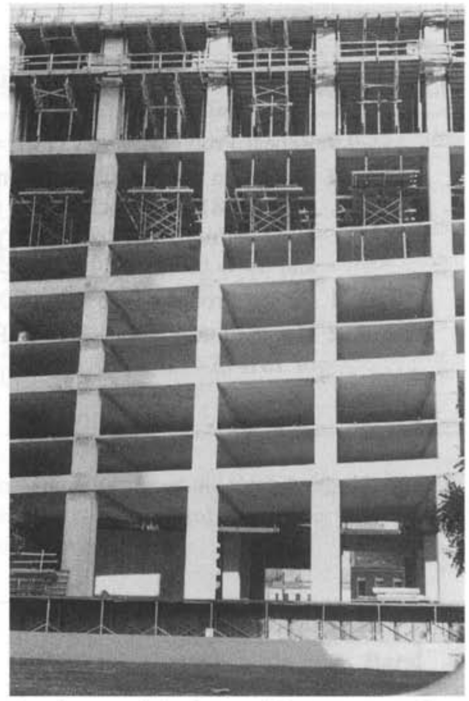
9.10 Design an interior rib of a concrete joist floor system with the following description: Span of ribbed slab is 18 ft (5.4 m) between simple supports; uniform dead load (excluding self-weight) is 30 psf (1.44 kN/m^2); live load is 100 psf (4.8 kN/m^2); support width is 14 in. (350 mm); $f_c' = 3$ ksi (21 MPa) and $f_y = 60$ ksi (420 MPa). Use 30-in.- (750-mm-)wide removable pans.

9.11 Repeat Problem 9.10 using 20-in.- (500-mm-)wide removable pans.

9.12 Use the information given in Problem 9.10 to design a continuous ribbed slab with three equal spans of 18 ft (5.4 m) each.

CHAPTER 10

AXIALLY LOADED COLUMNS



Continuous slabs in a parking structure, New Orleans, Louisiana.

10.1 INTRODUCTION

Columns are members used primarily to support axial compressive loads and have a ratio of height to the least lateral dimension of 3 or greater. In reinforced concrete buildings, concrete beams, floors, and columns are cast monolithically, causing some moments in the columns due to end restraint. Moreover, perfect vertical alignment of columns in a multistory building is not possible, causing loads to be eccentric relative to the center of columns. The eccentric loads will cause moments in columns. Therefore, a column subjected to pure axial loads does not exist in concrete buildings. However, it can be assumed that axially loaded columns are those with relatively small eccentricity, e, of about $0.1h$ or less, where h is the total depth of the column and e is the eccentric distance from the center of the column. Because concrete has a high compressive strength and is an inexpensive material, it can be used in the design of compression members economically. This chapter deals only with short columns; slender columns are covered in detail in Chapter 12.

10.2 TYPES OF COLUMNS

Columns may be classified based on the following different categories (Fig. 10.1):

1. Based on loading, columns may be classified as follows:
   a. Axially loaded columns, where loads are assumed acting at the center of the column section.

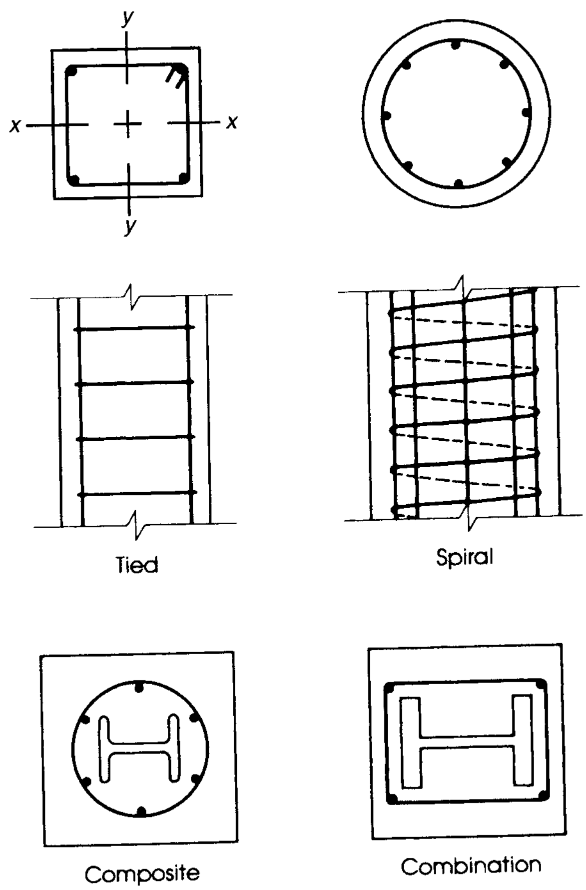


Figure 10.1 Types of columns.

b. Eccentrically loaded columns, where loads are acting at a distance e from the center of the column section. The distance e could be along the x- or y-axis, causing moments either about the x- or y-axis.

c. Biaxially loaded columns, where the load is applied at any point on the column section, causing moments about both the x- and y-axes simultaneously.

2. Based on length, columns may be classified as follows:

a. Short columns, where the column's failure is due to the crushing of concrete or the yielding of the steel bars under the full load capacity of the column.

b. Long columns, where buckling effect and slenderness ratio must be taken into consideration in the design, thus reducing the load capacity of the column relative to that of a short column.

3. Based on the shape of the cross-section, column sections may be square, rectangular, round, L-shaped, octagonal, or any desired shape with an adequate side width or dimensions.

4. Based on column ties, columns may be classified as follows:

a. Tied columns containing steel ties to confine the main longitudinal bars in the columns. Ties are normally spaced uniformly along the height of the column.

b. Spiral columns containing spirals (spring-type reinforcement) to hold the main longitudinal reinforcement and to help increase the column ductility before failure. In general, ties and spirals prevent the slender, highly stressed longitudinal bars from buckling and bursting the concrete cover.

5. Based on frame bracing, columns may be part of a frame that is braced against sidesway or unbraced against sidesway. Bracing may be achieved by using shear walls or bracings in the building frame. In braced frames, columns resist mainly gravity loads, and shear walls resist lateral loads and wind loads. In unbraced frames, columns resist both gravity and lateral loads, which reduce the load capacity of the columns.

6. Based on materials, columns may be reinforced, prestressed, composite (containing rolled steel sections such as I-sections), or a combination of rolled steel sections and reinforcing bars. Concrete columns reinforced with longitudinal reinforcing bars are the most common type used in concrete buildings.

10.3 BEHAVIOR OF AXIALLY LOADED COLUMNS

When an axial load is applied to a reinforced concrete short column, the concrete can be considered to behave elastically up to a low stress of about $(\frac{1}{3})f'_c$. If the load on the column is increased to reach its ultimate strength, the concrete will reach the maximum strength and the steel will reach its yield strength, f_y. The nominal load capacity of the column can be written as follows:

$$P_o = 0.85 f'_c A_n + A_{\mathrm{st}} f_y \tag{10.1}$$

where A_n and A_{st} = the net concrete and total steel compressive areas, respectively.

$$A_n = A_g - A_{\mathrm{st}}$$

$$A_g = \text{gross concrete area}$$

Two different types of failure occur in columns, depending on whether ties or spirals are used. For a tied column, the concrete fails by crushing and shearing outward, the longitudinal steel bars fail by buckling outward between ties, and the column failure occurs suddenly, much like the failure of a concrete cylinder.

A spiral column undergoes a marked yielding, followed by considerable deformation before complete failure. The concrete in the outer shell fails and spalls off. The concrete inside the spiral is confined and provides little strength before the initiation of column failure. A hoop tension develops in the spiral, and for a closely spaced spiral, the steel may yield. A sudden failure is not expected. Figure 10.2 shows typical load deformation curves for tied and spiral columns. Up to point a, both columns behave similarly. At point a, the longitudinal steel bars of the column yield, and the spiral column shell spalls off. After the factored load is reached, a tied column fails suddenly (curve b), whereas a spiral column deforms appreciably before failure (curve c).

10.4 ACI CODE LIMITATIONS

The ACI Code presents the following limitations for the design of compression members:

1. For axially as well as eccentrically loaded columns, the ACI Code sets the strength-reduction factors at $\phi = 0.65$ for tied columns and $\phi = 0.75$ for spirally reinforced columns. The

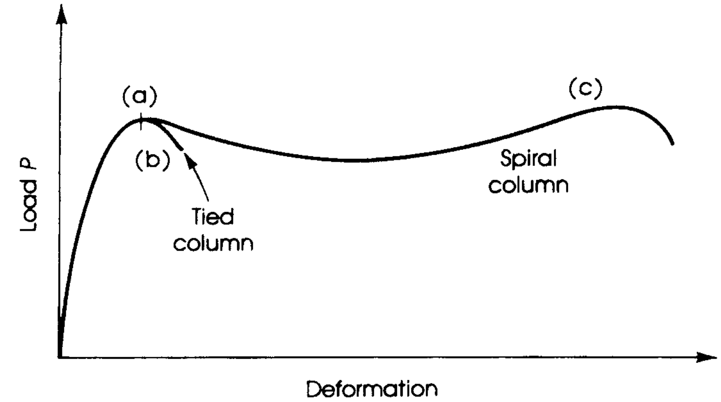


Figure 10.2 Behavior of tied and spiral columns.

difference of 0.05 between the two values shows the additional ductility of spirally reinforced columns.

The strength-reduction factor for columns is much lower than those for flexure ($\phi = 0.9$) and shear ($\phi = 0.75$). This is because in axially loaded columns, the strength depends mainly on the concrete compression strength, whereas the strength of members in bending is less affected by the variation of concrete strength, especially in the case of an under-reinforced section. Furthermore, the concrete in columns is subjected to more segregation than in the case of beams. Columns are cast vertically in long, narrow forms, but the concrete in beams is cast in shallow, horizontal forms. Also, the failure of a column in a structure is more critical than that of a floor beam.

2. The minimum longitudinal steel percentage is 1%, and the maximum percentage is 8% of the gross area of the section (ACI Code, Section 10.9.1). Minimum reinforcement is necessary to provide resistance to bending, which may exist, and to reduce the effects of creep and shrinkage of the concrete under sustained compressive stresses. Practically, it is very difficult to fit more than 8% of steel reinforcement into a column and maintain sufficient space for concrete to flow between bars.
3. At least four bars are required for tied circular and rectangular members and six bars are needed for circular members enclosed by spirals (ACI Code, Section 10.9.2). For other shapes, one bar should be provided at each corner, and proper lateral reinforcement must be provided. For tied triangular columns, at least three bars are required. Bars shall not be located at a distance greater than 6 in. clear on either side from a laterally supported bar. Figure 10.3 shows the arrangement of longitudinal bars in tied columns and the distribution of ties. Ties shown in dotted lines are required when the clear distance on either side from laterally supported bars exceeds 6 in. The minimum concrete cover in columns is 1.5 in.
4. The minimum ratio of spiral reinforcement, ρ_s, according to the ACI Code, Eq. 10.5, and as explained in Section 10.9.3, is limited to

$$\rho_s = 0.45\left(\frac{A_g}{A_{ch}} - 1\right)\frac{f'_c}{f_{yt}} \tag{10.2}$$

where

A_g = gross area of section

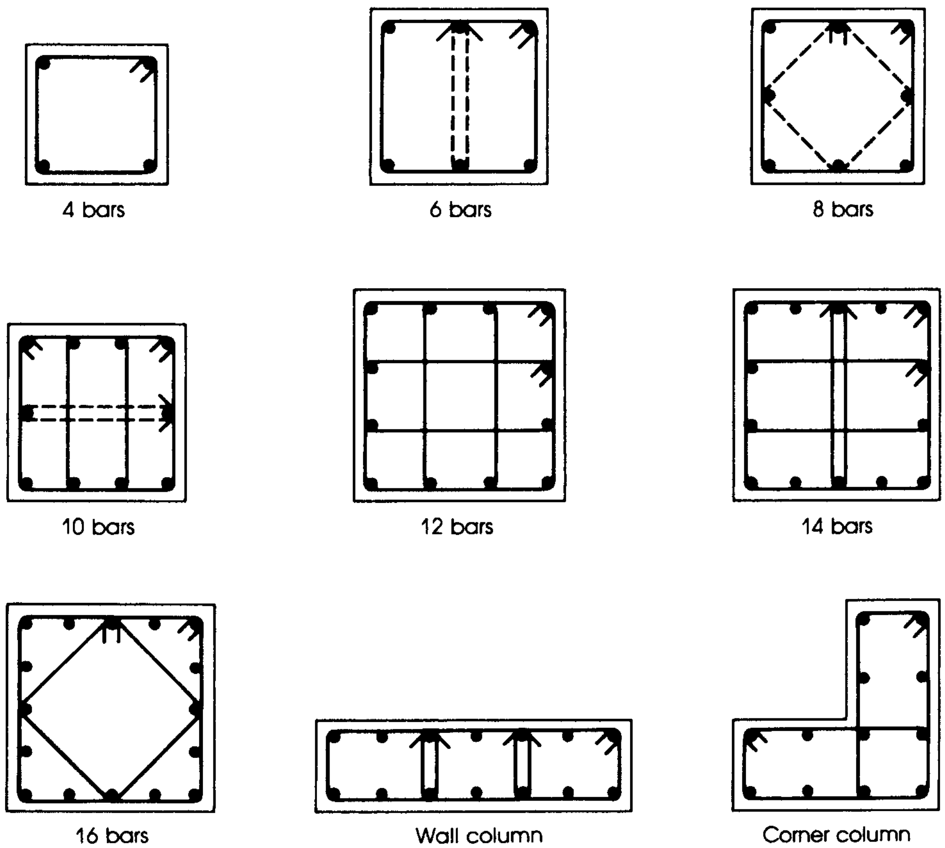


Figure 10.3 Arrangement of bars and ties in columns.

A_{ch} =area of core of spirally reinforced column measured to the outside diameter of spiral

f_{yt} = yield strength of spiral reinforcement (60 ksi; ACI Code, Section 10.9.3)

5. The minimum diameter of spirals is $\frac{3}{8}$ in., and their clear spacing should not be more than 3 in. nor less than 1 in., according to the ACI Code, Section 7.10.4. Splices may be provided by welding or lapping the deformed uncoated spiral bars by 48 diameters or a minimum of 12 in. Lap splices for plain uncoated bar or wire = $72d_p \leq 12$ in. The same applies for epoxy-coated deformed bar or wire. The Code also allows full mechanical splices.
6. Ties for columns must have a minimum diameter of $\frac{3}{8}$ in. to enclose longitudinal bars of no. 10 size or smaller and a minimum diameter of $\frac{1}{2}$ in. for larger bar diameters (ACI Code, Section 7.10.5).
7. Spacing of ties shall not exceed the smallest of 48 times the tie diameter, 16 times the longitudinal bar diameter, or the least dimension of the column. Table 10.1 gives spacings for no. 3 and no. 4 ties. The Code does not give restrictions on the size of columns to allow wider utilization of reinforced concrete columns in smaller sizes.

Table 10.1 Maximum Spacings of Ties

Column Least Side or Diameter (in.)	Spacings of Ties (in.) for Bar					
	No. 6	No. 7	No. 8	No. 9	No. 10	No. 11
12	12	12	12	12	12	12
14	12	14	14	14	14	14
16	12	14	16	16	16	16
18	12	14	16	18	18	18
20	12	14	16	18	18	20
22–40	12	14	16	18	18	22
Ties	No. 3	No. 3	No. 3	No. 3	No. 3	No. 4

10.5 SPIRAL REINFORCEMENT

Spiral reinforcement in compression members prevents a sudden crushing of concrete and buckling of longitudinal steel bars. It has the advantage of producing a tough column that undergoes gradual and ductile failure. The minimum spiral ratio required by the ACI Code is meant to provide an additional compressive capacity to compensate for the spalling of the column shell. The strength contribution of the shell is

$$P_u(\text{shell}) = 0.85 f'_c(A_g - A_{ch}) \tag{10.3}$$

where A_g is the gross concrete area and A_{ch} is the core area (Fig. 10.4).

In spirally reinforced columns, spiral steel is at least twice as effective as longitudinal bars; therefore, the strength contribution of spiral equals $2\rho_s A_{ch} f_{yt}$, where ρ_s is the ratio of volume of spiral reinforcement to total volume of core.

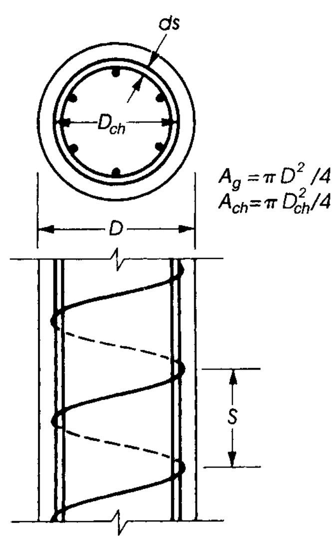


Figure 10.4 Dimensions of a column spiral.

Table 10.2 Spirals for Circular Columns ($f_y = 60$ ksi)

	$f'_c = 4$ ksi No. 3 Spirals	$f'_c = 5$ ksi No. 3 and no. 4 Spirals		$f'_c = 6$ ksi No. 4 Spirals
Column Diameter (in.)	***Spacing (in.)***	***Spiral No.***	***Spacing (in.)***	***Spacing (in.)***
12	2.0	4	2.75	2.25
14	2.0	4	3.00	2.25
16	2.0	4	3.00	2.50
18	2.0	4	3.00	2.50
20	2.0	4	3.00	2.50
22	2.0	4	3.00	2.50
24	2.0	3	1.75	2.50
26 to 40	2.25	3	1.75	2.75

If the strength of the column shell is equated to the spiral strength contribution, then

$$0.85f'_c(A_g - A_{ch}) = 2\rho_s A_{ch} f_{yt} \tag{10.4}$$

$$\rho_s = 0.425\left(\frac{A_g}{A_{ch}} - 1\right)\frac{f'_c}{f_{yt}}$$

The ACI Code adopted a minimum ratio of ρ_s according to the following equation:

$$\text{Minimum}\,\rho_s = 0.45\left(\frac{A_g}{A_{ch}} - 1\right)\frac{f'_c}{f_{yt}} \tag{10.2}$$

The design relationship of spirals may be obtained as follows (Fig. 10.4):

$$\rho_s = \frac{\text{volume of spiral in one loop}}{\text{volume of core for a spacing } S}$$

$$= \frac{a_s\pi(D_{ch} - d_s)}{\left(\frac{\pi}{4}D_{ch}^2\right)S} = \frac{4a_s(D_{ch} - d_s)}{D_{ch}^2 S} \tag{10.5}$$

where

a_s = area of spiral reinforcement
D_{ch} =diameter of the core measured to the outside diameter of spiral
D = diameter of the column
d_s = diameter of the spiral
S = spacing of the spiral

Table 10.2 gives spiral spacings for no. 3 and no. 4 spirals with $f_y = 60$ ksi.

10.6 DESIGN EQUATIONS

The nominal load strength of an axially loaded column was given in Eq. 10.1. Because a perfect axially loaded column does not exist, some eccentricity occurs on the column section, thus reducing its load capacity, P_o. To take that into consideration, the ACI Code specifies that the maximum nominal load, P_o, should be multiplied by a factor equal to 0.8 for tied columns and 0.85 for spirally reinforced columns. Introducing the strength reduction factor, the axial load strength of columns according to the ACI Code, Section 10.3.6, are as follows:

$$P_u = \phi P_n = \phi(0.80)[0.85f'_c(A_g - A_{st}) + A_{st}f_y] \quad (10.6)$$

for tied columns and

$$P_u = \phi P_n = \phi(0.85)[0.85f'_c(A_g - A_{st}) + A_{st}f_y] \quad (10.7)$$

for spiral columns, where

A_g = gross concrete area
A_{st} = total steel compressive area
ϕ = 0.65 for tied columns and 0.70 for spirally reinforced columns

Equations 10.8 and 10.9 may be written as follows:

$$P_u = \phi P_n = \phi K[0.85f'_c A_g + A_{st}(f_y - 0.85f'_c)] \quad (10.8)$$

where $\phi = 0.65$ and $K = 0.8$ for tied columns and $\phi = 0.75$ and $K = 0.85$ for spiral columns.

If the gross steel ratio is $\rho_g = A_{st}/A_g$, or $A_{st} = \rho_g A_g$, then Eq. 10.8 may be written as follows:

$$P_u = \phi P_n = \phi K A_g[0.85f'_c + \rho_g(f_y - 0.85f'_c)] \quad (10.9)$$

Equation 10.8 can be used to calculate the axial load strength of the column, whereas Eq. 10.9 is used when the external factored load is given and it is required to calculate the size of the column section, A_g, based on an assumed steel ratio, ρ_g, between a minimum of 1% and a maximum of 8%.

It is a common practice to use grade 60 reinforcing steel bars in columns with a concrete compressive strength of 4 ksi or greater to produce relatively small concrete column sections.

10.7 AXIAL TENSION

Concrete will not crack as long as stresses are below its tensile strength; in this case, both concrete and steel resist the tensile stresses, but when the tension force exceeds the tensile strength of concrete (about one-tenth of the compressive strength), cracks develop across the section, and the entire tension force is resisted by steel. The nominal load that the member can carry is that due to tension steel only:

$$T_n = A_{st}f_y \quad (10.10)$$

$$T_u = \phi A_{st}f_y \quad (10.11)$$

where $\phi = 0.9$ for axial tension.

Tie rods in arches and similar structures are subjected to axial tension. Under working loads, the concrete cracks and the steel bars carry the whole tension force. The concrete acts as a fire and corrosion protector. Special provisions must be taken for water structures, as in the case of water tanks. In such designs, the concrete is not allowed to crack under the tension caused by the fluid pressure.

10.8 LONG COLUMNS

The equations developed in this chapter for the strength of axially loaded members are for short columns. In the case of long columns, the load capacity of the column is reduced by a reduction factor.

A long column is one with a high slenderness ratio, h/r, where h is the effective height of the column and r is the radius of gyration. The design of long columns is explained in detail in Chapter 12.

Example 10.1

Determine the allowable design axial load on a 12-in. square, short tied column reinforced with four no. 9 bars. Ties are no. 3 spaced at 12 in. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. Using Eq. 10.9,

$$P_u = \phi P_n = \phi K[0.85 f'_c A_g + A_{st}(f_y - 0.85 f'_c)]$$

For a tied column, $\phi = 0.65$, $K = 0.8$, and $A_{st} = 4.0\,\text{in.}^2$

$$P_u = \phi P_u = 0.65(0.8)[0.85(4)(12 \times 12) + 4(60 - 0.85 \times 4)] = 372 \text{ K}$$

2. Check steel percentage: $\rho_g = \frac{4}{144} = 0.02778 = 2.778\%$. This is less than 8% and greater than 1%.
3. Check tie spacings: Minimum tie diameter is no. 3. Spacing is the smallest of the 48-tie diameter, 16-bar diameter, or least column side. $S_1 = 48(\frac{3}{8}) = 18$ in., $S_2 = 16(\frac{9}{8}) = 18$ in., $S_3 = 12.0$ in. Ties are adequate (Table 10.1).

Example 10.2

Design a square tied column to support an axial dead load of 400 K and a live load of 232 K using $f'_c = 5$ksi, $f_y = 60$ ksi and a steel ratio of about 5%. Design the necessary ties.

Solution

1. Calculate $P_u = 1.2P_D + 1.6P_L = 1.2(400) + 1.6(232) = 851$ K. Using Eq. 10.10, $P_u = 851 = 0.65(0.8)A_g[0.85 \times 5 + 0.05(60 - 0.85 \times 5)]$, $A_g = 232.5\,\text{in.}^2$, and column side = 15.25 in., so use 16 in. (Actual $A_g = 256\,\text{in.}^2$)
2. Because a larger section is adopted, the steel percentage may be reduced by using $A_g = 256\,\text{in.}^2$ in Eq. 10.8:

$$851 = 0.65(0.8)[0.85 \times 5 \times 256 + A_{st}(60 - 0.85 \times 5)]$$

$$A_{st} = 9.84 \text{ in.}^2$$

Use eight no. 11 bars ($A_{st} = 12.50\,\text{in.}^2$). See Fig. 10.5.
3. Design of ties (by calculation or from Table 10.1): Choose no. 3 ties with spacings equal to the least of $S_1 = 16(\frac{11}{8}) = 22$ in., $S_2 = 48(\frac{3}{8}) = 18$ in., or S_3 = column side = 16 in. Use no. 3 ties spaced at 16 in. Clear distance between bars is 4.23 in., which is less than 6 in. Therefore, no additional ties are required.

Example 10.3

Repeat Example 10.2 using a rectangular section that has a width of $b = 14$ in.

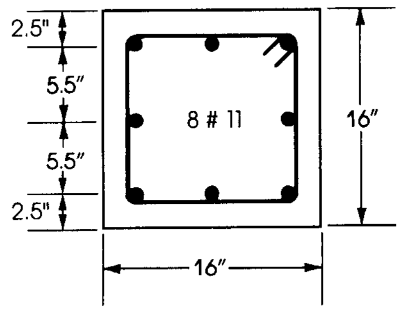


Figure 10.5 Example 10.2.

Solution

1. $P_u = 851$ K and calculated $A_g = 232.5\,\text{in.}^2$ For $b = 14$ in., $h = 232.5/14 = 16.6$ in. Choose a column 14×18 in.; actual $A_g = 252\,\text{in.}^2$.
2. $P_u = 851 = 0.65(0.8)[0.85 \times 5 \times 252 + A_{st}(60 - 0.85 \times 5)]$

$$A_{st} = 10.14\ \text{in.}^2$$

Use eight no. 10 bars. ($A_{st} = 10.16\,\text{in.}^2$)

3. Design of ties: Choose no. 3 ties, $S_1 = 20$ in., $S_2 = 18$ in., and $S_3 = 14$ in. (least side). Use no. 3 ties spaced at 14 in. Clear distance between bars in the long direction is $(18 - 5)/2$ − bar diameter of $1.27 = 5.23$ in. < 6 in. No additional ties are needed. Clear distance in the short direction is $(14 - 5)/2 - 1.27 = 3.23$ in. < 6 in.

Example 10.4

Design a circular spiral column to support an axial dead load of 475 K and a live load of 250 K using $f_c' = 4$ ksi, $f_y = 60$ ksi, and a steel ratio of about 3%. Also, design the necessary spirals.

Solution

1. Calculate $P_u = 1.2P_D + 1.6P_L = 1.2(475) + 1.6(250) = 970$ K. Using Eq. 10.10 and spiral columns,

$$P_u = 970 = 0.75(0.85)A_g[0.85 \times 4 + 0.03(60 - 0.85 \times 4)]$$

$A_g = 299\,\text{in.}^2$ and column diameter = 19.5 in., so use 20 in. Actual $A_g = 314.2\,\text{in.}^2$

2. Calculate A_{st} needed from Eq. 10.8:

$$P_u = 970 = 0.75(0.85)[0.85 \times 4 \times 314.2 + A_{st}(60 - 0.85 \times 4)]$$

$$A_{st} = 8\ \text{in.}^3$$

Use eight no. 10 bars. ($A_{st} = 10.16\,\text{in.}^2$)

3. Design of spirals: The diameter of core is $20 - 2(1.5) = 17$ in. The area of core is

$$A_{ch} = \frac{\pi}{4}(17)^2 \quad A_g = \frac{\pi}{4}(20)^2$$

$$\text{Minimum } \rho_s = 0.45\left(\frac{A_g}{A_{ch}} - 1\right)\frac{f_c'}{f_{yt}} = 0.45\left(\frac{20^2}{17^2} - 1\right)\left(\frac{4}{60}\right) = 0.01152$$

Assume no. 3 spiral, $a_s = 0.11\text{ in.}^2$, and $d_s = 0.375\text{ in.}$

$$\rho_s = 0.01152 = \frac{4a_s(D_{ch} - d_b)}{SD_{ch}^2} = \frac{4(0.11)(17 - 0.375)}{S(17)^2}$$

Spacing s is equal to 2.2 in; use no. 3 spiral at $s = 2\text{ in.}$ (as shown in Table 10.2).

Example 10.5

Design a rectangular tied short column to carry a factored axial load of 1765 kN. Use $f'_c = 30$ MPa, $f_y = 400$ MPa, column width $(b) = 300$ mm, and a steel ratio of about 2%.

Solution SI Units

1. Using Eq. 10.9,

$$P_u = 0.8\phi A_g[0.85f'_c + \rho_g(f_y - 0.85f'_c)]$$

Assuming a steel percentage of 2%,

$$1765 \times 10^3 = 0.8 \times 0.65A_g[0.85 \times 30 + 0.02(400 - 0.85 \times 30)]$$

$$A_g = 102{,}887\text{ mm}^2$$

For $b = 300$ mm, the other side of the rectangular column is 343 mm. Therefore, use a section of 300 by 350 mm ($A_g = 105{,}000\text{ mm}^2$).

2. $A_s = 0.02 \times 102{,}887 = 2057\text{ mm}^2$. Choose six bars, 22 mm in diameter ($A_s = 2280\text{ mm}^2$).
3. Check the axial load strength of the section using Eq. 10.6:

$$\begin{aligned}\phi P_n &= 0.8\phi[0.85f'_c(A_g - A_{st}) + A_{st}f_y]\\ &= 0.8 \times 0.65[0.85 \times 30(105{,}000 - 2280) + 2280 \times 400] \times 10^{-3}\\ &= 1836\text{ kN}\end{aligned}$$

This meets the required P_u of 1765 kN.

4. Choose ties 10 mm in diameter. Spacing is the least of (1) $16 \times 22 = 352$ mm, (2) $48 \times 10 = 480$ mm, or (3) 300 mm. Choose 10-mm ties spaced at 300 mm.

SUMMARY

Sections 10.1–10.4

Columns may be tied or spirally reinforced.

$$\phi = 0.65 \text{ for tied columns}$$

$$\phi = 0.75 \text{ for spirally reinforced columns}$$

ρ_g must be $\leq 8\%$ and $\geq 1\%$.

Section 10.5

Minimum ratio of spirals is

$$\rho_s = 0.45\left(\frac{A_g}{A_{ch}} - 1\right)\frac{f'_c}{f_{yt}} \qquad (10.2)$$

$$\rho_s = \frac{4a_s(D_{ch} - d_s)}{D_{ch}^2 S} \tag{10.5}$$

The minimum diameter of spirals is $\frac{3}{8}$ in., and their clear spacings should be not more than 3 in. or less than 1 in.

Section 10.6

For tied columns,

$$P_u = \phi P_n = 0.8\phi[0.85f_c'(A_g - A_{st}) + A_{st}f_y] \tag{10.6}$$

or

$$P_u = \phi P_n = 0.8\phi A_g[0.85f_c' + \rho_g(f_y - 0.85f_c')]$$

For spiral columns,

$$P_u = \phi P_n = 0.85\phi[0.85f_c'(A_g - A_{st}) + A_{st}f_y] \tag{10.7}$$

or

$$P_u = \phi P_n = 0.85\phi A_g[0.85f_c' + \rho_g(f_y - 0.85f_c')]$$

where $\rho_g = A_{st}/A_g$.

Section 10.7

1. For axial tension,

$$T_u = \phi A_{st} f_y \quad (\phi = 0.9) \tag{10.11}$$

2. Arrangements of vertical bars and ties in columns are shown in Fig. 10.3.

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PROBLEMS

10.1 For each problem, determine the allowable design load-bearing strength ($0.8\phi P_o$) for each of the following short rectangular columns according to the ACI Code limitations. Assume $f_y = 60$ ksi and properly tied columns (b = width of column, in., and h = total depth, in.).

Number	f'_c(ksi)	b (in.)	h (in.)	Bars	Answer ($\phi\ kP_o$) K
(a)	4	16	16	8 no. 9	688
(b)	4	20	20	16 no. 11	1442
(c)	4	12	12	8 no. 8	439
(d)	4	12	24	12 no. 10	955
(e)	5	14	14	10 no. 9	722
(f)	5	16	16	4 no. 10	712
(g)	5	14	26	12 no. 10	1244
(h)	5	18	32	8 no. 11	1634
(i)	6	16	16	8 no. 10	968
(j)	6	12	20	6 no. 10	852

10.2 For each problem, determine the allowable design load-bearing strength of each of the following short, spirally reinforced circular columns according to the ACI Code limitations. Assume $f_y = 60$ ksi and the spirals are adequate (D = diameter of column, in.).

Number	f'_c(ksi)	D (in.)	Bars	Answer ($\phi\ kP_o$) K
(a)	4	14	8 no. 9	581
(b)	4	16	6 no. 10	663
(c)	5	18	8 no. 10	980
(d)	5	20	12 no. 10	1300
(e)	6	15	8 no. 9	797

10.3 For each problem, design a short square, rectangular, or circular column, as indicated, for each set of axial loads given, according to ACI limitations. Also, design the necessary ties or spirals and draw sketches of the column sections showing all bar arrangements. Use $f_y = 60$ ksi and a steel ratio close to the ρ_g given (P_D = dead load, P_L = live load, b = width of a rectangular column, and = $\rho_g = A_{st}/A_g$).

Number	f'_c(ksi)	P_D (K)	P_L (K)	ρ_g%	Section	One Solution
(a)	4	200	200	4	Square	14 × 14, 8 no. 9
(b)	4	750	400	3.5	Square	24 × 24, 16 no. 10
(c)	4	220	165	7	Square	12 × 12, 8 no. 10
(d)	5	330	230	3	Square	16 × 16, 8 no. 9
(e)	4	190	170	2	Rectangular, b = 12 in.	12 × 18, 6 no. 8
(f)	4	280	315	4.5	Rectangular, b = 14 in.	14 × 20, 10 no. 10
(g)	4	210	150	3	Rectangular, b = 12 in.	12 × 16, 6 no. 9
(h)	5	690	460	2	Rectangular, b = 18 in.	18 × 32, 8 no. 10
(i)	4	350	130	4	Circular—spiral	16, 7 no. 9
(j)	4	475	220	3.25	Circular—spiral	20, 7 no. 10
(k)	4	400	260	5	Circular—spiral	18, 9 no. 10
(l)	5	285	200	4.25	Circular—spiral	15, 6 no. 10

For SI units, use 1 psi = 0.0069 MPa, 1 K = 4.45 kN, and 1 in. = 25.4 mm.

CHAPTER 11

MEMBERS IN COMPRESSION AND BENDING

Residential building, Minneapolis, Minnesota.

11.1 INTRODUCTION

Vertical members that are part of a building frame are subjected to combined axial loads and bending moments. These forces develop due to external loads, such as dead, live, and wind loads. The forces are determined by manual calculations or computer applications that are based on the principles of statics and structural analysis. For example, Fig. 11.1 shows a two-hinged portal frame that carries a uniform factored load on BC. The bending moment is drawn on the tension side of the frame for clarification. Columns AB and CD are subjected to an axial compressive force and a bending moment. The ratio of the moment to the axial force is usually defined as the eccentricity, e, where $e = M_n/P_n$ (Fig. 11.1). The eccentricity, e, represents the distance from the plastic centroid of the section to the point of application of the load. The plastic centroid is obtained by determining the location of the resultant force produced by the steel and the concrete, assuming that both are stressed in compression to f_y and 0.85 f'_c, respectively. For symmetrical sections, the plastic centroid coincides with the centroid of the section. For nonsymmetrical sections, the plastic centroid is determined by taking moments about an arbitrary axis, as explained in Example 11.1.

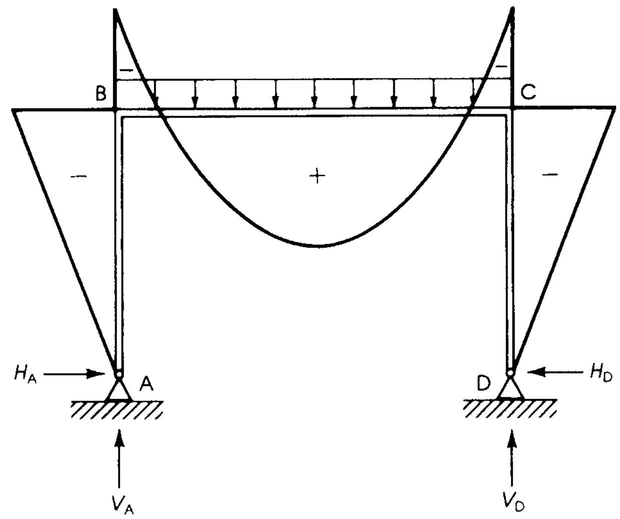


Figure 11.1 Two-hinged portal frame with bending moment diagram drawn on the tension side.

Example 11.1

Determine the plastic centroid of the section shown in Fig. 11.2. Given: $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. It is assumed that the concrete is stressed in compression to 0.85 f'_c:

$$F_c = \text{force in concrete} = (0.85\, f'_c)\, A_g$$
$$= (0.85 \times 4) \times 14 \times 20 = 952 \text{ K}$$

F_c is located at the centroid of the concrete section (at 10 in. from axis A-A).

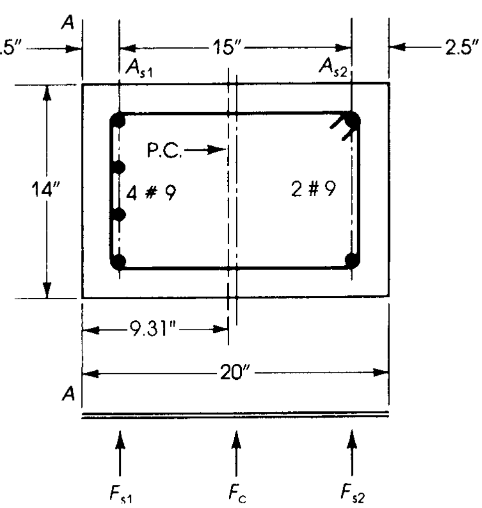


Figure 11.2 Example 11.1: Plastic centroid (P.C.) of section.

2. Forces in steel bars:

$$F_{s1} = A_{s1} f_y = 4 \times 60 = 240 \text{ K}$$

$$F_{s2} = A_{s2} f_y = 2 \times 60 = 120 \text{ K}$$

3. Take moments about A-A:

$$x = \frac{(952 \times 10) + (240 \times 2.5) + (120 \times 17.5)}{952 + 240 + 120} = 9.31 \text{ in.}$$

Therefore, the plastic centroid lies at 9.31 in. from axis A-A.

4. If $A_{s1} = A_{s2}$ (symmetrical section), then $x = 10$ in. from axis A-A.

11.2 DESIGN ASSUMPTIONS FOR COLUMNS

The design limitations for columns, according to the ACI Code, Section 10.2, are as follows:

1. Strains in concrete and steel are proportional to the distance from the neutral axis.
2. Equilibrium of forces and strain compatibility must be satisfied.
3. The maximum usable compressive strain in concrete is 0.003.
4. Strength of concrete in tension can be neglected.
5. The stress in the steel is $f_s = \varepsilon E_s \leq f_y$.
6. The concrete stress block may be taken as a rectangular shape with concrete stress of 0.85 f'_c that extends from the extreme compressive fibers a distance $a = \beta_1 c$, where c is the distance to the neutral axis and β_1 is 0.85 when $f'_c \leq 4000$ psi (30 MPa); β_1 decreases by 0.05 for each 1000 psi above 4000 psi (0.008 per 1 MPa above 30 MPa) but is not less than 0.65. (Refer to Fig. 3.6, Chapter 3.)

11.3 LOAD-MOMENT INTERACTION DIAGRAM

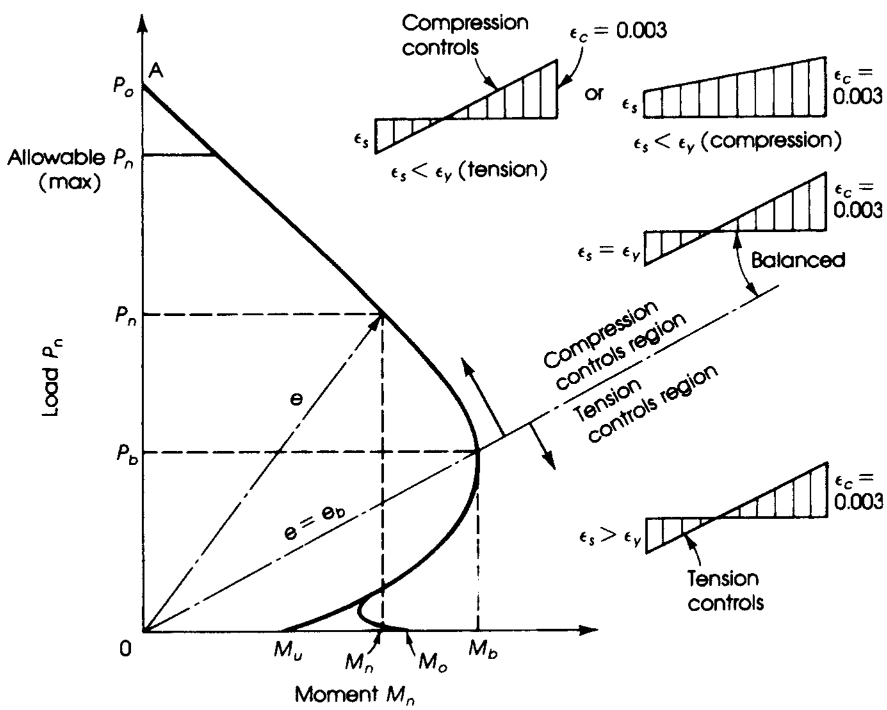
When a normal force is applied on a short reinforced concrete column, the following cases may arise, according to the location of the normal force with respect to the plastic centroid. Refer to Fig. 11.3*a* and 11.3*b*:

Axial compression (P_0). This is a theoretical case assuming that a large axial load is acting at the plastic centroid; $e = 0$ and $M_n = 0$. Failure of the column occurs by crushing of the concrete and yielding of steel bars. This is represented by P_0 on the curve of Fig. 11.3*a*.

1. *Maximum nominal axial load $P_{n\,\max}$:* This is the case of a normal force acting on the section with minimum eccentricity. According to the ACI Code, $P_{n\,\max} = 0.80P_0$ for tied columns and $0.85P_0$ for spirally reinforced columns, as explained in Chapter 10. In this case, failure occurs by crushing of the concrete and the yielding of steel bars.
2. *Compression failure:* This is the case of a large axial load acting at a small eccentricity. The range of this case varies from a maximum value of $P_n = P_{n\,\max}$ to a minimum value of $P_n = P_b$ (balanced load). Failure occurs by crushing of the concrete on the compression side with a strain of 0.003, whereas the stress in the steel bars (on the tension side) is less than the yield strength, f_y ($f_s < f_y$). In this case $P_n > P_b$ and $e < e_b$.

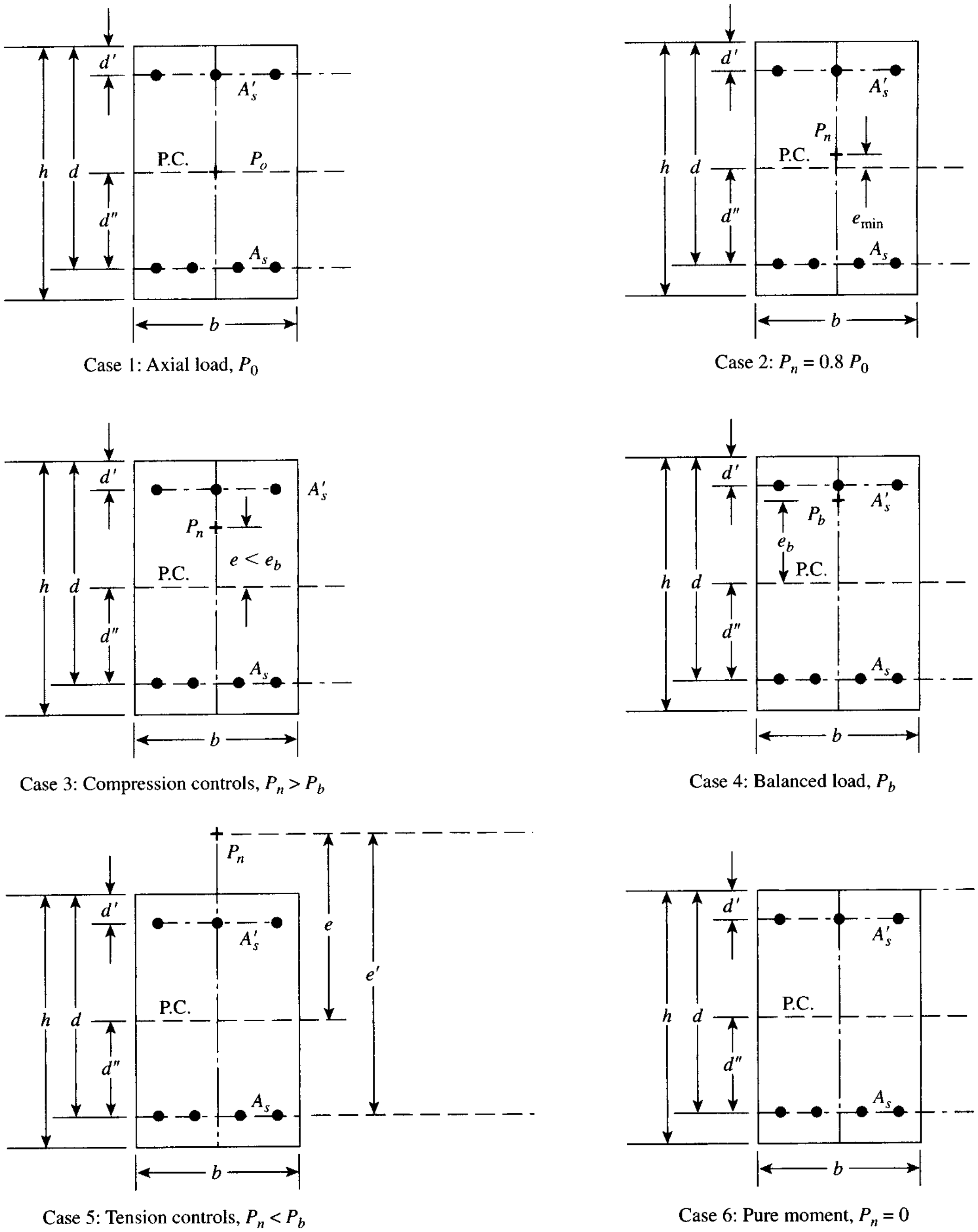
3. *Balanced condition* (P_b): A balanced condition is reached when the compression strain in the concrete reaches 0.003 and the strain in the tensile reinforcement reaches $\varepsilon_y = f_y/E_s$ simultaneously; failure of concrete occurs at the same time as the steel yields. The moment that accompanies this load is called the *balanced moment*, M_b, and the relevant balanced eccentricity is $e_b = M_b/P_b$.
4. *Tension failure:* This is the case of a small axial load with large eccentricity, that is, a large moment. Before failure, tension occurs in a large portion of the section, causing the tension steel bars to yield before actual crushing of the concrete. At failure, the strain in the tension steel is greater than the yield strain, ε_y, whereas the strain in the concrete reaches 0.003. The range of this case extends from the balanced to the case of pure flexure (Fig. 11.3). When tension controls, $P_n < P_b$ and $e > e_b$.
5. *Pure flexure:* The section in this case is subjected to a bending moment, M_n, whereas the axial load is $P_n = 0$. Failure occurs as in a beam subjected to bending moment only. The eccentricity is assumed to be at infinity. Note that radial lines from the origin represent constant ratios of $M_n/P_n = e =$ eccentricity of the load P_n from the plastic centroid.

Cases 1 and 2 were discussed in Chapter 10, and Case 6 was discussed in detail in Chapter 3. The other cases are discussed in this chapter.



(a)

Figure 11.3 (*a*) Load-moment strength interaction diagram showing ranges of cases discussed in text, and (*b*) column sections showing the location of P_n for different load conditions.



(b)

Figure 11.3 (*continued*)

11.4 SAFETY PROVISIONS

The safety provisions for load factors were discussed earlier in Section 3.6. For columns, the safety provisions may be summarized as follows:

1. Load factors for gravity and wind loads are

$$U = 1.4D$$
$$U = 1.2D + 1.6L$$
$$U = 1.2D + 1.6L + 0.8W$$
$$U = 1.2D + 1.0L + 1.6W$$
$$U = 0.9D + 1.6W$$

The most critical factored load should be used.

2. The strength reduction factor, ϕ, to be used for columns may vary according to the following cases:

a. When $P_u = \phi P_n \geq 0.1 f'_c A_g$, ϕ is 0.65 for tied columns and 0.75 for spirally reinforced columns. This case occurs generally when compression failure is expected. A_g is the gross area of the concrete section.

b. The sections in which the net tensile strain, ε_t, at the extreme tension steel, at nominal strength, is between 0.005 and 0.002 (transition region) ϕ varies linearly between 0.90 and 0.65 (or 0.75), respectively (Fig. 11.4). Refer to Section 3.7. For spiral sections,

$$\phi = 0.75 + (\varepsilon_t - 0.002)(50) \quad \text{or} \quad \phi = 0.75 + 0.15\left[\frac{1}{c/d_t} - \frac{5}{3}\right] \tag{11.1}$$

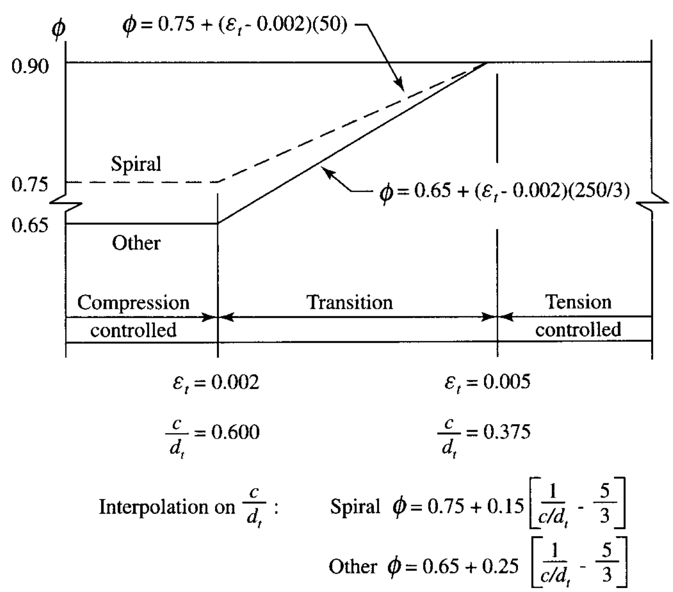


Figure 11.4 Variation in ϕ with NTS for grade 60 steel 7. Courtesy of ACI.

For spiral sections

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) \quad \text{or} \quad \phi = 0.65 + 0.25\left[\frac{1}{c/d_t} - \frac{5}{3}\right] \tag{11.2}$$

c. When $P_u = 0$, the case of pure flexure, then $\phi = 0.90$ for tension-controlled sections and varies between 0.90 and 0.65 (or 0.75) in the transition region.

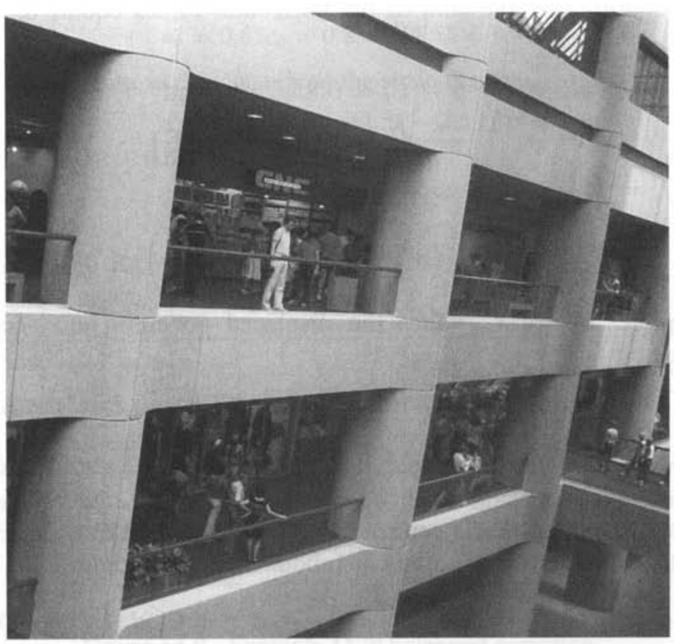
11.5 BALANCED CONDITION — RECTANGULAR SECTIONS

A balanced condition occurs in a column section when a load is applied on the section and produces, at nominal strength, a strain of 0.003 in the compressive fibers of concrete and a strain $\varepsilon_y = f_y/E_s$ in the tension steel bars simultaneously. This is a special case where the neutral axis can be determined from the strain diagram with known extreme values. When the applied eccentric load is greater than P_b, compression controls; if it is smaller than P_b, tension controls in the section.

The analysis of a balanced column section can be explained in steps (Fig. 11.5):

1. Let c equal the distance from the extreme compressive fibers to the neutral axis. From the strain diagram,

$$\frac{c_b(\text{balanced})}{d_t} = \frac{0.003}{0.003 + \dfrac{f_y}{E_s}} \qquad (\text{where } E_s = 29{,}000 \text{ ksi}) \tag{11.3}$$



Columns supporting 52-story building, Minneapolis, Minnesota.
(Columns are 96 × 64 in. with round ends.)

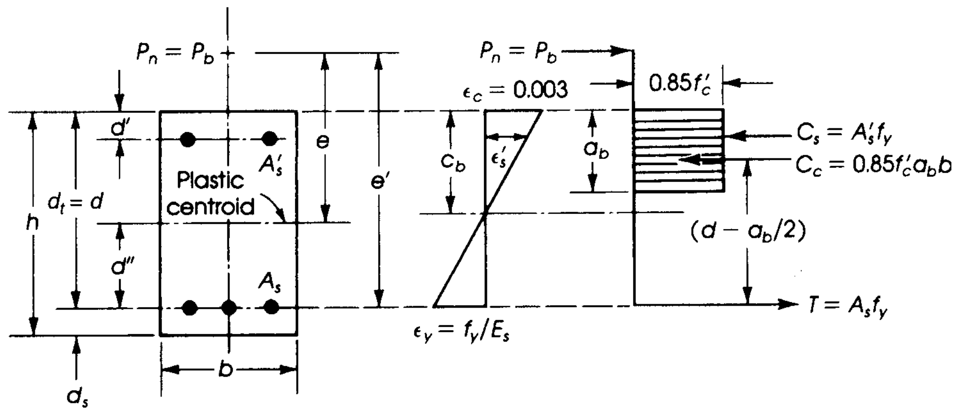


Figure 11.5 Balanced condition (rectangular section).

and

$$c_b = \frac{87d_t}{87 + f_y} \qquad \text{(where } f_y \text{ is in ksi)}$$

The depth of the equivalent compressive block is

$$a_b = \beta_1 c_b = \left(\frac{87}{87 + f_y}\right)\beta_1 d_t \tag{11.4}$$

where $\beta_1 = 0.85$ for $f_c' \leq 4000$ psi and decreases by 0.05 for each 1000-psi increase in f_c'.

2. From equilibrium, the sum of the horizontal forces equals 0: $P_b - C_c - C_s + T = 0$, where

$$C_c = 0.85f_c'ab \quad \text{and} \quad T = A_s f_y \tag{11.5}$$

$$C_s = A_s'(f_s' - 0.85f_c')$$

(Use $f_s' = f_y$ if compression steel yields.)

$$f_s' = 87\left(\frac{c - d'}{c}\right) \leq f_y$$

The expression of C_s takes the displaced concrete into account. Therefore, Eq. 11.5 becomes

$$P_b = 0.85\, f_c'ab + A_s'(f_s' - 0.85\, f_c') - A_s f_y \tag{11.6}$$

3. The eccentricity e_b is measured from the plastic centroid and e' is measured from the centroid of the tension steel: $e' = e + d''$ (in this case $e = eb + d''$), where d'' is the distance from the plastic centroid to the centroid of the tension steel. The value of e_b can be determined by taking moments about the plastic centroid.

$$P_b e_b = C_c\left(d - \frac{a}{2} - d''\right) + C_s(d - d' - d'') + Td'' \tag{11.7}$$

or

$$P_b e_b = M_b = 0.85\, f_c' ab\left(d - \frac{a}{2} - d''\right) + A_s'(f_y - 0.85\, f_c')(d - d' - d'') + A_s f_y d'' \tag{11.8}$$

The balanced eccentricity is

$$e_b = \frac{M_b}{P_b} \tag{11.9}$$

For nonrectangular sections, the same procedure applies, taking into consideration the actual area of concrete in compression.

The strength reduction factor, ϕ, for the balanced condition with $f_y = 60$ ksi, can be assumed $= 0.65$ (or 0.75). This is because $\varepsilon_s = \varepsilon_t = f_y/E_s = 0.00207$ (or 0.002), for which $\phi = 0.65$ (Fig. 11.4).

Example 11.2

Determine the balanced compressive force P_b; then determine e_b and M_b for the section shown in Fig. 11.6. Given: $f_c' = 4$ ksi and $f_y = 60$ ksi.

Solution

1. For a balanced condition, the strain in the concrete is 0.003 and the strain in the tension steel is

$$\varepsilon_y = \frac{f_y}{E_s} = \frac{60}{29{,}000} = 0.00207$$

2. Locate the neutral axis:

$$c_b = \frac{87}{87 + f_y} d_t = \frac{87}{87 + 60}(19.5) = 11.54 \text{ in.}$$

$$a_b = 0.85 c_b = 0.85 \times 11.54 = 9.81 \text{ in.}$$

3. Check if compression steel yields. From the strain diagram,

$$\frac{\varepsilon_s'}{0.003} = \frac{c - d'}{c} = \frac{11.54 - 2.5}{11.54} \qquad \varepsilon_s' = 0.00235$$

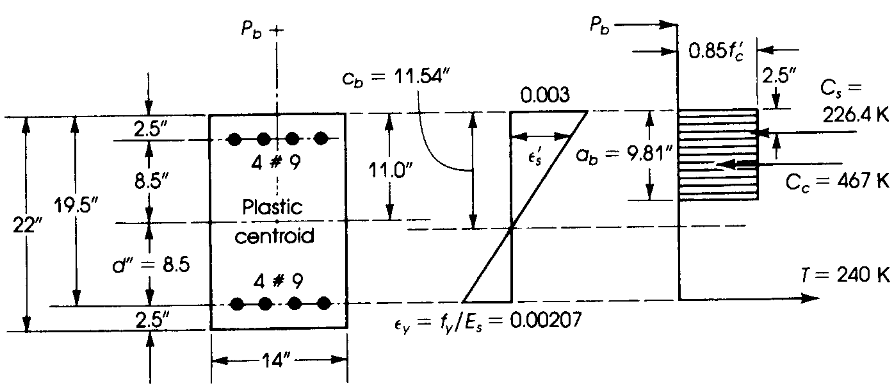


Figure 11.6 Example 11.2: balanced condition.

which exceeds ε_y of 0.00207; thus, compression steel yields. Or check that

$$f'_s = 87\left(\frac{c - d''}{c}\right) \le f_y$$

$$f'_s = \frac{87(11.54 - 2.5)}{11.54} = 68 \text{ ksi} > 60 \text{ ksi}$$

Then $f'_s = f_y = 60$ ksi.

4. Calculate the forces acting on the section:

$$C_c = 0.85\, f'_c ab = 0.85 \times 4 \times 9.81 \times 14 = 467 \text{ K}$$
$$T = A_s f_y = 4 \times 60 = 240 \text{ K}$$
$$C_s = A'_s(f_y - 0.85\, f'_c) = 4(60 - 3.4) = 226.4 \text{ K}$$

5. Calculate P_b and e_b:

$$P_b = C_c + C_s - T = 467 + 226.4 - 240 = 453.4 \text{ K}$$

From Eq. 11.7,

$$M_b = P_b e_b = C_c\left(d - \frac{a}{2} - d''\right) + C_s(d - d' - d'') + Td''$$

The plastic centroid is at the centroid of the section, and d'' = 8.5 in.

$$M_b = 453.4 e_b = 467\left(19.5 - \frac{9.81}{2} - 8.5\right) + 226.4(19.5 - 2.5 - 8.5) + 240 \times 8.5$$
$$= 6810.8 \text{ K·in.} = 567.6 \text{ K·ft}$$
$$e_b = \frac{M_b}{P_b} = \frac{6810.8}{453.4} = 15.0 \text{ in.}$$

6. For a balanced condition, $\phi = 0.65$, $\phi P_b = 294.7$ K, and $\phi M_b = 368.9$ K·ft.

11.6 COLUMN SECTIONS UNDER ECCENTRIC LOADING

For the two cases when compression or tension failure occurs, two basic equations of equilibrium can be used in the analysis of columns under eccentric loadings: (1) the sum of the horizontal or vertical forces = 0, and (2) the sum of moments about any axis = 0. Referring to Fig. 11.7, the following equations may be established.

1. $$P_n - C_c - C_s + T = 0 \qquad (11.10)$$

where

$C_c = 0.85\, f'_c ab$

$C_s = A'_s(f'_s - 0.85\, f'_c)$ (If compression steel yields, then $f'_s = f_y$.)

$T = A_s f_s$ (If tension steel yields, then $f_s = f_y$.)

2. Taking moments about A_s,

$$P_n e' - C_c\left(d - \frac{a}{2}\right) - C_s(d - d') = 0 \qquad (11.11)$$

Reinforced concrete tied columns under construction. The two columns are separated by an expansion joint.

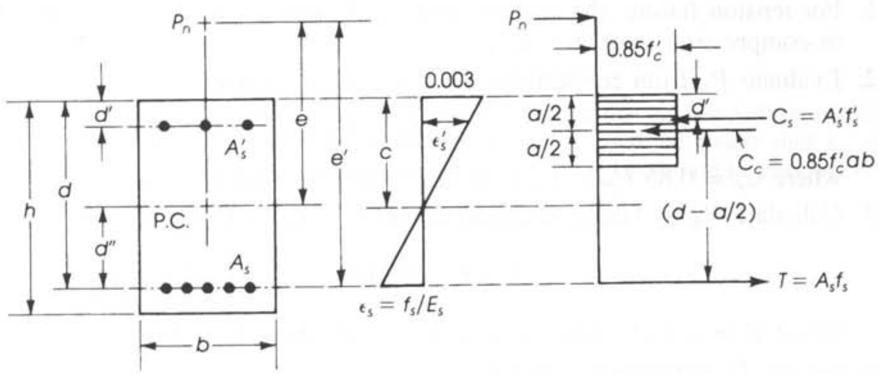


Figure 11.7 General case, rectangular section.

The quantity $e' = e + d''$, and $e' = (e + d - h/2)$ for symmetrical reinforcement (d'' is the distance from the plastic centroid to the centroid of the tension steel.)

$$P_n = \frac{1}{e'}\left[C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right] \tag{11.12}$$

Taking moments about C_c,

$$P_n\left[e' - \left(d - \frac{a}{2}\right)\right] - T\left(d - \frac{a}{2}\right) - C_s\left(\frac{a}{2} - d'\right) = 0 \tag{11.13}$$

$$P_n = \frac{T\left(d - \frac{a}{2}\right) + C_s\left(\frac{a}{2} - d'\right)}{\left(e' + \frac{a}{2} - d\right)} \tag{11.14}$$

If $A_s = A'_s$ and $f_s = f'_s = f_y$, then

$$P_n = \frac{A_s f_y (d - d')}{\left(e' + \frac{a}{2} - d\right)} = \frac{A_s f_y (d - d')}{\left(e - \frac{h}{2} + \frac{a}{2}\right)} \tag{11.15}$$

$$A_s = A'_s = \frac{P_n \left(e - \frac{h}{2} + \frac{a}{2}\right)}{f_y (d - d')} \tag{11.16}$$

11.7 STRENGTH OF COLUMNS FOR TENSION FAILURE

When a column is subjected to an eccentric force with large eccentricity e, tension failure is expected. The column section fails due to the yielding of steel and crushing of concrete when the strain in the steel exceeds $\varepsilon_y (\varepsilon_y = f_y / E_s)$. In this case the nominal strength, P_n, will be less than P_b or the eccentricity, $e = M_n / P_n$, is greater than the balanced eccentricity, e_b. Because it is difficult in some cases to predict if tension or compression controls, it can be assumed (as a guide) a tension failure will occur when $e > d$. This assumption should be checked later.

The general equations of equilibrium, Eqs. 11.10 and 11.11, may be used to calculate the nominal strength of the column. This is illustrated in steps as follows:

1. For tension failure, the tension steel yields and its stress is $f_s = f_y$. Assume that stress in compression steel is $f'_s = f_y$.
2. Evaluate P_n from equilibrium conditions (Eq. 11.10):

$$P_n = C_c + C_s - T$$

where $C_c = 0.85\, f'_c ab$, $C_s = A'_s(f_y - 0.85\, f'_c)$, and $T = A_s f_y$.

3. Calculate P_n by taking moments about A_s (Eq. 11.11):

$$P_n \cdot e' = C_c \left(d - \frac{a}{2}\right) + C_s (d - d')$$

where $e' = e + d''$ and $e' = e + d - h/2$ when $A_s = A'_s$.

4. Equate P_n from steps 2 and 3:

$$C_c + C_s - T = \frac{1}{e'} \left[C_c \left(d - \frac{a}{2}\right) + C_s (d - d')\right]$$

This is a second-degree equation in a. Substitute the values of C_c, C_s, and T and solve for a.

5. The second-degree equation, after the substitution of C_c, C_s, and T, is reduced to the following equation:

$$Aa^2 + Ba + C = 0$$

where

$$A = 0.425\, f'_c b$$
$$B = 0.85\, f'_c b(e' - d) = 2A(e' - d)$$
$$C = A'_s(f'_s - 0.85\, f'_c)(e' - d + d') - A_s f_y e'$$

Solve for a to get

$$a = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Note that the value of $(f_s' - 0.85\,f_c')$ must be a positive value. If this value is negative, then let $(f_s' - 0.85\,f_c') = 0$.

6. Substitute a in the equation of step 2 to obtain P_n. The moment M_n can be calculated:

$$M_n = P_n \cdot e$$

7. Check if compression steel yields as assumed. If $\varepsilon_s' \geq \varepsilon_y$, then compression steel yields; otherwise, $f_s' = E_s \varepsilon_s'$. Repeat steps 2 through 5. Note that $\varepsilon_s' = [(c - d'/c]0.003$, $\varepsilon_y = f_y/E_s$ and $c = a\beta_1$.
8. Check that tension controls. Tension controls when $e > e_b$ or $P_n < P_b$. Example 11.3 illustrates this procedure.
9. The net tensile strain, ε_t, in this section, is normally greater than the limit strain of 0.002 for a compression-controlled section (Fig. 11.4). Consequently, the value of the strength reduction factor, ϕ, will vary between 0.65 (or 0.75) and 0.90. Equation 11.1 or 11.2 can be used to calculate ϕ.

Example 11.3

Determine the nominal compressive strength, P_n, for the section given in Example 11.2 if $e = 20$ in. (See Fig. 11.8.)

Solution

1. Because $e = 20$ in. is greater than $d = 19.5$ in., assume that tension failure condition controls (to be checked later). The strain in the tension steel, ε_s, will be greater than ε_y and its stress is f_y. Assume that compression steel yields $f_s' = f_y$, which should be checked later.
2. From the equation of equilibrium (Eq. 11.10),

$$P_n = C_c + C_s - T$$

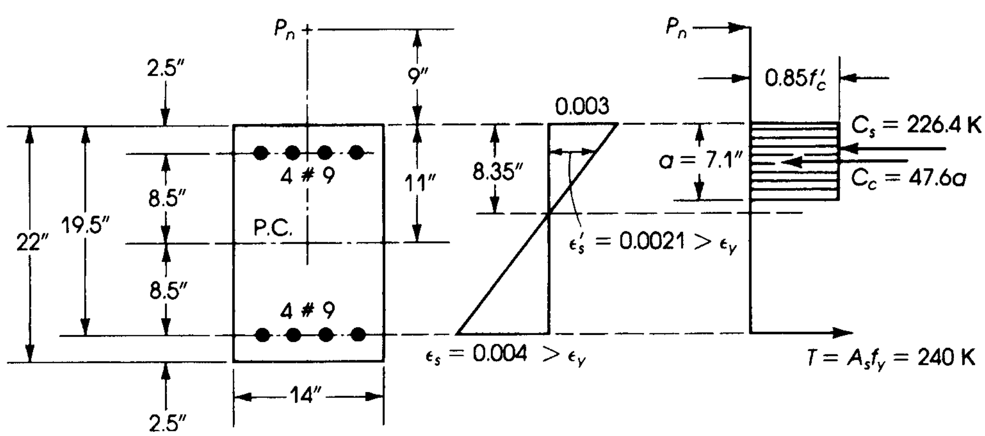


Figure 11.8 Example 11.3: tension failure.

where

$$C_c = 0.85\, f'_c ab = 0.85 \times 4 \times 14a = 47.6a$$

$$C_s = A'_s(f_y - 0.85\, f'_c) = 4(60 - 0.85 \times 4) = 226.4 \text{ K}$$

$$T = A_s f_y = 4 \times 60 = 240 \text{ K}$$

$$P_n = 47.6a + 226.4 - 240 = (47.6a - 13.6) \qquad \text{(I)}$$

3. Taking moments about A_s (Eq. 11.12),

$$P_n = \frac{1}{e'}\left[C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right]$$

Note that for the plastic centroid at the center of the section, $d'' = 8.5$ in.

$$e' = e + d'' = 20 + 8.5 = 28.5 \text{ in.}$$

$$P_n = \frac{1}{28.5}\left[47.6a\left(19.5 - \frac{a}{2}\right) + 226.4 \times 17\right]$$

$$P_n = 32.56a - 0.835a^2 + 135.0$$

4. Equating Eqs. I and II,

$$P_n = (47.6a - 13.6) = 32.56a - 0.835a^2 + 135.0 \qquad \text{(II)}$$

or

$$a^2 + 18a - 178.0 = 0 \qquad a = 7.1 \text{ in.}$$

5. From Eq. I:

$$P_n = 47.6 \times 7.1 - 13.6 = 324.4 \text{ K}$$

$$M_n = P_n e = 324.4 \times \frac{20}{12} = 540.67 \text{ K·ft}$$

6. Check if compression steel has yielded:

$$c = \frac{a}{0.85} = \frac{7.1}{0.85} = 8.35 \text{ in.} \qquad \varepsilon_y = \frac{60}{29{,}000} = 0.00207$$

$$\varepsilon'_s = \frac{(8.35 - 2.5)}{8.35}(0.003) = 0.0021 > \varepsilon_y$$

Compression steel yields. Check strain in tension steel:

$$\varepsilon_s = \left(\frac{19.5 - 8.35}{8.35}\right) \times 0.003 = 0.004 > \varepsilon_y$$

If compression steel does not yield, use f'_s as calculated from $f'_s = \varepsilon'_s E_s$ and revise the calculations.

7. Calculate ϕ: Since $\varepsilon_t = 0.004$, the section is in the transition region.

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) = 0.817$$

$$\phi P_n = 0.817(324.4) = 264.9 \text{ K}$$

$$\phi M_n = 0.817(540.67) = 441.7 \text{ K·ft}$$

8. Because $e = 20\,\text{in.} > e_b = 15\,\text{in.}$ (Example 11.2), there is a tension failure condition.
9. The same results can be obtained using the values of A, B, and C given earlier.

$$Aa^2 + Ba + C = 0$$

$$A = 0.425\, f'_c b = 0.425(4)(14) = 23.8$$

$$B = 2A(e' - d) = 2(23.8)(28.5 - 19.5) = 428.4$$

$$C = 4(60 - 0.85 \times 4)(28.5 - 19.5 + 2.5) - 4(60)(28.5)$$

$$= -4236.4$$

Solve for a to get $a = 7.1\,\text{in.}$ and $P_n = 324.4\,\text{K}$.

11.8 STRENGTH OF COLUMNS FOR COMPRESSION FAILURE

If the compressive applied force, P_n, exceeds the balanced force, P_b, or the eccentricity, $e = M_n/P_n$, is less than e_b, compression failure is expected. In this case compression controls, and the strain in the concrete will reach 0.003, whereas the strain in the steel is less than ε_y (Fig. 11.9). A large part of the column will be in compression. The neutral axis moves toward the tension steel, increasing the compression area, and therefore the distance to the neutral axis c is greater than the balanced c_b (Fig. 11.9).

Because it is difficult to predict compression or tension failure whenever a section is given, compression failure can be assumed when $e < 2d/3$, which should be checked later. The nominal load strength, P_n, can be calculated using the principles of statics. The analysis of column sections for compression failure can be achieved using Eqs. 11.10 and 11.11 given earlier and one of the following solutions.

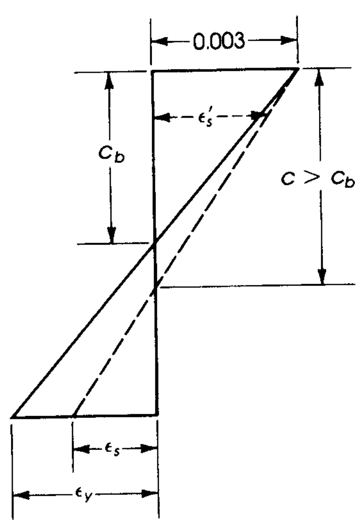


Figure 11.9 Strain diagram when compression controls. When $\varepsilon_s < \varepsilon_y$, $c > c_b$ and $\varepsilon'_s \geq \varepsilon_y$.

11.8.1 Trial Solution

This solution can be summarized as follows:

1. Calculate the distance to the neutral axis for a balanced section, c_b:

$$c_b = \left(\frac{87d_t}{87 + f_y}\right) \tag{11.17}$$

where f_y is in ksi.

2. Evaluate P_n using equilibrium conditions:

$$P_n = C_c + C_s - T \tag{11.18}$$

3. Evaluate P_n by taking moments about the tension steel, A_s:

$$P_n \cdot e' = C_c\left(d - \frac{a}{2}\right) + C_s(d - d') \tag{11.19}$$

where $e' = e + d - h/2$ when $A_s = A_s'$ or $e' = e + d''$ in general, $C_c = 0.85\, f_c'ab$, $C_s = A_s'(f_s' - 0.85\, f_c')$, and $T = A_s f_s$.

4. Assume a value for c such that $c > c_b$ (calculated in step 1). Calculate $a = \beta_1 c$. Assume $f_s' = f_y$.
5. Calculate f_s based on the assumed c:

$$f_s = \varepsilon_s E_s = 87\left(\frac{d_t - c}{c}\right) \text{ ksi} \le f_y$$

6. Substitute the preceding values in Eq. 11.10 to calculate P_{n1} and in Eq. 11.11 to calculate P_{n2}. If P_{n1} is close to P_{n2}, then choose the smaller or average of P_{n1} and P_{n2}. If P_{n1} is not close to P_{n2}, assume a new c or a and repeat the calculations starting from step 4 until P_{n1} is close to P_{n2}. (1% is quite reasonable.)
7. Check that compression steel yields by calculating $\varepsilon_s' = 0.003[(c - d')/c]$ and comparing it with $\varepsilon_y = f_y/E_s$. When $\varepsilon_s' \ge \varepsilon_y$, compression steel yields; otherwise, $f_s' = \varepsilon_s' E_s$ or, directly,

$$f_s' = 87\left(\frac{c - d'}{c}\right) \le f_y \text{ ksi}$$

8. Check that $e < e_b$ or $P_n > P_b$ for compression failure. Example 11.4 illustrates the procedure.
9. The net tensile strain, ε_t, in the section is normally less than 0.002 for compression-controlled sections (Fig. 11.4). Consequently, the strength reduction factor $(\phi) = 0.65$ (or 0.70 for spiral columns).

Example 11.4

Determine the nominal compressive strength, P_n, for the section given in Example 11.2 if $e = 10$ in. (See Fig. 11.10.)

Solution

1. Because $e = 10$ in. $< (2/3)d = 13$ in., assume compression failure. This assumption will be checked later. Calculate the distance to the neutral axis for a balanced section, c_b:

$$c_b = \frac{87}{87 + f_y} d_t = \frac{87}{87 + 60}(19.5) = 11.54 \text{ in.}$$

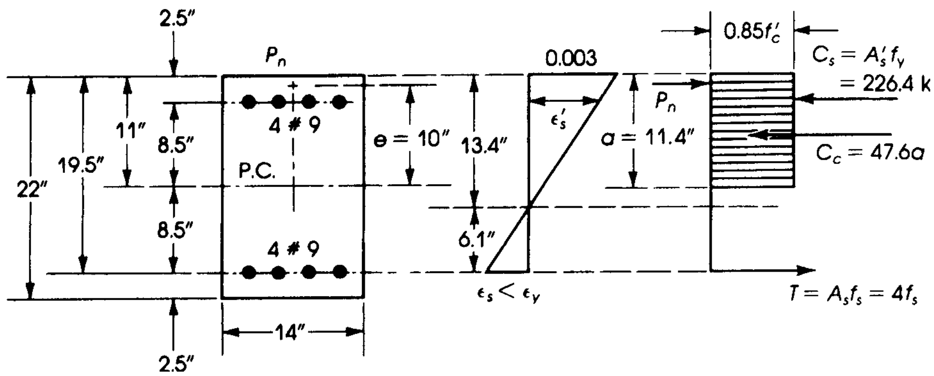


Figure 11.10 Example 11.4: compression controls.

2. From the equations of equilibrium,

$$P_n = C_c + C_s - T \tag{11.10}$$

where

$$C_c = 0.85\, f'_c ab = 0.85 \times 4 \times 14a = 47.6a$$

$$C_s = A'_s(f_y - 0.85\, f'_c) = 4(60 - 0.85 \times 4) = 226.4 \text{ K}$$

Assume compression steel yields. (This assumption will be checked later.)

$$T = A_s f_s = 4 f_s \qquad (f_s < f_y) \tag{I}$$

$$P_n = 47.6a + 226.4 - 4 f_s$$

3. Taking moments about A_s,

$$P_n = \frac{1}{e'}\left[C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right] \tag{11.11}$$

The plastic centroid is at the center of the section and $d'' = 8.5$ in.

$$e' = e + d'' = 10 + 8.5 = 18.5 \text{ in.}$$

$$P_n = \frac{1}{18.5}\left[47.6a\left(19.5 - \frac{a}{2}\right) + 226.4(19.5 - 2.5)\right] \tag{II}$$

$$P_n = 50.17a - 1.29a^2 + 208$$

4. Assume $c = 13.45$ in., which exceeds c_b (11.54 in.).

$$a = 0.85 \times 13.45 = 11.43 \text{ in.}$$

Substitute $a = 11.43$ in Eq. II:

$$P_{n1} = 50.17 \times 11.43 - 1.29(11.43)^2 + 208 = 612.9 \text{ K}$$

5. Calculate f_s from the strain diagram when $c = 13.45$ in.

$$f_s = \left(\frac{19.5 - 13.45}{13.45}\right) 87 = 39.13 \text{ ksi} \quad \varepsilon_s = \varepsilon_t = f_s/E_s = 0.00135$$

6. Substitute $a = 11.43$ in. and $f_s = 39.13$ ksi in Eq. I to calculate P_{n2}:

$$P_{n2} = 47.6(11.43) + 226.4 - 4(39.13) = 613.9 \text{ K}$$

which is very close to the calculated P_{n1} of 612.9 K (less than 1% difference).

$$M_n = P_n \cdot e = 612.9\left(\frac{10}{12}\right) = 510.8 \text{ K·ft}$$

7. Check if compression steel yields. From the strain diagram,

$$\varepsilon_s' = \frac{13.45 - 2.5}{13.45}(0.003) = 0.00244 > \varepsilon_y = 0.00207$$

Compression steel yields, as assumed.

8. $P_n = 612.9$ K is greater than $P_b = 453.4$ K, and $e = 10$ in. $< e_b = 15$ in., both calculated in the previous example, indicating that compression controls, as assumed. Note that it may take a few trials to get P_{n1} close to P_{n2}.
9. Calculate ϕ:

$$d_t = d = 19.5 \text{ in.} \qquad c = 13.45 \text{ in.}$$

ε_t (at the tension steel level) $= 0.003(d_t - c)/c$.

$$\varepsilon_t = 0.003(19.5 - 13.45)/13.45 = 0.00135$$

Since $\varepsilon_t < 0.002$, then $\phi = 0.65$.

$$\phi P_n = 0.65(612.9) = 398.4 \text{ K}$$

$$\phi M_n = 0.65(510.8) = 332 \text{ K·ft.}$$

11.8.2 Numerical Analysis Solution

The analysis of columns when compression controls can also be performed by reducing the calculations into one cubic equation in the form

$$Aa^3 + Ba^2 + Ca + D = 0$$

and then solving for a by a numerical method, or a can be obtained directly by using one of many inexpensive scientific calculators with built-in programs that are available. From the equations of equilibrium,

$$\begin{aligned} P_n &= C_c + C_s - T \\ &= (0.85 f_c'ab) + A_s'(f_y - 0.85 f_c') - A_s f_s \end{aligned} \tag{11.10}$$

Taking moments about the tension steel, A_s,

$$\begin{aligned} P_n &= \frac{1}{e'}\left[C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right] \\ &= \frac{1}{e'}\left[0.85 f_c'ab\left(d - \frac{a}{2}\right) + A_s'(f_y - 0.85 f_c')(d - d')\right] \end{aligned} \tag{11.11}$$

From the strain diagram,

$$\varepsilon_s = \left(\frac{d_t - c}{c}\right)(0.003) = \frac{\left(d - \dfrac{a}{\beta_1}\right)}{\dfrac{a}{\beta_1}}(0.003)$$

The stress in the tension steel is

$$f_s = \varepsilon_s E_s = 29{,}000\varepsilon_s = \frac{87}{a}(\beta_1 d - a)$$

Substituting this value of f_s in Eq. 11.10 and equating Eqs. 11.10 and 11.11 and simplifying gives

$$\left(\frac{0.85\,f_c'b}{2}\right)a^3 + [0.85f_c'b(e'-d)]a^2 + [A_s'(f_y - 0.85\,f_c')(e'-d+d') + 87A_se']a$$
$$-\,87\,A_se'\beta_1 d = 0$$

This is a cubic equation in terms of a:

$$Aa^3 + Ba^2 + Ca + D = 0$$

where

$$A = \frac{0.85\,f_c'b}{2}$$
$$B = 0.85\,f_c'b(e'-d)$$
$$C = A_s'(f_y - 0.85\,f_c')(e'-d+d') + 87\,A_se'$$
$$D = -87\,A_se'\beta_1 d$$

Once the values of A, B, C, and D are calculated, a can be determined by trial or directly by a scientific calculator. Also, the solution of the cubic equation can be obtained by using the well known Newton-Raphson method. This method is very powerful for finding a root of $f(x) = 0$. It involves a simple technique, and the solution converges rapidly by using the following steps:

1. Let $f(a) = Aa^3 + Ba^2 + Ca + D$, and calculate A, B, C, and D.
2. Calculate the first derivative of $f(a)$:
$$f'(a) = 3Aa^2 + 2Ba + C$$
3. Assume any initial value of a, say, a_0, and compute the next value:
$$a_1 = a_0 - \frac{f(a_0)}{f'(a_0)}$$
4. Use the obtained value a_1 in the same way to get
$$a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}$$
5. Repeat the same steps to get the answer up to the desired accuracy. In the case of the analysis of columns when compression controls, the value a is greater than the balanced $a(a_b)$. Therefore, start with $a_0 = a_b$ and repeat twice to get reasonable results.

Example 11.5

Repeat Example 11.4 using numerical solution.

Solution

1. Calculate A, B, C, and D and determine $f(a)$.
$$A = 0.85 \times 4 \times \frac{14}{2} = 23.8$$
$$B = 0.85 \times 4 \times 14(18.5 - 19.5) = -47.6$$

$$C = 4(60 - 0.85 \times 4)(18.5 - 19.5 + 2.5) + 87 \times 4 \times 18.5$$
$$= 6777.6$$
$$D = -87 \times 4 \times 18.5 \times (0.85 \times 19.5) = -106{,}710$$
$$f(a) = 23.8a^3 - 47.6a^2 + 6777.6a - 106{,}710$$

2. Calculate the first derivative:

$$f'(a) = 71.4a^2 - 95.2a + 6777.6$$

3. Let $a_0 = a_b = 9.81$ in. For a balanced section, $c_b = 11.54$ in. and $a_b = 9.81$ in.

$$a_1 = 9.81 - \frac{f(9.81)}{f'(9.81)} = 9.81 - \frac{-22{,}334}{12{,}715} = 11.566 \text{ in.}$$

4. Calculate a_2:

$$a_2 = 11.566 - \frac{f(11.566)}{f'(11.566)} = 11.566 - \frac{2136}{15{,}228} = 11.43 \text{ in.}$$

This value of a is similar to that obtained earlier in Example 11.3. Substitute the value of a in Eq. 11.10 or 11.11 to get $P_n = 612.9$ K.

11.8.3 Approximate Solution

An approximate equation was suggested by Whitney to estimate the nominal compressive strength of short columns when compression controls, as follows [15]:

$$P_n = \frac{bhf'_c}{\dfrac{3he}{d^2} + 1.18} + \frac{A'_s f_y}{\dfrac{e}{(d - d')} + 0.5} \tag{11.17}$$

This equation can be used only when the reinforcement is symmetrically placed in single layers parallel to the axis of bending.

A second approximate equation was suggested by Hsu [16]:

$$\frac{P_n - P_b}{P_o - P_b} + \left(\frac{M_n}{M_b}\right)^{1.5} = 1.0 \tag{11.18}$$

where

P_n = nominal axial strength of the column section
P_b, M_b = nominal load and moment of the balanced section
M_n = nominal bending moment = $P_n \cdot e$
P_o = nominal axial load at $e = 0$
$= 0.85 f'_c(A_g - A_{st}) + A_{st} f_y$
A_g = gross area of the section = bh
A_{st} = total area of nonprestressed longitudinal reinforcement

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Example 11.6

Determine the nominal compressive strength, P_n, for the section given in Example 11.4 by Eqs. 11.17 and 11.18 using the same eccentricity, $e = 10$ in., and compare results.

Solution

1. Solution by Whitney equation (Eq. 11.29):
   a. Properties of the section shown in Fig. 11.10 are $b = 14$ in., $h = 22$ in., $d = 19.5$ in., $d' = 2.5$ in., $A'_s = 4.0$ in.^{2}, and $(d - d') = 17$ in.

b. Apply the Whitney equation:

$$P_n = \frac{14 \times 22 \times 4}{(3 \times 22 \times 10)/(19.5)^2 + 1.18} = \frac{4 \times 60}{(\frac{10}{17}) + 0.5} = 643 \text{ K}$$

$$\phi P_n = 0.65 P_n = 418 \text{ K}$$

c. P_n calculated by the Whitney equation is not a conservative value in this example, and the value of $P_n = 643$ K is greater than the more accurate value of 612.9 K calculated by statics in Example 11.4.

2. Solution by Hsu equation (Eq. 11.18):

a. For a balanced condition, $P_b = 453.4$ K and $M_b = 6810.8$ K·in. (Example 11.2).

b. $P_0 = 0.85 f'_c(A_g - A_{st}) + A_{st} f$
$= 0.85(4)(14 \times 22 - 8) + 8(60) = 1500$ K

c. $\dfrac{P_n - 453.4}{1500 - 453.4} + \left(\dfrac{10P_n}{6810.8}\right)^{1.5} = 1$
Multiply by 1000 and solve for P_n.

$$0.9555 P_n + 0.05626 P_n^{1.5} = 1433.2 \text{ K}$$

By trial, $P_n = 611$ K, which is very close to 612.9 K, as calculated by statics.

11.9 INTERACTION DIAGRAM EXAMPLE

In Example 11.2, the balanced loads P_b, M_b, and e_b were calculated for the section shown in Fig. 11.6 ($e_b = 15$ in.). Also, in Examples 11.3 and 11.4, the load capacity of the same section was calculated for the case when $e = 20$ in. (tension failure) and when $e = 10$ in. (compression failure). These values are shown in Table 11.1.

To plot the load–moment interaction diagram, different values of ϕP_n and ϕM_n were calculated for various e values that varied between $e = 0$ and $e =$ maximum for the case of

Table 11.1 Summary of the Load Strength of the Column Section in the Previous Examples

e (in.)	a (in.)	ϕ	P_n (K)	ϕP_n (K)	ϕM_n (K·ft)	Notes
0	—	0.65	1500	975	0.0	ϕP_{n0}
2.25	19.39	0.65	1200	780	146.3	0.8 ϕP_{n0}
4	16.82	0.65	1018	661.7	220.6	Compression
6	14.19	0.65	843.3	548.1	274.0	Compression
10*	11.43	0.65	612.9	398.4	332.0	Compression
12	10.63	0.65	538.0	349.7	349.7	Compression
15*	9.81	0.65	453.4	294.7	368.9	Balanced
20*	7.10	0.81	324.4	263.4	439.0	Transition
30	5.06	0.90	189.4	170.5	426.2	Tension
50	4.01	0.90	100.6	90.5	377.2	Tension
80	3.59	0.90	58.8	52.9	352.0	Tension
P.M.	3.08	0.90	0.0	0.0	352.0	Tension
P.M.	3.08	0.65	0.0	0.0	254.2	P.M. (X)

*= values calculated in Examples 11.2,11.3, and 11.4.
P.M. = pure moment.
X = Not applicable, for comparison only.

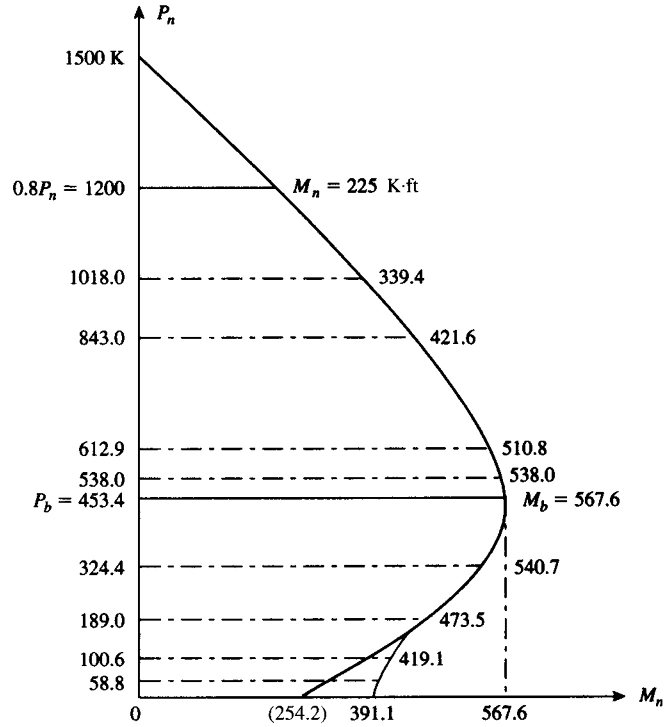


Figure 11.11 Interaction diagram of the column section shown in Fig. 11.10.

pure moment when $P_n = 0$. These values are shown in Table 11.1. The interaction diagram is shown in Fig. 11.11. The load $\phi P_{n0} = 975\,\text{K}$ represents the theoretical axial load when $e = 0$, whereas $0.8\ \phi P_{n0} = 780\,\text{K}$ represents the maximum axial load allowed by the ACI Code based on minimum eccentricity. Note that for compression failure, $e < e_b$ and $P_n > P_b$, and for tension failure, $e > e_b$ and $P_n < P_b$. The last two cases in the table represent the pure moment (P.M.) or beam-action case for $\phi = 0.9$ and $\phi = 0.65$ ($M_n = 391$ K·ft). To be consistent with the design of beams due to bending moments, the ACI Code allows the use of $\phi = 0.9$ with pure moment, so $\phi M_n = 352$ K·ft instead of 254.2 K·ft. Also note that ϕ varies between 0.65 and 0.9 according to Eq. 11.2 for tied columns. Note that $M_n = 391.1$ K·ft.

11.10 RECTANGULAR COLUMNS WITH SIDE BARS

In some column sections, the steel reinforcement bars are distributed around the four sides of the column section. The side bars are those placed on the sides along the depth of the section in addition to the tension and compression steel, A_s and A_s', and can be denoted by A_{ss} (Fig. 11.12). In this case the same procedure explained earlier can be applied, taking into consideration the strain variation along the depth of the section and the relative force in each side bar either in the compression or tension zone of the section. These are added to those of C_c, C_s, and T to determine P_n Equation 11.10 becomes

$$P_n = C_c + \sum C_s - \Sigma T \tag{11.10a}$$

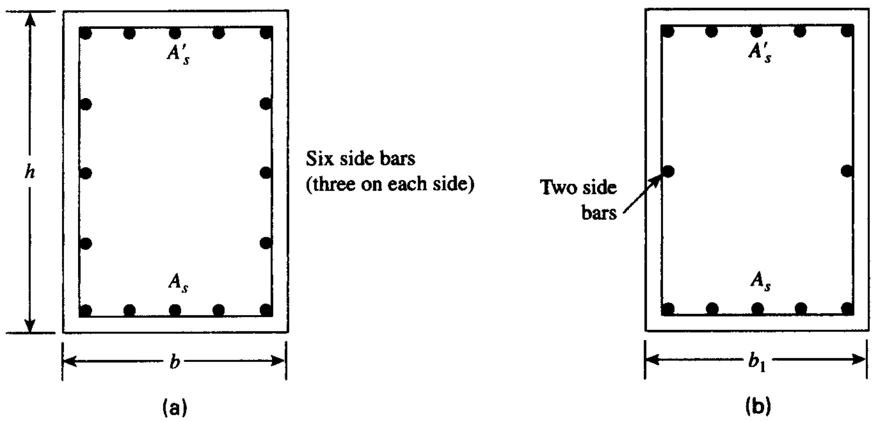


Figure 11.12 Side bars in rectangular sections: (*a*) six side bars and (*b*) two side bars (may be neglected).

Example 11.7 explains this analysis. Note that if the side bars are located near the neutral axis (Fig. 11.12*b*), the strains—and, consequently, the forces—in these bars are very small and can be neglected. Those bars close to A_s and A'_s have appreciable force and increase the load capacity of the section.

Example 11.7

Determine the balanced load, P_b moment, M_b, and e_b for the section shown in Fig. 11.13. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

The balanced section is similar to Example 11.2. Given: $b = h = 22$ in., $d = 19.5$ in., $d' = 2.5$ in., $A_s = A'_s = 6.35$ in^2. (five no. 10 bars), and six no. 10 side bars (three on each side).

1. Calculate the distance to the neutral axis:

$$c_b = \left(\frac{87}{87 + f_y}\right) d_t = \left(\frac{87}{87 + 60}\right) 19.5 = 11.54 \text{ in.}$$

$$a_b = 0.85(11.54) = 9.81 \text{ in.}$$

2. Calculate the forces in concrete and steel bars; refer to Fig. 11.13*a*. In the compression zone, $C_c = 0.85 f'_c ab = 0.85(4)(9.81)(22) = 733.8$ K.

$$f'_s = 87\left(\frac{c - d'}{c}\right) = 87\left(\frac{11.54 - 2.5}{11.54}\right) = 68.15 \text{ ksi} > 60 \text{ ksi}$$

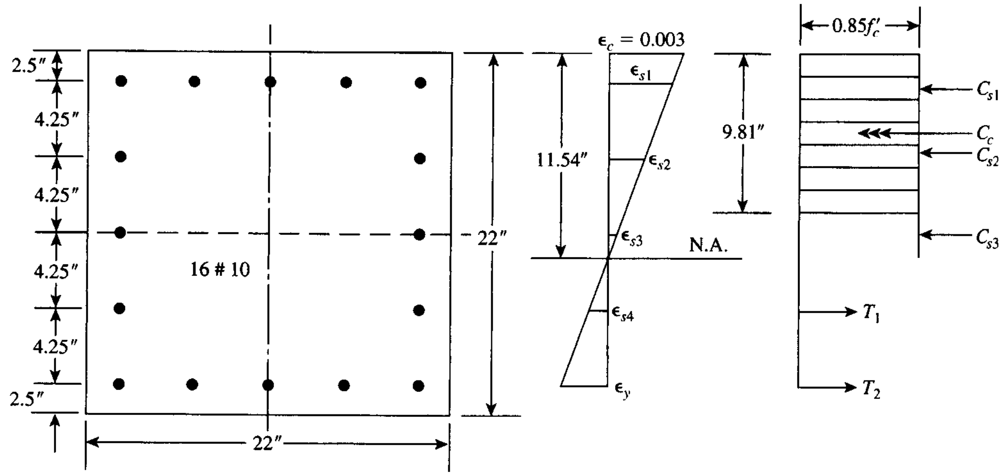
Then $f'_s = 60$ ksi.

$$C_{s1} = A'_s(f_y - 0.85 f'_c) = 6.35(60 - 0.85 \times 4) = 359.4 \text{ K}$$

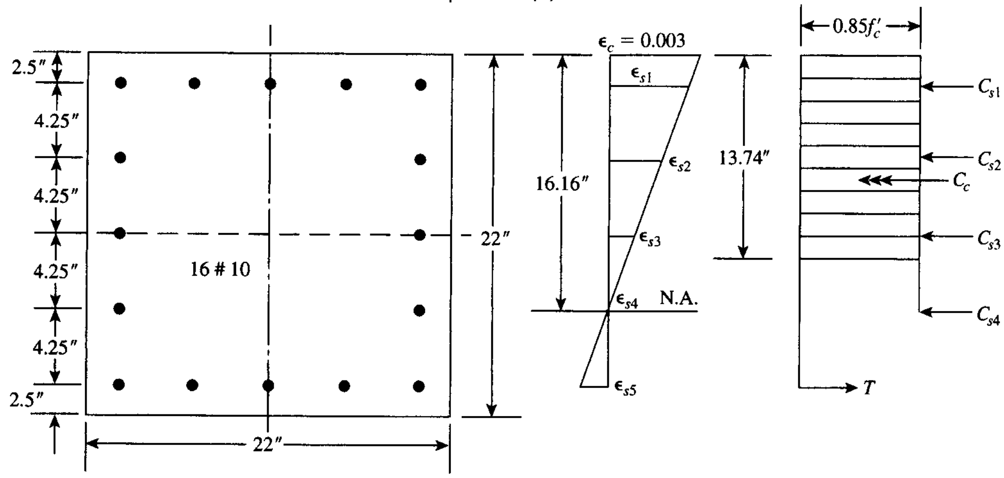
$$f_{s2} = 87\left(\frac{11.54 - 2.5 - 4.25}{11.54}\right) = 36.11 \text{ ksi}$$

$$C_{s2} = 2(1.27)(36.11 - 0.85 \times 4) = 83.1 \text{ K}$$

Similarly, $f_{s3} = 4.07$ ksi and $C_{s3} = 2(1.27)(4.07 - 0.85 \times 4) = 1.7$ K.



Example 11.7 (a)



Example 11.8 (b)

Figure 11.13 Example 11.7: (*a*) balanced section. Example 11.8: (*b*) for compression failure, $e = 6$ in.

In the tension zone,

$$\varepsilon_{s4} = 964.50 \times 10^{-6} \qquad f_{s4} = 28 \text{ ksi}$$

$$T_1 = 2(1.27)(28) = 71 \text{ K}$$

$$T_2 = A_s f_y = 6.35(60) = 381 \text{ K}$$

3. Calculate $P_b = C_c + \Sigma C_s - \Sigma T$.

$$P_b = 733.8 + (359.4 + 83.1 + 1.7) - (71 + 381)$$

$$= 726 \text{ K}$$

4. Taking moments about the plastic centroid,

$$M_b = 733.8(6.095) + 359.4(8.5) + 83.1(4.25) + 71(4.25) + 381(8.5)$$

$$= 11{,}421 \text{ K·in.} = 952 \text{ K·ft}$$

$$e_b = \frac{M_b}{P_b} = 15.735 \text{ in.}$$

5. Determine ϕ: For a balanced section, $\varepsilon_t = \varepsilon_y = 0.002$, $\phi = 0.65$,

$$\phi P_b = 0.65\, P_b = 472 \text{ K, and } \phi M_b = 0.65\, M_b = 618.8 \text{ K·ft.}$$

Example 11.8

Repeat the previous example when $e = 6.0$ in.

Solution

1. Because $e = 6$ in. $< e_b = 15.735$ in., this is a compression failure condition. Assume $c = 16.16$ in. (by trial) and $a = 0.85(16.16) = 13.74$ in. (Fig. 11.13*b*).
2. Calculate the forces in concrete and steel bars:

$$C_c = 0.85(4)(13.74)(22) = 1027.75 \text{ K}$$

In a similar approach to the balanced case, $f_{s1} = 60$ ksi and $C_{s1} = 359.41$.

$$f_{s2} = 50.66 \text{ ksi} \quad C_{s2} = 120.0 \text{ K}$$

$$f_{s3} = 27.78 \text{ ksi} \quad C_{s3} = 61.92 \text{ K}$$

$$f_{s4} = 4.9 \text{ ksi} \quad C_{s4} = 3.81 \text{ K}$$

$$f_{s5} = 18 \text{ ksi} \quad T = 6.35(18) = 114.2 \text{ K}$$

3. Calculate $P_n = C_c + \Sigma C_s - \Sigma T = 1458.7$ K.

$$M_n = P_n \cdot e = 729.35 \text{ K·ft} \quad (e = 6 \text{ in.})$$

4. Check P_n by taking moments about A_s,

$$P_n = \frac{1}{e'}\left[C_c\left(d - \frac{a}{2}\right) + C_{s1}(d - d') + C_{s2}(d - d' - s) + C_{s3}(d - d' - 2s) + C_{s4}(d - d' - 3s)\right]$$

$$e' = e + d - \frac{h}{2} = 6 + 19.5 - \frac{22}{2} = 14.5 \text{ in.}$$

s = distance between side bars

= 4.25 in. (s = constant in this example.)

$$P_n = \frac{1}{14.5}\left[1027.75\left(19.5 - \frac{13.74}{2}\right) + 359.41(17) + 120(17 - 4.25) + 61.92(17 - 8.5) + 3.81(17 - 12.75)\right] = 1459 \text{ K}$$

5. Calculate ϕ:

$$d_t = d = 19.5 \text{ in.} \quad c = 16.16 \text{ in.}$$

ε_t (at the tension steel level) $= 0.003(d_t - c)/c$

$$\varepsilon_t = 0.003(19.5 - 16.16)/16.16 = 0.00062$$

Since $\varepsilon_t < 0.002$, then $\phi = 0.65$.

$$\phi P_n = 0.65(1459) = 948.3 \text{ K}$$

$$\phi M_n = 0.65(729.5) = 474 \text{ K·ft}$$

Note: If side bars are neglected, then

$$P_b = 733.8 + 359.4 - 381 = 712.2 \text{ K}$$

$$P_n \text{ (at } e = 6 \text{ in.)} = 1027.75 + 359.4 - 114.2 = 1273 \text{ K}$$

If side bars are considered, the increase in P_b is about 2% and that in P_n is about 14.6%.

11.11 LOAD CAPACITY OF CIRCULAR COLUMNS

11.11.1 Balanced Condition

The values of the balanced load P_b and the balanced moment M_b for circular sections can be determined using the equations of equilibrium, as was done in the case of rectangular sections. The bars in a circular section are arranged in such a way that their distance from the axis of plastic centroid varies, depending on the number of bars in the section. The main problem is to find the depth of the compressive block a and the stresses in the reinforcing bars. The following example explains the analysis of circular sections under balanced conditions. A similar procedure can be adopted to analyze sections when tension or compression controls.

Example 11.9

Determine the balanced load P_b and the balanced moment M_b for the 16-in. diameter circular spiral column reinforced with eight no. 9 bars shown in Fig. 11.14. Given: $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. Because the reinforcement bars are symmetrical about the axis A-A passing through the center of the circle, the plastic centroid lies on that axis.
2. Determine the location of the neutral axis:

$$d_t = 13.1 \text{ in.} \qquad \varepsilon_y = \frac{f_y}{E_s} \qquad (E_s = 29{,}000 \text{ ksi})$$

$$\frac{c_b}{d_t} = \frac{0.003}{0.003 + \varepsilon_y} = \frac{0.003}{0.003 + \frac{f_y}{E_s}} = \frac{87}{87 + f_y}$$

$$c_b = \frac{87}{87 + 60}(13.1) = 7.75 \text{ in.}$$

$$a_b = 0.85 \times 7.75 = 6.59 \text{ in.}$$

3. Calculate the properties of a circular segment (Fig. 11.15):

$$\text{Area of segment} = r^2(\alpha - \sin\alpha \cos\alpha) \tag{11.19}$$

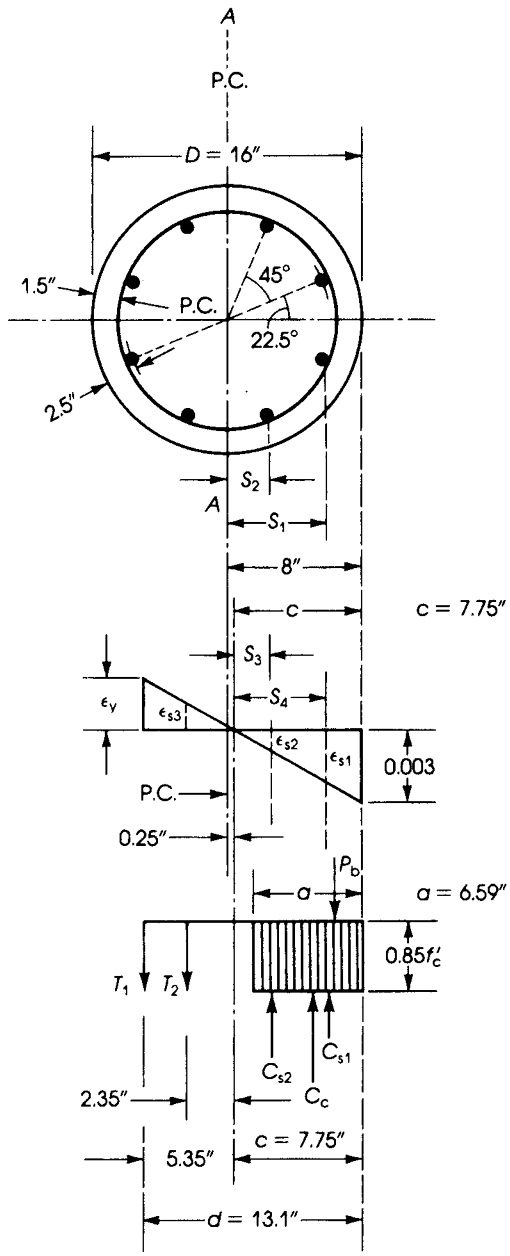


Figure 11.14 Example 11.9: eight no. 9 bars

$$S = 8 - 2.5 = 5.5 \text{ in.}$$
$$S_1 = S \cos 22.5^\circ = 5.1 \text{ in.}$$
$$S_2 = S \cos 67.5^\circ = 2.1 \text{ in.}$$
$$d = 8 + 5.1 = 13.1 \text{ in.}$$
$$S_3 = 1.85 \text{ in.}$$
$$S_4 = 4.85 \text{ in.}$$

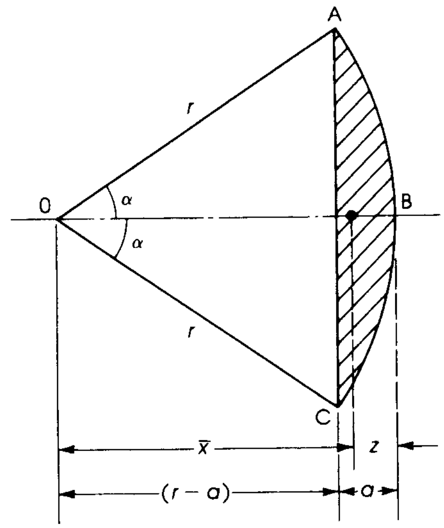


Figure 11.15 Example 11.9: Properties of circular segments.

Location of centroid $\bar{x}$ (from the circle center 0):

$$\bar{x} = \frac{2}{3}\frac{(r\sin^3\alpha)}{(\alpha - \sin\alpha\cos\alpha)} \quad (11.20)$$

$$Z = r - \bar{x} \quad (11.21)$$

$$r\cos\alpha = (r - a) \quad \text{or} \quad \cos\alpha = \left(1 - \frac{a}{r}\right) \quad (11.22)$$

$$\cos\alpha = \left(1 - \frac{6.59}{8}\right) = 0.176$$

and $\alpha = 79.85°$, $\sin\alpha = 0.984$, and $\alpha = 1.394$ rad.

$$\text{Area for segment} = (8)^2(1.394 - 0.984 \times 0.176)$$
$$= 78.12 \text{ in.}^2$$

$$\bar{x} = \left(\frac{2}{3}\right)\frac{8(0.984)^3}{(1.394 - 0.984 \times 0.176)} = 4.16 \text{ in.}$$

$$Z = r - \bar{x} = 8 - 4.16 = 3.84 \text{ in.}$$

4. Calculate the compressive force C_c:

$$C_c = 0.85 f_c' \times \text{area of segment}$$
$$= 0.85 \times 4 \times 78.12 = 265.6 \text{ K}$$

It acts at 4.16 in. from the center of the column.

5. Calculate the strains, stresses, and forces in the tension and the compression steel. Determine the strains from the strain diagram. For T_1,

$$\varepsilon = e_y = 0.00207 \qquad f_s = f_y = 60 \text{ ksi}$$

$$T_1 = 2 \times 60 = 120 \text{ K}$$

For T_2,

$$\varepsilon_{s3} = \frac{2.35}{5.35}\varepsilon_y = \frac{2.35}{5.35} \times 0.00207 = 0.00091$$

$$f_{s3} = 0.00091 \times 29{,}000 = 26.4 \text{ ksi}$$

$$T_2 = 26.4 \times 2 = 52.8 \text{ K}$$

For C_{s1},

$$\varepsilon_{s1} = \frac{4.85}{7.75} \times 0.003 = 0.00188$$

$$f_{s1} = 0.00188 \times 29{,}000 = 54.5 \text{ ksi} < 60 \text{ ksi}$$

$$C_{s1} = 2(54.5 - 3.4) = 102.2 \text{ K}$$

For C_{s2},

$$\varepsilon_{s2} = \frac{1.85}{7.75} \times 0.003 = 0.000716$$

$$f_{s2} = 0.000716 \times 29{,}000 = 20.8 \text{ ksi}$$

$$C_{s2} = 2(20.8 - 3.4) = 34.8 \text{ K}$$

The stresses in the compression steel have been reduced to take into account the concrete displaced by the steel bars.

6. The balanced force is $P_b = C_c + \Sigma C_s - \Sigma T$ ($\phi = 0.75$).

$$P_b = 265.6 + (102.2 + 34.8) - (120 + 52.8) = 230 \text{ K}$$

For a balanced section,

$$\varepsilon_t = 0.002 \quad \text{and} \quad \phi = 0.65$$

$$\phi P_b = 149.5 \text{ K}$$

7. Take moments about the plastic centroid (axis A-A through the center of the section) for all forces:

$$M_b = P_b e_b = [C_c \times 4.16 + C_{s1} \times 5.1 + C_{s2} \times 2.1 + T_1 \times 5.1 + T_2 \times 2.1]$$

$$= 2422.1 \text{ K·in.} = 201.9 \text{ K·ft}$$

$$\phi M_b = 131.2 \text{ K·ft}$$

$$e_b = \frac{2422.1}{230} = 10.5 \text{ in.}$$

11.11.2 Strength of Circular Columns for Compression Failure

A circular column section under eccentric load can be analyzed in similar steps as the balanced section. This is achieved by assuming a value for $c > c_b$ or $a > a_b$ and calculating the forces in concrete and steel at different locations to determine P_{n1} $P_{n1} = C_c + \Sigma C_s - \Sigma T$. Also, M_n can be calculated by taking moments about the plastic centroid (center of the section) and determining $P_{n2} = M_n/e$. If they are not close enough, within about 1%, assume a new c or a and repeat the calculations. (See also Section 11.8.) Compression controls when $e < e_b$ or $P_n > P_b$.

For example, if it is required to determine the load capacity of the column section of Example 11.9 when $e = 6$ in., P_n can be determined in steps similar to those of Example 11.9:

1. Because $e = 6$ in. is less than $e_b = 10.5$ in., compression failure condition occurs.
2. Assume $c = 9.0$ in. (by trial) $> c_b = 7.75$ in. and $a = 7.65$ in.
3. Calculate $\overline{x} = 3.585$ in., $Z = 4.415$ in., and the area of concrete segment = 94.93 in.^{2}

4–5. Calculate forces: and $C_c = 322.7$ K, $C_{s1} = 110.7$ K, $C_{s2} = 53.1$ K, $T_1 = 21.6$ K, and $T_2 = 78.9$ K.

6. Calculate $P_{n1} = C_c + \Sigma C_s - \Sigma T = 386$ K.
7. Taking moments about the center of the column (plastic centroid): $M_n = 191$ K·ft, $P_{n2} = M_n/6 = 382$ K, which is close to P_{n1} (the difference is about 1%). Therefore, $P_n = 382$ K. Note that if the column is spirally reinforced, $\phi = 0.70$.

An approximate equation for estimating P_n in a circular section when compression controls was suggested by Whitney [15]:

$$P_n = \frac{A_g f'_c}{\left[\dfrac{9.6he}{(0.8h + 0.67D_s)^2} + 1.18\right]} + \frac{A_{st} f_y}{\left(\dfrac{3e}{D_s} + 1\right)} \tag{11.23}$$

where

A_g = gross area of the section
h = diameter of section
D_s = diameter measured through the centroid of the bar arrangement
A_{st} = total vertical steel area
e = eccentricity measured from the plastic centroid

Example 11.10

Calculate the nominal compressive strength P_n for the section of Example 11.9 using the Whitney equation if the eccentricity is $e = 6$ in.

Solution

1. $e = 6$ in. is less than $e_b = 10.5$ in., calculated earlier; thus, compression controls.
2. Using the Whitney equation,

$$A_g = \frac{\pi}{4} h^2 = \frac{\pi}{4} (16)^2 = 20.1 \text{ in.}^2$$

$$h = 16 \text{ in.} \quad D_s = 16 - 5 = 11.0 \text{ in.} \quad A_{st} = 8 \times 1 = 8 \text{ in.}^2$$

$$P_n = \frac{(201.1 \times 4)}{\left[\dfrac{(9.6 \times 16 \times 6)}{(0.8 \times 16 + 0.67 \times 11)^2} + 1.18\right]} + \frac{8 \times 60}{\left(\dfrac{3 \times 6}{11}\right) + 1}$$

$$= 415.5 \text{ K}$$

3. $M_n = P_n e = 415.5 \times \frac{6}{12} = 207.8$ K·ft. The value of P_n here is greater than $P_n = 382$ K calculated earlier by statics.

11.11.3 Strength of Circular Columns for Tension Failure

Tension failure occurs in circular columns when the load is applied at an eccentricity $e > e_b$, or $P_n < P_b$. In this case, the column section can be analyzed in steps similar to those of the balanced section and Example 11.8. This is achieved by assuming $c < c_b$ or $a < a_b$ and then following the steps explained in Section 11.11.1. Note that because the steel bars are uniformly distributed along the perimeter of the circular section, the tension steel A_s provided could be relatively low, and the load capacity becomes relatively small. Therefore, it is advisable to avoid the use of circular columns for tension failure cases.

11.12 ANALYSIS AND DESIGN OF COLUMNS USING CHARTS

The analysis of column sections explained earlier is based on the principles of statics. For preliminary analysis or design of columns, special charts or tables may be used either to determine ϕP_n and ϕM_n for a given section or determine the steel requirement for a given load P_u and moment M_u. These charts and tables are published by the American Concrete Institute (ACI) [7], the Concrete Reinforcing Steel Institute (CRSI), and the Portland Cement Association (PCA). Final design of columns must be based on statics by using manual calculations or computer programs. The use of the ACI charts is illustrated in the following examples. The charts are given in Figs. 11.16 and 11.17 [7]. These data are limited to the column sections shown on the top right corner of the charts.

Example 11.11

Determine the necessary reinforcement for a short tied column shown in Fig. 11.18*a* to support a factored load of 483 K and a factored moment of 322 K·ft. The column section has a width of 14 in. and a total depth, h, of 20 in. Use $f'_c = 4$ ksi, $f_y = 60$ ksi.

Solution

1. The eccentricity $e = M_u/P_u = 322 \times 12/483 = 8$ in. Let $d = 20 - 2.5 = 17.5$ in., $\gamma h = 20 - 5 = 15$ in., and $\gamma = 15/20 = 0.75$.
2. Since $e = 8$ in. $< d$, assume compression-controlled section with $\phi = 0.65$.

$$P_n = 483/0.65 = 743 \text{ K} \quad \text{and} \quad M_n = 322/0.65 = 495.4 \text{ K·ft.}$$

$$K_n = \frac{743}{(4 \times 14 \times 20)} = 0.663$$

$$R_n = K_n \left(\frac{e}{h}\right) = 0.663 \left(\frac{8}{20}\right) = 0.265$$

3. From the charts of Fig. 11.16, for $\gamma = 0.7$, $\rho = 0.034$. Also, for $\gamma = 0.8$, $\rho = 0.039$. By interpolation, for $\gamma = 0.75$, $\rho = 0.0365$.

$$A_s = 0.0365\,(14 \times 20) = 10.22 \text{ in.}^2$$

Use eight no. 10 bars ($A_s = 10.16$ in.$_2$), four on each short side. Use no. 3 ties spaced at 14 in. (Fig. 11.18*a*).

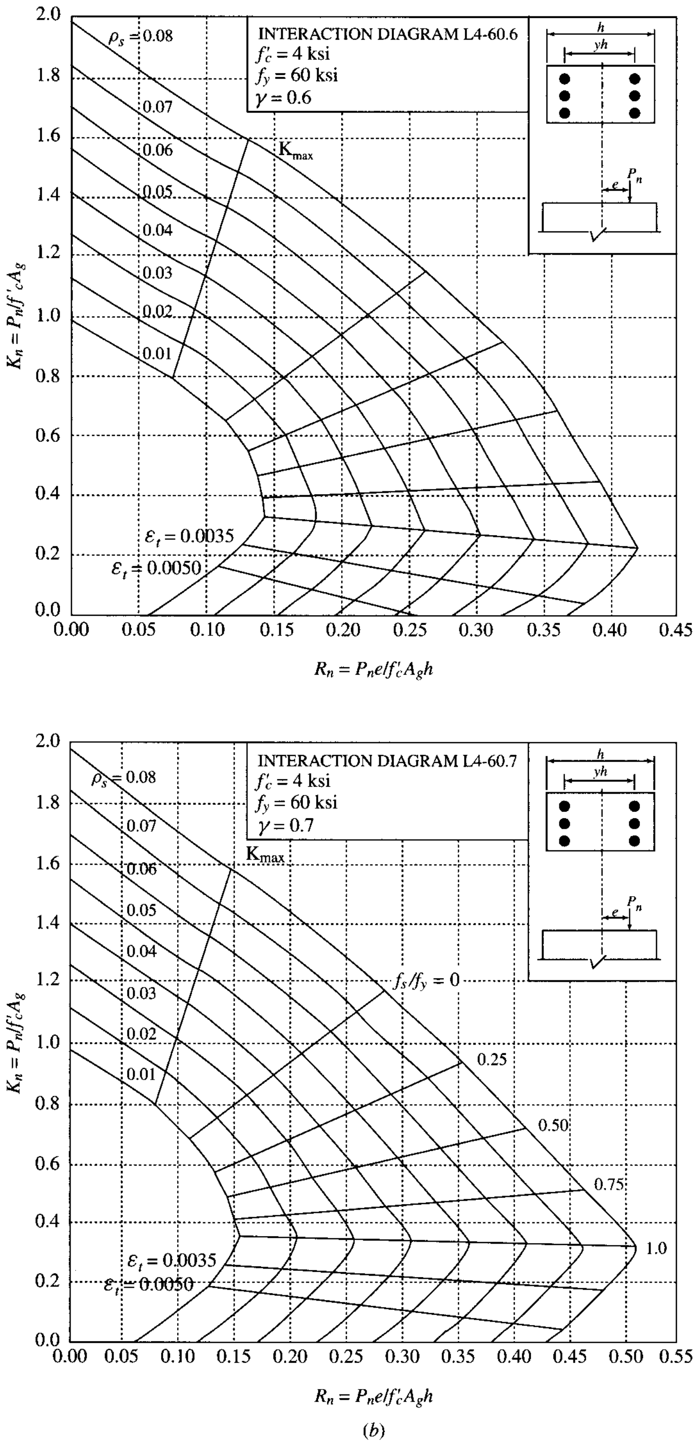
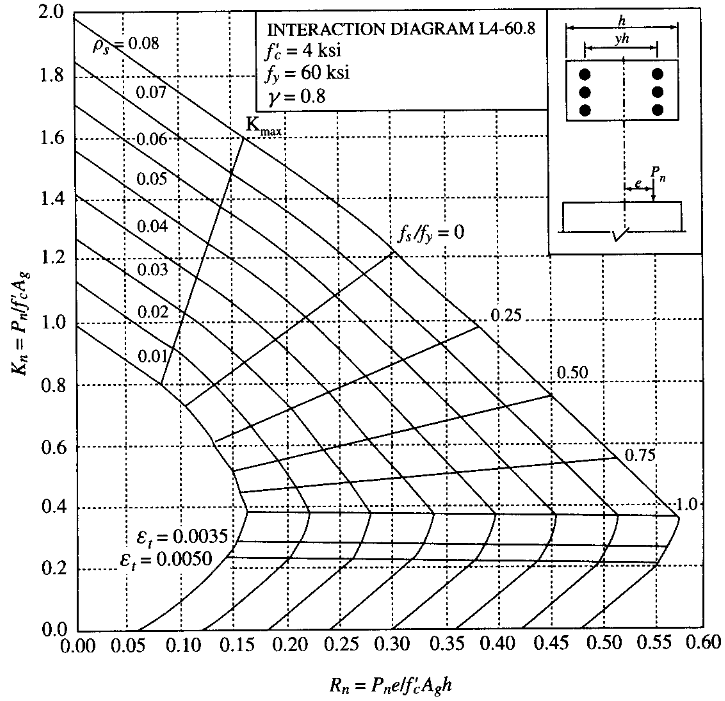
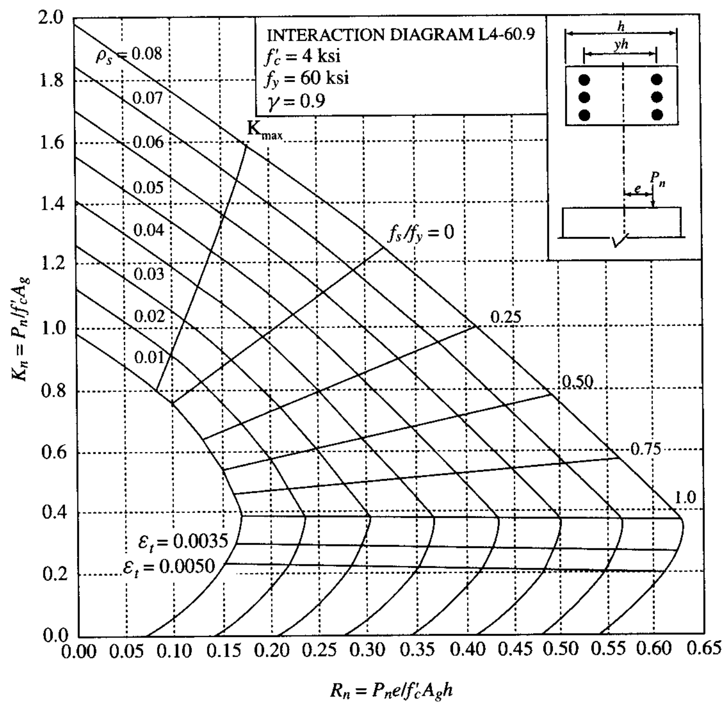


Figure 11.16 Load-moment strength interaction diagram for rectangular columns where $f'_c = 4$ ksi, $f_y = 60$ ksi, and (*a*) $\gamma = 0.60$, (*b*) $\gamma = 0.70$, (*c*) $\gamma = 0.80$, and (*d*) $\gamma = 0.90$. Courtesy of American Concrete Institute [7].

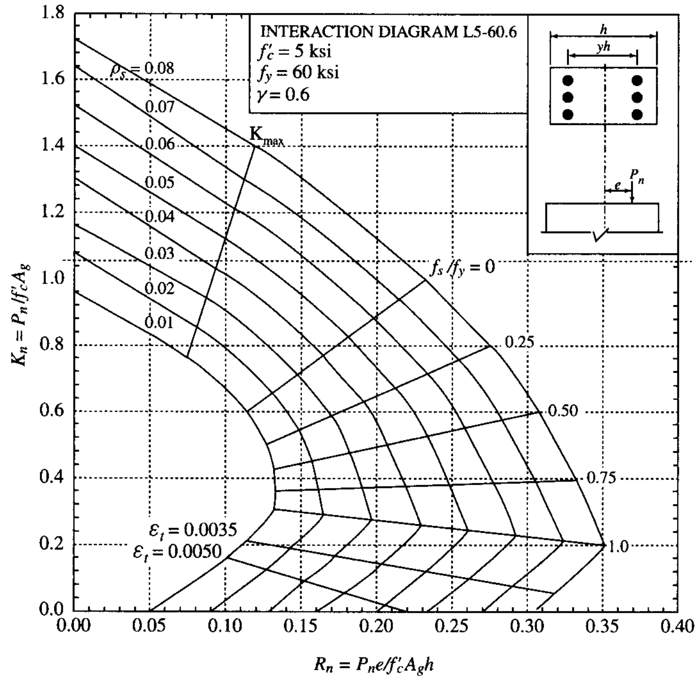


(*c*)

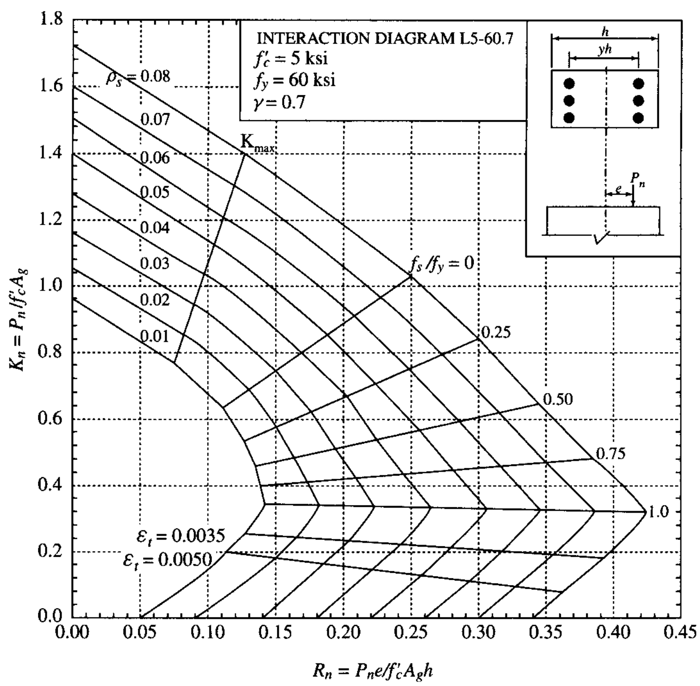


(*d*)

Figure 11.16 (*continued*)

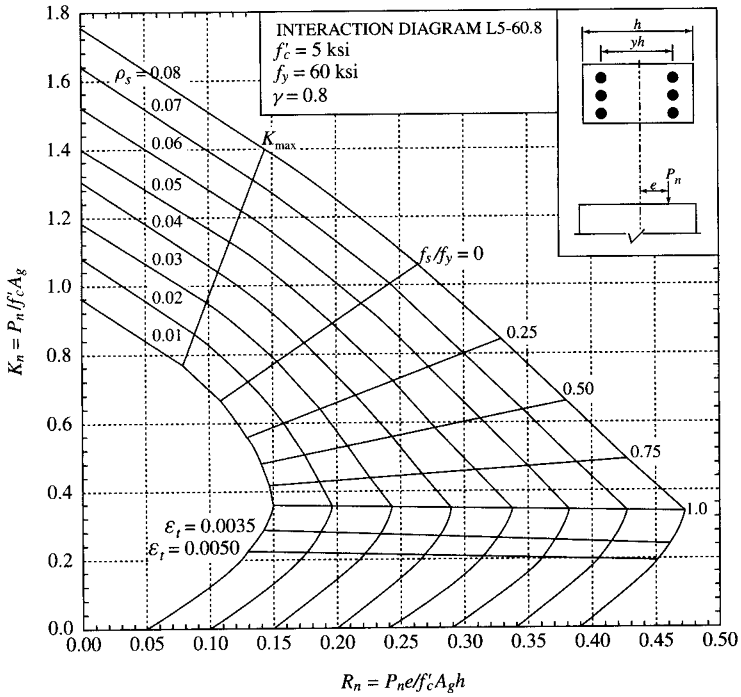


(a)

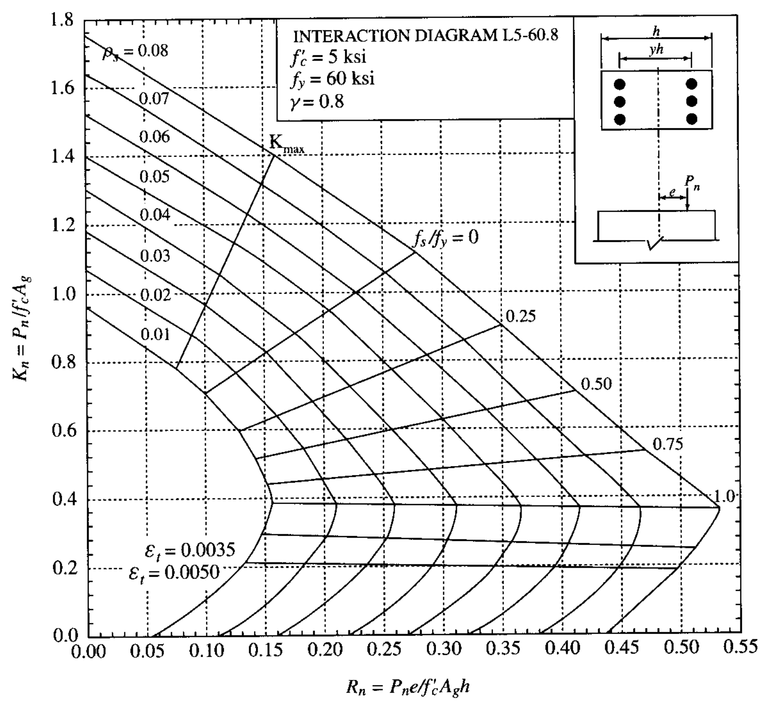


(b)

Figure 11.17 Load–moment strength interaction diagram for rectangular columns where f'_c = 5 ksi, f_y = 60 ksi, and (a) γ = 0.60, (b) γ = 0.70, (c) γ = 0.80, and (d) γ = 0.90. Courtesy of American Concrete Institute [7].



(c)



(d)

Figure 11.17 *(continued)*

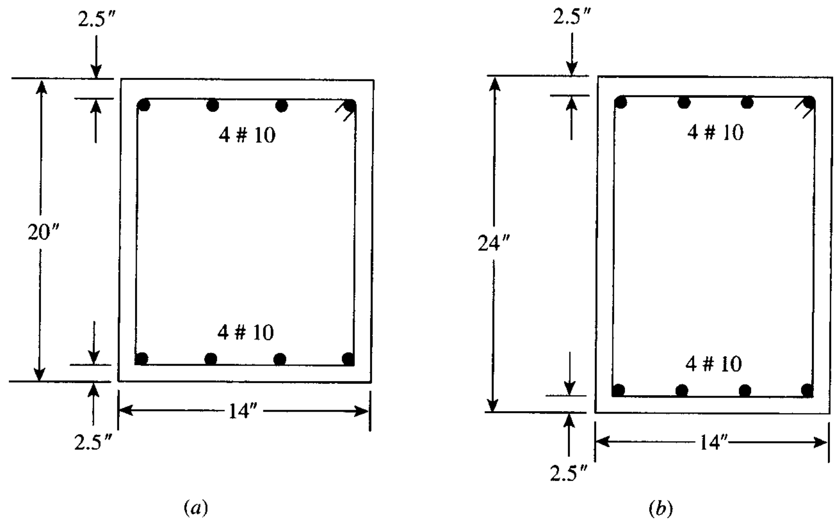


Figure 11.18 Column sections of (*a*) Example 11.11 and (*b*) Example 11.12.

Example 11.12

Use the charts to determine the column strength, ϕP_n, of the short column shown in Fig. 11.18*b* acting at an eccentricity $e = 12$ in. Use $f_c' = 5$ ksi and $f_y = 60$ ksi.

Solution

1. Properties of the section: $H = 24$ in., $\gamma h = 24 - 5 = 19$ in. (distance between tension and compression steel). $\gamma = 19/24 = 0.79$, and $\rho = 8(1.27)/(14 \times 24) = 0.03$.
2. Since $e < d$, assume compression-controlled section. Let $\varepsilon_t = 0.002$, $f_s/f_y = 1.0$ and $\phi = 0.65$. From the charts of Fig. 11.17, get $K_n = 0.36 = P_n/(5 \times 14 \times 24)$. Then $P_n = 605$ K.
3. Check assumption for compression-controlled section: For $K_n = 0.36$, $R_n = K_n\,(e/h) = 0.36\,(12/24) = 0.18$. From charts, get $\rho = 0.018 < 0.03$. Therefore, $P_n > 605$ K (to use $\rho = 0.003$).
4. Second trial: Let $\varepsilon_t = 0.0015$, $f_s = 0.0015\,(29{,}000) = 43.5$ ksi.

$$f_s/f_y = 43.5/60 = 0.725 \quad \rho = 0.03 \quad K_n = 0.44$$

$$0.44 = P_n/(5 \times 14 \times 24) \quad P_n = 740 \text{ K}$$

5. Check assumption: For $K_n = 0.44$, $R_n = 0.44\,(12/24) = 0.22$. From charts, $\rho = 0.03$ as given. Therefore, $P_n = 740$ K.

$$\phi P_n = 0.65(740) = 480 \text{ K} \quad \text{and} \quad \phi M_n = 0.65(740) = 480 \text{ K·ft}$$

By analysis, $\phi P_n = 485$ K(which is close to 480 K·ft).

11.13 DESIGN OF COLUMNS UNDER ECCENTRIC LOADING

In the previous sections, the analysis, behavior, and the load–interaction diagram of columns subjected to an axial load and bending moment were discussed. The design of columns is more complicated, because the external load and moment, P_u and M_u, are given and it is required to determine many unknowns, such as b, h, A_s, and A'_s, within the ACI Code limitations. It is a common practice to assume a column section first and then determine the amount of reinforcement needed. If the designer needs to change the steel reinforcement calculated, then the cross-section may be adjusted accordingly. The following examples illustrate the design of columns.

11.13.1 Design of Columns for Compression Failure

For compression failure, it is preferable to use $A_s = A'_s$ for rectangular sections. The eccentricity, e, is equal to M_u/P_u. Based on the magnitude of e, two cases may develop.

1. When e is relatively very small (say, $e \leq 4$ in.), a minimum eccentricity case may develop that can be treated by using Eq. 10.8, as explained in the examples of Chapter 10. Alternatively, the designer may proceed as in Case 2. This loading case occurs in the design of the lower-floor columns in a multistory building, where the moment, M_u, develops from one floor system and the load, P_u, develops from all floor loads above the column section.
2. The compression failure zone represents the range from the axial to the balanced load, as shown in Figs. 11.3 and 11.11. In this case, a cross-section (bh) may be assumed and then the steel reinforcement is calculated for the given P_u and M_u. The steps can be summarized as follows:
   - **a.** Assume a square or rectangular section (bh); then determine d, d', and $e = M_u/P_u$.
   - **b.** Assuming $A_s = A'_s$, calculate A'_s from Eq. 11.17 using the dimensions of the assumed section, and $\phi = 0.65$ for tied columns. Let $A_s = A'_s$ and then choose adequate bars. Determine the actual areas used for A_s and A'_s. Alternatively, use the ACI charts.
   - **c.** Check that $\rho_g = (A_s + A'_s)/bh$ is less than or equal to 8% and greater or equal to 1%. If ρ_g is small, reduce the assumed section, but increase the section if less steel is required.
   - **d.** Check the adequacy of the final section by calculating ϕP_n from statics; as explained in the previous examples, ϕP_n should be greater than or equal to P_u.
   - **e.** Determine the necessary ties.

A simple approximate formula for determining the initial size of the column bh or the total steel ratio ρ_g is

$$P_n = K_c bh^2 \quad \text{or} \quad P_u = \phi P_n = \phi K_c bh^2 \tag{11.24}$$

where K_c has the values shown in Table 11.2 and plotted in Fig. 11.19 for $f_y = 60$ ksi and $A_s = A'_s$. Units for K_c are in lb/in.^{3}

The values of K_c shown in Table 11.2 are approximate and easy to use, because K_c increases by 0.02 for each increase of 1 ksi in f'_c. For the same section, as the eccentricity,

Table 11.2 Values of K_c ($f_y = 60$ ksi)

	K_c		
ρ_g (%)	$f'_c = 4$ ksi	$f'_c = 5$ ksi	$f'_c = 6$ ksi
1%	0.090	0.110	0.130
4%	0.137	0.157	0.177
8%	0.200	0.220	0.240

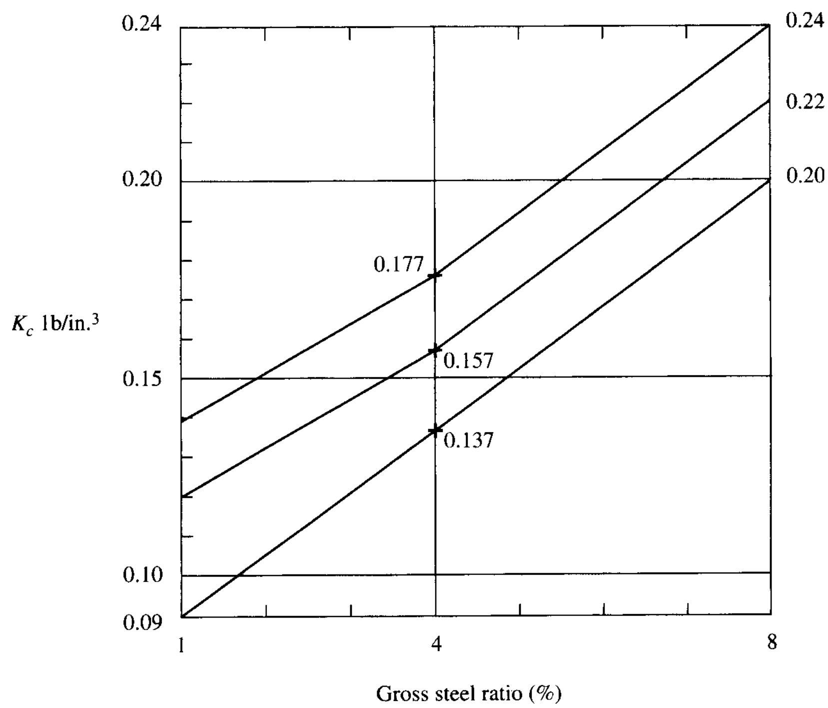


Figure 11.19 Values of K_c versus ρ_g (%).

$e = M_u/P_u$, increases, P_n decreases, and, consequently, K_c decreases. Thus, K_c values represent a load P_n on the interaction diagram between 0.8 P_{n_0} and P_b as shown in Fig. 11.3 or 11.11.

Linear interpolation can be used. For example, $K_c = 0.1685$ for $\rho_g = 6\%$ and $f'_c = 4$ ksi. The steps in designing a column section can be summarized as follows:

1. Assume an initial size of the column section bh.
2. Calculate $K_c = P_u/(\phi bh^2)$.
3. Determine ρ_g from Table 11.2 for the given f'_c.
4. Determine $A_s = A'_s = \rho_g bh/2$ and choose bars and ties.
5. Determine ϕP_n of the final section by statics (accurate solution). The value of ϕP_n should be greater than or equal to P_u. If not, adjust bh or ρ_g.

Alternatively, if a specific steel ratio is desired, say $\rho_g = 6\%$, then proceed as follows:

1. Assume ρ_g as required and then calculate $e = M_u/P_u$.
2. Based on the given f'_c and ρ_g, determine K_c from Table 11.2.
3. Calculate $bh^2 = P_u/\phi K_c$; then choose b and h. Repeat steps 4 and 5. It should be checked that ρ_g is less than or equal to 8% and greater than or equal to 1%. Also, check that c calculated by statics is greater than $c_b = 87d_t/(87 + f_y)$ for compression failure to control.

Example 11.13

Determine the tension and compression reinforcement for a 16 × 24-in. rectangular tied column to support $P_u = 780$ K and $M_u = 390$ K·ft. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. Calculate $e = M_u/P_u = 420(12)/840 = 6.0$ in. We have $h = 24$ in; let $d = 21.5$ in. and $d' = 2.5$ in. Because e is less than $\frac{2}{3}d = 14.38$ in., assume compression failure.
2. Assume $A_s = A'_s$ and use Eq. 11.17 to determine the initial value of $A'_s P_n = P_u/\phi = 780/0.65 = 1200$ K.

$$P_n = \frac{bhf'_c}{\left(\frac{3he}{d^2}\right) + 1.18} + \left[\frac{A'_s f_y}{\left(\frac{e}{d - d'}\right) + 0.5}\right] \tag{11.17}$$

For $P_n = 1200$ K, $e = 6$ in., $d = 21.5$ in., $d' = 2.5$ in., and $h = 24$ in., calculate $A'_s = 6.44$ in.$^2 = A_s$. Choose five no. 10 bars ($A_s = 6.35$ in.^{2}) for A_s and A'_s (Fig. 11.20).
3. $\rho_g = 2(6.35)/(16 \times 24) = 0.033$, which is less than 0.08 and >0.01.
4. Check the section by statics following the steps of Example 11.4 to get

$$a = 16.64 \text{ in.} \quad c = 19.58 \text{ in.} \qquad C_c = 905.2 \text{ K}$$

$$C_s = 6.35(60 - 0.85 \times 4) = 359.4 \text{ K}$$

$$f_s = 87\left(\frac{d - c}{c}\right) = 8.55 \text{ ksi}$$

$$T = A_s f_s = 6.35(8.55) = 54.3 \text{ K}$$

$$P_n = C_c + C_s - T = 1210.3 \text{ K} > 1200 \text{ K}$$

Note that if $\phi P_n < P_u$, increase A_s and A'_s, for example, to six no. 10 bars, and check the section again.
5. Check P_n based on moments about A_s (Eq. 11.12) to get $P_n = 1210$ K.
6. For a balanced section,

$$c_b = \left(\frac{87}{87 + f_y}\right) d_t = \left(\frac{87}{147}\right) 21.5 = 12.7 \text{ in.}$$

Because $c = 19.58$ in. $> c_b = 12.7$ in., this is a compression failure case, as assumed.
7. Use no. 3 ties spaced at 16 in. (Refer to Chapter 10.)

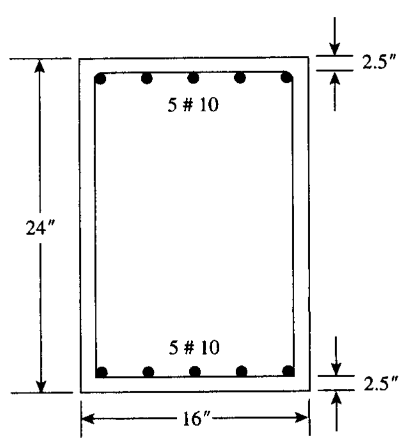


Figure 11.20 Example 11.13.

Example 11.14

Repeat Example 11.13 using Eq. 11.24.

Solution

1. The column section is given: 16 × 24 in.
2. Determine K_c from Eq. 11.24:

$$K_c = \frac{P_u}{\phi bh^2} = \frac{780}{0.65 \times 16 \times 24^2} = 0.13 \text{ lb/in.}^3$$

3. From Table 11.2 or Fig. 11.19, for $K_c = 0.13$, $f'_c = 4$ ksi, by interpolation, get $\rho_g = 3.5\%$.
4. Calculate $A_s = A'_s = \rho bh/2 = 0.035(16)(24)/2 = 6.77$ in.^{2} Choose five no. 10 bars ($A_s = 6.35$ in.^{2}) for the first trial.
5. Determine ϕP_n using steps 4–7 in Example 11.13. $\phi P_n = 1210.3$ K $> P_n = 1200$ K, so the section is adequate.
6. If the section is not adequate, or $\phi P_n < P_u$, increase A_s and A'_s and check again to get closer values.

Example 11.15

Design a rectangular column section to support $P_u = 696$ K and $M_u = 465$ K·ft with a total steel ratio ρ_g of about 4%. Use $f'_c = 4$ ksi, $f_y = 60$ ksi, and $b = 18$ in.

Solution

1. Calculate $e = M_u/P_u = 465(12)/696 = 8$ in. Assume compression failure ($\phi = 0.65$) (to be checked later) and $A_s = A'_s$.
2. For $\rho_s = 4\%$ and $f'_c = 4$ ksi, $K_c = 0.137$ (Table 11.2).
3. Calculate bh^2 from Eq. 11.24: $P_u = \phi K_c bh^2$, or $696 = 0.65(0.137)(18)h^2$. Thus, $h = 20.84$ in. Let $h = 22$ in.
4. Calculate $A_s = A'_s = 0.04(18 \times 22)/2 = 7.92$ in.^{2} Choose five no. 11 bars ($A_s = 7.8$ in.^{2}) in one row for A_s and A'_s (Fig. 11.21). Choose no. 4 ties spaced at 18 in.

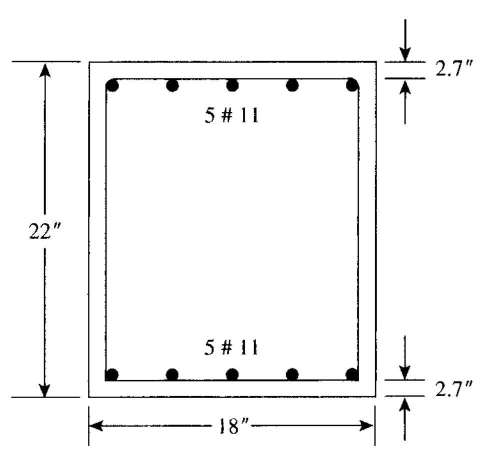


Figure 11.21 Example 11.15.

5. Check the final section by analysis, similar to Example 11.4, to get $a = 13.15$ in., $c = 15.47$ in., $C_c = 0.85\ f'_c ab = 804.8$ K, $f'_s = 60$ ksi, $C_s = A'_s(f_y - 0.85 f'_c) = 441.5$ K, $f_s = 87[(d - c)/c] = 21.24$ ksi, and $T = A_s f_s = 168$ K. Also, $P_n = C_c + C_s - T = 1078.3$ K and $\phi P_n = 0.65 P_n = 701\text{ K} > 696\text{ K}$. The section is adequate.
6. For a balanced section,

$$c_b = \left(\frac{87}{87 + f_y}\right) \qquad d_t = \left(\frac{87}{147}\right) 19.3 = 11.42 \text{ in.} < c = 15.47 \qquad (d = 19.3 \text{ in.})$$

Therefore, this is a compression failure case, as assumed.

11.13.2 Design of Columns for Tension Failure

Tension failure occurs when $P_n < P_b$ or the eccentricity $e > e_b$, as explained in Section 11.7. In the design of columns, P_u and M_u are given, and it is required to determine the column size and its reinforcement. It may be assumed (as a guide) that tension controls when the ratio of M_u (K·ft) to P_u (kips) is greater than 1.75 for sections of $h < 24$ in. and 2.0 for $h \geq 24$ in. In this case, a section may be assumed, and then A_s and A'_s are determined. The ACI charts may be used to determine ρ_g for a given section with $A_s = A'_s$. Note that ϕ varies between 0.65 (0.75) and 0.9, as explained in Section 11.4.

When tension controls, the tension steel yields, whereas the compression steel may or may not yield. Assuming initially $f'_s = f_y$ and $A_s = A'_s$, Eq. 11.16 (Section 11.6) may be used to determine the initial values of A_s and A'_s:

$$A_s = A'_s = \frac{P_n\left(e - \frac{h}{2} + \frac{a}{2}\right)}{f_y(d - d')} \tag{11.16}$$

Because a is not known yet, assume $a = 0.4d$ and $P_u = \phi P_n$; then

$$A_s = A'_s = \frac{P_u(e - 0.5h + 0.2d)}{\phi f_y(d - d')} \tag{11.25}$$

The final column section should be checked by statics to prove that $\phi P_n \geq P_u$. Example 11.16 explains this approach.

When the load P_u is very small relative to M_u, the section dimensions may be determined due to M_u only, assuming $P_u = 0$. The final section should be checked by statics. This case occurs in single- or two-story building frames used mainly for exhibition halls or similar structures. In this case, A'_s may be assumed to be less than A_s. A detailed design of a one-story, two-hinged frame exhibition hall is given in Chapter 16.

Example 11.16

Determine the necessary reinforcement for a 16 × 22-in. rectangular tied column to support a factored load $P_u = 257$ K and a factored moment $M_u = 643$ K·ft. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

Solution

1. Calculate $e = M_u/P_u = 643(12)/257 = 30$ in; let $d = 22 - 2.5 = 19.5$ in. Because $M_u/P_u = 500/200 = 2.5 > 1.75$, or because $e > d$, assume tension failure case, $\phi = 0.9$ (to be checked later).

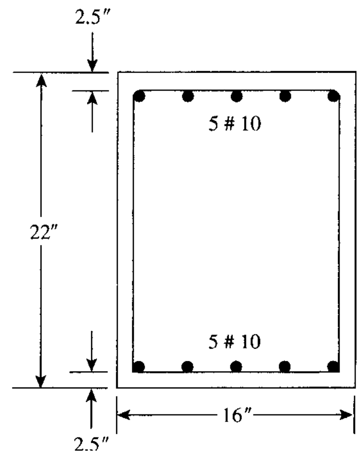


Figure 11.22 Example 11.16.

2. Assume $A_s = A'_s$ and $f'_s = f_y$ and use Eq. 11.25 to determine A_s and A'_s. Let $P_u = 257.0\,\text{K}$, $e = 30\,\text{in.}$, $h = 22\,\text{in.}$, $d = 19.5\,\text{in.}$, and $d' = 2.5\,\text{in.}$

$$A_s = A'_s = \frac{257(30 - 0.5 \times 22 + 0.2 \times 19.5)}{0.9(60)(17.0)} = 6.41\ \text{in.}^2$$

Choose five no. 10 bars ($6.35\,\text{in.}^2$) in one row for each of A_s and A'_s (Fig. 11.22).

3. Check $\rho_g = 2(6.35)(16 \times 22) = 0.036$, which is less than 0.08 and greater than 0.01.
4. Check the chosen section by statics similar to Example 11.3.
   a. Determine the value of a using the general equation $Aa^2 + Ba + C = 0$ with $e' = e + d - h/2 = 38.5\,\text{in.}$, $A = 0.425\ f'_c b = 27.2$, $B = 2A(e' - d) = 1033.6$, $C = A'_s(f_y - 0.85f'_c)(e' - d + d') - A_s f_y e' = -6941.2$. Solve to get $a = 5.82$ in. and $c = a/0.85 = 6.85$.
   b. Check f'_s:

$$f'_s = 87\left(\frac{c - d'}{c}\right) = 87\left(\frac{6.85 - 2.5}{6.85}\right) = 55.26\ \text{ksi}$$

   Let $f'_s = 57$ ksi.
   c. Recalculate a:

$$C = A'_s(f'_s - 0.85f'_c)(e' - d + d') - A_s f_y e' = -7351$$

   Solve now for a to get $a = 6.13$ and $c = 7.21\,\text{in.}$
   d. Check f'_s:

$$f'_s = 87\left(\frac{c - 2.5}{c}\right) = 56.83\ \text{ksi}$$

   Calculate

$$C_c = 0.85(4)(6.13)(16) = 333.5\ \text{K},\ C_s = A'_s(f'_s - 0.85f'_c) = 6.35(57 - 0.85 \times 4)$$
$$= 340.4\ \text{K},\ T = A_s f_y = 6.35(60) = 381\ \text{K}.$$

   e. $P_n = C_c + C_s - T = 292.9$ K.
5. Determine ϕ: $\varepsilon_t = [(d_t - c)/c]\ 0.003 = 0.00511$. Because $\varepsilon_t = 0.00511 > 0.005$, $\phi = 0.9$.
6. $\phi P_n = 0.9(292.9) = 263.6\,\text{K} > 257\,\text{K}$, the section is adequate.

11.14 BIAXIAL BENDING

The analysis and design of columns under eccentric loading was discussed earlier in this chapter, considering a uniaxial case. This means that the load P_n was acting along the y-axis (Fig. 11.23), causing a combination of axial load P_n and a moment about the x-axis equal to $M_{nx} = P_n e_y$ or acting along the x-axis (Fig. 11.24) with an eccentricity e_x, causing a combination of an axial load P_n and a moment $M_{nx} = P_n e_x$.

If the load P_n is acting anywhere such that its distance from the x-axis is e_y and its distance from the y-axis is e_x, then the column section will be subjected to a combination of forces: an axial load P_n a moment about the x-axis $= M_{nx} = P_n e_y$ and a moment about the y-axis $= M_{nx} = P_n e_x$ (Fig. 11.25). The column section in this case is said to be subjected to *biaxial bending*. The analysis and design of columns under this combination of forces is not simple when the principles of statics are used. The neutral axis is at an angle with respect to both axes, and lengthy calculations are needed to determine the location of the neutral axis, strains, concrete compression area, and internal forces and their point of application. Therefore, it was necessary

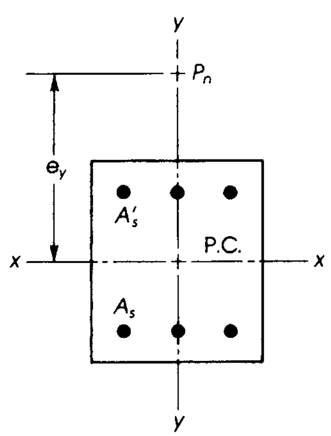


Figure 11.23 Uniaxial bending with load P_n along the y-axis with eccentricity e_y.

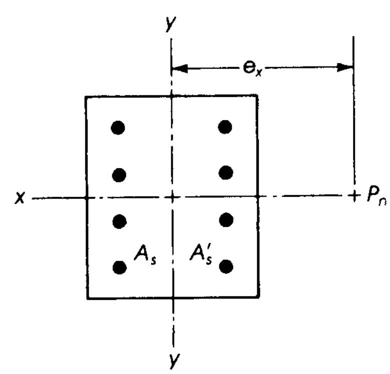


Figure 11.24 Uniaxial bending with load P_n along the x-axis, with eccentricity e_x.

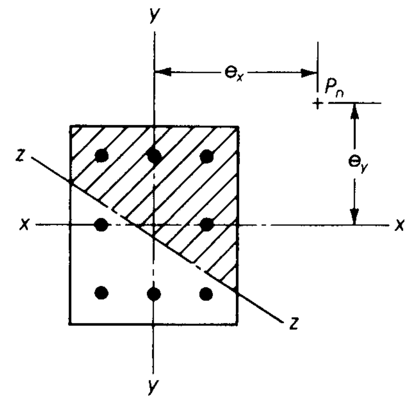


Figure 11.25 Biaxial bending.

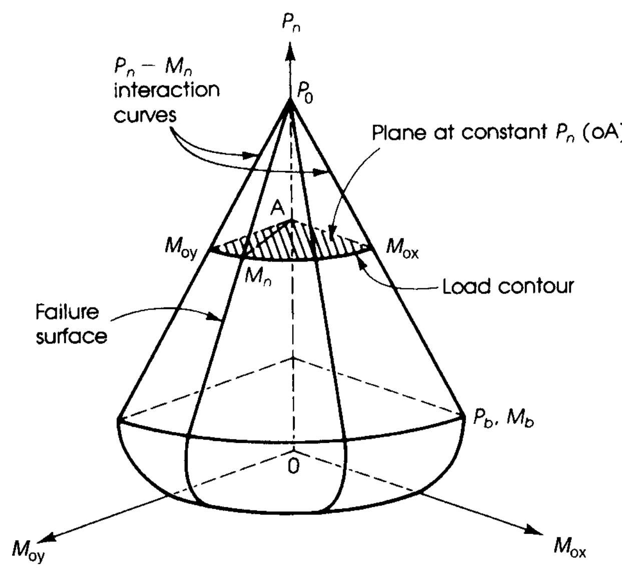


Figure 11.26 Biaxial interaction surface.

to develop practical solutions to estimate the strength of columns under axial load and biaxial bending. The formulas developed relate the response of the column in biaxial bending to its uniaxial strength about each major axis.

The biaxial bending strength of an axially loaded column can be represented by a three-dimensional interaction curve, as shown in Fig. 11.26. The surface is formed by a series of uniaxial interaction curves drawn radially from the P_n-axis. The curve M_{ox} represents the interaction curve in uniaxial bending about the x-axis, and the curve M_{oy} represents the curve in uniaxial bending about the y-axis. The plane at constant axial load P_n shown in Fig. 11.26 represents the contour of the bending moment M_n about any axis.

Different shapes of columns may be used to resist axial loads and biaxial bending. Circular, square, or rectangular column cross-sections may be used with equal or unequal bending capacities in the x- and y-directions.

11.15 CIRCULAR COLUMNS WITH UNIFORM REINFORCEMENT UNDER BIAXIAL BENDING

Circular columns with reinforcement distributed uniformly about the perimeter of the section have almost the same moment capacity in all directions. If a circular column is subjected to biaxial bending about the x- and y-axes, the equivalent uniaxial M_u moment can be calculated using the following equations:

$$M_u = \sqrt{(M_{\text{ux}})^2 + (M_{\text{uy}})^2} = P_u \cdot e \tag{11.26}$$

and

$$e = \sqrt{(e_x)^2 + (e_y)^2} = \frac{M_u}{P_u} \tag{11.27}$$

where

$M_{\text{ux}} = P_u e_y$ = factored moment about the x-axis
$M_{\text{uy}} = P_u e_x$ = factored moment about the y-axis
$M_u = P_u e$ = equivalent uniaxial factored moment of the section due to M_{ux} and M_{uy}

In circular columns, a minimum of six bars should be used, and these should be uniformly distributed in the section.

Example 11.17: Circular Column

Determine the load capacity P_n of a 20-in.-diameter column reinforced with 10 no. 10 bars when $e_x = 4$ in. and $e_y = 6$ in. Use $f'_c = 4$ ksi and f_y and 60 ksi.

Solution

1. Calculate the eccentricity that is equivalent to uniaxial loading by using Eq. 11.41.

$$e(\text{for uniaxial loading}) = \sqrt{e_x^2 + e_y^2} = \sqrt{(4)^2 + (6)^2} = 7.211 \text{ in.}$$

2. Determine the load capacity of the column based on $e = 7.211$ in. Proceed as in Example 11.9:

$$d = 17.12 \text{ in.} \qquad a = 9.81 \text{ in.} \qquad c = 11.54 \text{ in. (by trial)}$$

$$C_c = 521.2 \text{ K} \qquad \sum C_s = 269.8 \text{ K} \qquad \sum T = 132.1 \text{ K}$$

$$P_n = C_c + \sum C_s - \sum T = 650 \text{ K}$$

3. For a balanced condition,

$$c_b = \left(\frac{87}{87 + f_y}\right) d_t = \left(\frac{87}{147}\right) 17.12 = 10.13 \text{ in.}$$

$c = 11.54$ in. $> c_b$, which is a compression failure case.

11.16 SQUARE AND RECTANGULAR COLUMNS UNDER BIAXIAL BENDING

11.16.1 Bresler Reciprocal Method

Square or rectangular columns with unequal bending moments about their major axes will require a different amount of reinforcement in each direction. An approximate method of analysis of such sections was developed by Boris Bresler and is called the Bresler reciprocal method [9,12]. According to this method, the load capacity of the column under biaxial bending can be determined by using the following expression:

$$\frac{1}{P_u} = \frac{1}{P_{ux}} + \frac{1}{P_{uy}} - \frac{1}{P_{u0}} \tag{11.28}$$

or

$$\frac{1}{P_n} = \frac{1}{P_{nx}} + \frac{1}{P_{ny}} - \frac{1}{P_{n0}} \tag{11.29}$$

where

P_u = factored load under biaxial bending
P_{ux} = factored uniaxial load when the load acts at an eccentricity e_y and $e_x = 0$
P_{uy} = factored uniaxial load when the load acts at an eccentricity e_x and $e_y = 0$
P_{u0} = factored axial load when $e_x = e_y = 0$

$$P_n = \frac{P_u}{\phi} \quad P_{nx} = \frac{P_{ux}}{\phi} \quad P_{ny} = \frac{P_{uy}}{\phi} \quad P_{n0} = \frac{P_{u0}}{\phi}$$

The uniaxial load strengths P_{nx}, P_{ny}, and P_{n0} can be calculated according to the equations and method given earlier in this chapter. After that, they are substituted into Eq. 11.29 to calculate P_n.

The Bresler equation is valid for all cases when P_n is equal to or greater than $0.10P_{n0}$. When P_n is less than $0.10P_{n0}$, the axial force may be neglected and the section can be designed as a member subjected to pure biaxial bending according to the following equations:

$$\frac{M_{ux}}{M_x} + \frac{M_{uy}}{M_y} \leq 1.0 \tag{11.30}$$

or

$$\frac{M_{nx}}{M_{ox}} + \frac{M_{ny}}{M_{oy}} \leq 1.0 \tag{11.31}$$

where

$M_{ux} = P_u e_y$ = design moment about the x-axis
$M_{uy} = P_u e_x$ = design moment about the y-axis
M_x and M_y = uniaxial moment strengths about the x-and y-axes

$$M_{nx} = \frac{M_{ux}}{\phi} \quad M_{ny} = \frac{M_{uy}}{\phi} \quad M_{ox} = \frac{M_x}{\phi} \quad M_{oy} = \frac{M_y}{\phi}$$

The Bresler equation is not recommended when the section is subjected to axial tension loads.

11.16.2 Bresler Load Contour Method

In this method, the failure surface shown in Fig. 11.26 is cut at a constant value of P_n, giving the related values of M_{nx} and M_{ny}. The general nondimension expression for the load contour method is

$$\left(\frac{M_{nx}}{M_{ox}}\right)^{\alpha 1} + \left(\frac{M_{ny}}{M_{oy}}\right)^{\alpha 2} = 1.0 \tag{11.32}$$

Bresler indicated that the exponent α can have the same value in both terms of this expression ($\alpha_1 = \alpha_2$). Furthermore, he indicated that the value of α varies between 1.15 and 1.55 and can be assumed to be 1.5 for rectangular sections. For square sections, α varies between 1.5 and 2.0, and an average value of $\alpha = 1.75$ may be used for practical designs. When the reinforcement is uniformly distributed around the four faces in square columns, α may be assumed to be 1.5.

$$\left(\frac{M_{nx}}{M_{ox}}\right)^{1.5} + \left(\frac{M_{ny}}{M_{oy}}\right)^{1.5} = 1.0 \tag{11.33}$$

The British Code assumed $\alpha = 1.0$, 1.33, 1.67, and 2.0 when the ratio $P_u/1.1P_{u0}$ is equal to 0.2, 0.4, 0.6, and ≥ 0.8, respectively.

11.17 PARME LOAD CONTOUR METHOD

The load contour approach, proposed by the Portland Cement Association (PCA), is an extension of the method developed by Bresler. In this approach, which is also called the *Parme method* [11], a point B on the load contour (of a horizontal plane at a constant P_n shown in Fig. 11.26) is defined such that the biaxial moment capacities M_{nx} and M_{ny} are in the same ratio as the uniaxial moment capacities M_{ox} and M_{oy}; that is,

$$\frac{M_{nx}}{M_{ny}} = \frac{M_{ox}}{M_{oy}} \quad \text{or} \quad \frac{M_{nx}}{M_{ox}} = \frac{M_{ny}}{M_{oy}} = \beta$$

The ratio β is shown in Fig. 11.27 and represents that constant portion of the uniaxial moment capacities that may be permitted to act simultaneously on the column section.

For practical design, the load contour shown in Fig. 11.27 may be approximated by two straight lines, AB and BC. The slope of line AB is $(1 - \beta)/\beta$, and the slope of line BC is $\beta/(1 - \beta)$. Therefore, when

$$\frac{M_{ny}}{M_{oy}} > \frac{M_{nx}}{M_{ox}}$$

then

$$\frac{M_{ny}}{M_{oy}} + \frac{M_{nx}}{M_{ox}}\left(\frac{1-\beta}{\beta}\right) = 1 \tag{11.34}$$

and when

$$\frac{M_{ny}}{M_{oy}} < \frac{M_{nx}}{M_{ox}}$$

then

$$\frac{M_{nx}}{M_{ox}} + \frac{M_{ny}}{M_{oy}}\left(\frac{1-\beta}{\beta}\right) = 1 \tag{11.35}$$

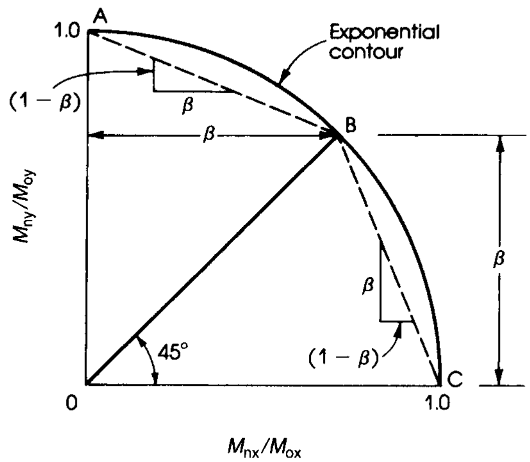


Figure 11.27 Nondimensional load contour at constant P_n (straight-line approximation).

The actual value of β depends on the ratio P_n/P_o as well as the material and properties of the cross-section. For lightly loaded columns, β will vary from 0.55 to 0.7. An average value of β = 0.65 can be used for design purposes.

When uniformly distributed reinforcement is adopted along all faces of rectangular columns, the ratio M_{oy}/M_{ox} is approximately b/h, where b and h are the width and total depth of the rectangular section, respectively. Substituting this ratio in Eqs. 11.34 and 11.35,

$$M_{ny} + M_{nx}\left(\frac{b}{h}\right)\left(\frac{1-\beta}{\beta}\right) \approx M_{oy} \tag{11.36}$$

and

$$M_{nx} + M_{ny}\left(\frac{h}{b}\right)\left(\frac{1-\beta}{\beta}\right) \approx M_{ox} \tag{11.37}$$

For $\beta = 0.65$ and $h/b = 1.5$,

$$M_{oy} \approx M_{ny} + 0.36M_{nx} \tag{11.38}$$

and

$$M_{ox} \approx M_{nx} + 0.80M_{ny} \tag{11.39}$$

From this presentation, it can be seen that direct explicit equations for the design of columns under axial load and biaxial bending are not available. Therefore, the designer should have enough experience to make an initial estimate of the section using the values of P_n, M_{nx} and M_{ny} and the uniaxial equations and then check the adequacy of the column section using the equations for biaxial bending or by computer.

Example 11.18

The section of a short tied column is 16 × 24 in. and is reinforced with eight no. 10 bars distributed as shown in Fig. 11.28. Determine the design load on the section ϕP_n if it acts at e_x = 8 in. and e_y = 12 in. Use f'_c = 5 ksi, f_y = 60 ksi, and the Bresler reciprocal equation.

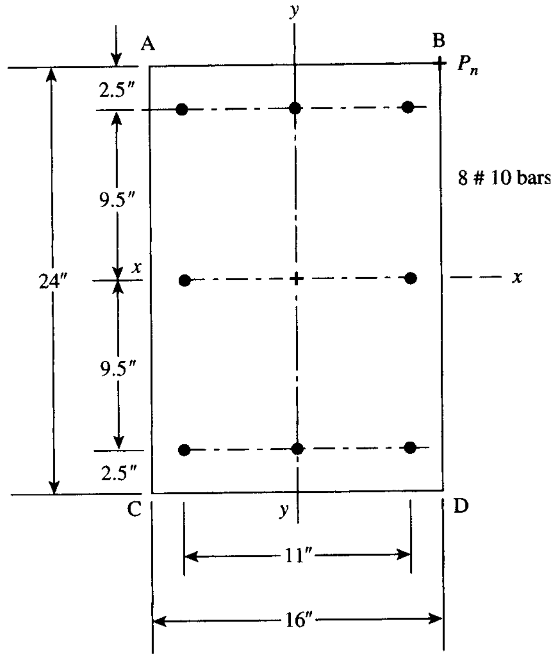


Figure 11.28 Example 11.18: biaxial load, Bresler method: $P_n = 421.5$ K.

Solution

1. Determine the uniaxial load capacity P_{nx} about the x-axis when $e_y = 12$ in. In this case, $b = 16$ in., $h = 24$ in., $d = 21.5$ in., $d' = 2.5$ in., and $A_s = A'_s = 3.81$ in.^{2} The solution will be performed using statics following the steps of Examples 11.2 and 11.4 for balanced and compression-control conditions.
   a. For the balanced condition,

$$c_b = \left(\frac{87}{87 + f_y}\right) \quad d = \left(\frac{87}{147}\right) 21.5 = 12.72 \text{ in.}$$

$$a_b = 0.80(12.72) = 10.18 \text{ in.} \qquad (\beta_1 = 0.8 \text{ when} f'_c = 5 \text{ ksi})$$

$$C_c = 0.85 f'_c ab = 692.3 \text{ K} \qquad f'_s = 87\left(\frac{c - d'}{c}\right) = 69.9 \text{ ksi}$$

Then $f'_s = 60$ ksi.

$$C_s = A'_s(f_y - 0.85 f'_c) = 212.4 \text{ K} \qquad T = A_s f_y = 228.6 \text{ K}$$

$$P_{ox} = C_c + C_s - T = 676.1 \text{ K}$$

$$\phi P_{bx} = 0.65 P_{bx} = 439.5 \text{ K} \qquad (\phi = 0.65 \text{ for } \varepsilon_t = 0.002)$$

   b. For $e_y = 12$ in. $< d = 21.5$ in., assume compression failure and follow the steps of Example 11.4 to get $a = 10.65$ in. and $c = a/0.8 = 13.31$ in. $> C_b = 12.72$ in. Thus, compression controls. Check

$$f'_s = 87\left(\frac{c - d'}{c}\right) = 70 \text{ ksi} > f_y$$

Therefore, $f'_s = 60$ ksi. Check

$$f_s = 87\left(\frac{d-c}{c}\right) = 53.53 \text{ ksi} < 60 \text{ ksi}$$

Calculate forces: $C_c = 0.85 f'_c ab = 724.2$ K, $C_s = A'_s(f_y - 0.85 f'_c) = 212.4$ K, $T = A_s f_s = 203.95$ K, $P_{\text{nx}} = C_c + C_s - T = 732.6$ K. $P_{\text{nx}} > P_{\text{bx}}$, so this is a compression failure case as assumed.

$$\varepsilon_t = \left(\frac{d-c}{c}\right) 0.003 = 0.00185$$

$$\varepsilon_t < 0.002 \quad \phi = 0.65$$

$$P_{\text{ux}} = \phi P_{\text{nx}} = 476.2 \text{ K}$$

c. Take moments about A_s using Eq. 11.11,

$$d'' = 9.5 \text{ in.} \qquad e' = 21.5 \text{ in.}$$

$$P_{\text{nx}} = \frac{1}{e'}\left[C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right] = 732.5 \text{ K}$$

2. Determine the uniaxial load capacity P_{ny} about the y-axis when $e_x = 8$ in. In this case, $b = 24$ in., $h = 16$ in., $d = 13.5$ in., $d' = 2.5$ in., and $A_s = A'_s = 3.81$ in.^{2} The solution will be performed using statics, as explained in step 1.

a. Balanced condition:

$$c_b = \left(\frac{87}{87 + f_y}\right) d = \left(\frac{87}{147}\right) 13.5 = 7.99 \text{ in.} \qquad a_b = 0.8(7.99) = 6.39 \text{ in.}$$

$$C_c = 0.85 f'_c ab = 651.8 \text{ K} \qquad f'_s = 87\left(\frac{c - d'}{c}\right) = 59.8 \text{ ksi}$$

$$C_s = A'_s(f'_s - 0.85 f'_c) = 211.6 \text{ K} \qquad T = A_s f_y = 228.6 \text{ K}$$

In a balanced load, $P_{\text{by}} = C_c + C_s - T = 634.8$ K, $\phi P_{\text{by}} = 0.65\ P_{\text{by}} = 444.4$ K.

b. For $e_x = 8$ in., assume compression failure case and follow the steps of Example 11.4 to get $a = 6.65$ in. and $c = a/0.8 = 8.31$ in. $> c_b$ (compression failure). Check

$$f'_s = 87\left(\frac{c - d'}{c}\right) = 60.8 \text{ ksi}$$

Therefore, $f'_s = 60$ ksi. Check

$$f_s = 87\left(\frac{d-c}{c}\right) = 54.3 \text{ ksi}$$

Calculate forces: $C_c = 0.85 f'_c ab = 678.3$ K, $C_s = A'_s(60 - 0.85 f'_c) = 212.4$ K, $T = A_s f_s = 206.9$ K, $P_{\text{ny}} = C_c + C_s - T = 683.3$ K, and $\phi P_{\text{ny}} = P_{\text{uy}} = 0.65\ P_{\text{ny}} = 444.5$ K. Because $P_{\text{ny}} > P_{\text{by}}$, compression failure occurs, as assumed.

$$\varepsilon_t = \left(\frac{d-c}{c}\right) 0.003 = 0.00187$$

$$\varepsilon_t < 0.002 \quad \phi = 0.65$$

$$P_{\text{uy}} = \phi P_{\text{ny}} = 444.5 \text{ K}$$

c. Take moments about A_s using Eq. 11.11:

$$d'' = 5.5 \text{ in.} \quad e' = 13.5 \text{ in.}$$

$$P_{\text{ny}} = \frac{1}{e'}\left[C_c\left(d - \frac{a}{2}\right) + C_s(d - d')\right] = 684 \text{ K}$$

3. Determine the theoretical axial load P_{n_0}:

$$P_{n_0} = 0.85 f_c' A_g + A_{\text{st}}(f_y - 0.85 f_c')$$

$$= 0.85(5)(16 \times 24) + 10.16(60 - 0.85 \times 5) = 2198.4 \text{ K} \quad \phi P_{n0} = 0.65 P_{n0} = 1429 \text{ K}$$

4. Using the Bresler equation (Eq. 11.42), multiply by 100:

$$\frac{100}{P_u} = \frac{100}{476.2} + \frac{100}{444.5} - \frac{100}{1429} = 0.365$$

$$P_u = 274 \text{ K} \quad \text{and} \quad P_n = \frac{P_u}{0.65} = 421.5 \text{ K}$$

Notes:

1. Approximate equations or the ACI charts may be used to calculate P_{nx} and P_{ny}. However, since the Bresler equation is an approximate solution, it is preferable to use accurate procedures, as was done in this example, to calculate P_{nx} and P_{ny}. Many approximations in the solution will produce inaccurate results. Computer programs based on statics are available and may be used with proper checking of the output.
2. In Example 11.18, the areas of the corner bars were used twice, once to calculate P_{nx} and once to calculate P_{ny}. The results obtained are consistent with similar solutions. A conservative solution is to use half of the corner bars in each direction, giving $A_s = A_s' = 2(1.27) = 2.54$ in.^{2}, which will reduce the values of P_{nx} and P_{ny}.

Example 11.19

Determine the nominal design load, P_n, for the column section of the previous example using the Parme load contour method; see Fig. 11.29.

Solution

1. Assume $\beta = 0.65$. The uniaxial load capacities in the direction of x- and y-axes were calculated in Example 11.18:

$$P_{\text{ux}} = 476.2 \text{ K} \quad P_{\text{uy}} = 444.5 \text{ K} \quad P_{\text{nx}} = 732.6 \text{ K} \quad P_{\text{ny}} = 683.8 \text{ K}$$

2. The moment capacity of the section about the x-axis is

$$M_{\text{ox}} = P_{\text{nx}} \cdot e_y = 732.6 \times 12$$

The moment capacity of the section about the y-axis is

$$M_{\text{oy}} = P_{\text{ny}} e_x = 683.8 \times 8 \text{ K·in.}$$

3. Let the nominal load capacity be P_n. The nominal design moment on the section about the x-axis is

$$M_{\text{nx}} = P_n e_y = P_n \times 12 \text{ K·in.}$$

and that about the y-axis is

$$M_{\text{ny}} = P_n e_x = 8 P_n$$

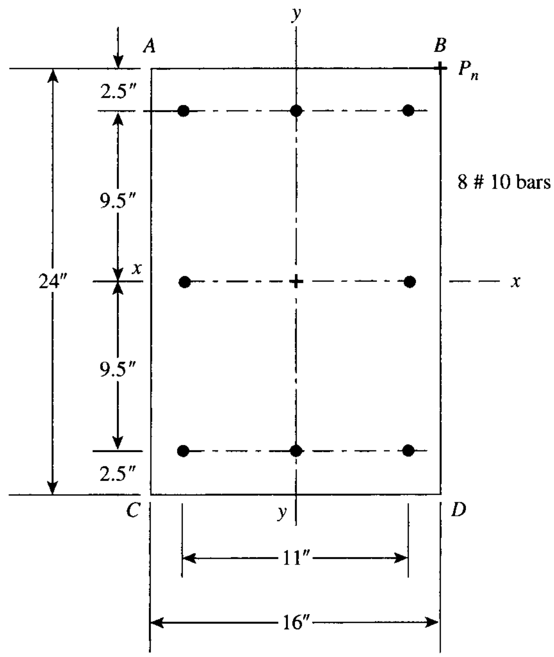


Figure 11.29 Example 11.19: biaxial load, PCA method: $P_n = 455$ K.

4. Check if $M_{ny}/M_{oy} > M_{nx}/M_{ox}$:

$$\frac{8P_n}{683.8 \times 8} > \frac{12P_n}{732.6 \times 12} \quad \text{or} \quad 1.463 \times 10^{-3}P_n > 1.365 \times 10^{-3}P_n$$

Then $M_{ny}/M_{oy} > M_{nx}/M_{ox}$. Therefore, use Eq. 11.48.

5. $$\frac{8P_n}{683.8 \times 8} + \frac{12P_n}{732.6 \times 12}\left(\frac{1-0.65}{0.65}\right) = 1$$

Multiply by 1000 to simplify calculations.

$$1.463P_n + 0.735P_n = 1000 \quad P_n = 455 \text{ K}$$

$$P_u = \phi P_n = 295.75 \text{ K} \quad (\phi = 0.65)$$

Note that P_u is greater than the value of 274 K obtained by the Bresler reciprocal method (Eq. 11.42) in the previous example by about 8%.

11.18 EQUATION OF FAILURE SURFACE

A general equation for the analysis and design of reinforced concrete short and tied rectangular columns was suggested by Hsu [16]. The equation is supposed to represent the failure surface and interaction diagrams of columns subjected to combined biaxial bending and axial load, as shown in Fig. 11.26. The axial load can be compressive or a tensile force. The equation is presented as follows:

$$\left(\frac{P_n - P_b}{P_o - P_b}\right) + \left(\frac{M_{\text{nx}}}{M_{\text{bx}}}\right)^{1.5} + \left(\frac{M_{\text{ny}}}{M_{\text{by}}}\right)^{1.5} = 1.0 \tag{11.40}$$

where

P_n = nominal axial strength (positive if compression and negative if tension) for a given eccentricity

P_o = nominal axial load (positive if compression and negative if tension) at zero eccentricity

P_b = nominal axial compressive load at balanced strain condition

$M_{\text{nx}}, M_{\text{ny}}$ = nominal bending moments about the x- and y-axes, respectively

$M_{\text{bx}}, M_{\text{by}}$ = nominal balanced bending moments about the x- and y-axes, respectively, at balanced strain conditions

To use Eq. 11.4, all terms must have a positive sign. The value of P_o was given earlier (Eq. 10.1):

$$P_o = 0.85 f_c'(A_g - A_{\text{st}}) + A_{\text{st}} \cdot f_y \tag{11.41}$$

The nominal balanced load, P_b, and the nominal balanced moment, $M_b = P_b e_b$, were given in Eq. 11.6 and 11.7, respectively, for sections with tension and compression reinforcement only. For other sections, these values can be obtained by using the principles of statics.

Note that the equation of failure surface can also be used for uniaxial bending representing the interaction diagram. In this case, the third term will be omitted when $e_x = 0$, and the second term will be omitted when $e_y = 0$.

When $e_x = 0$ (moment about the x-axis only),

$$\left(\frac{P_n - P_b}{P_o - P_b}\right) + \left(\frac{M_{\text{nx}}}{M_{\text{bx}}}\right)^{1.5} = 1.0 \tag{11.42}$$

(This is Eq. 11.18, given earlier.) When $e_y = 0$ (moment about the y-axis only),

$$\left(\frac{P_n - P_b}{P_o - P_b}\right) + \left(\frac{M_{\text{ny}}}{M_{\text{by}}}\right)^{1.5} = 1.0 \tag{11.43}$$

Applying Eq. 11.4 to Examples 11.2 and 11.4, $P_b = 453.4$ K, $M_{\text{bx}} = 6810.8$ K·in., $e_y = 10$ in., and $P_o = 0.85(4)(14 \times 22 - 8) + 8(60) = 1500$ K.

$$\frac{P_n - 453.4}{1500 - 453.4} + \left(\frac{10P_n}{6810.8}\right)^{1.5} = 1.0$$

Multiply by 1000 and solve for P_n:

$$(0.9555P_n - 433.2) + 0.05626P_n^{1.5} = 1000$$

$$0.9555P_n + 0.05626P_n^{1.5} = 1433.2$$

$P_n = 611$ K, which is close to that obtained by analysis.

Example 11.20

Determine the nominal design load, P_n, for the column section of Example 11.18 using the equation of failure surface.

Solution

1. Compute

$$\begin{aligned} P_o &= 0.85f'_c(A_g - A_{st}) + A_{st}f_y \\ &= 0.85(5)(16 \times 24 - 10.16) + (10.16 \times 60) \\ &= 2198.4\ \text{K} \end{aligned}$$

2. Compute P_b and M_b using Eqs. 11.6 and 11.8 about the x- and y-axes, respectively.

a. About the x-axis,

$$a_{\text{bx}} = \frac{87d_t}{87 + f_y} = \frac{87(21.5)}{87 + 60} = 12.72\ \text{in.}$$

$$a_{\text{bx}} = 0.8(12.72) = 10.18\ \text{in.}$$

$$f'_s = 87\left(\frac{c - d'}{c}\right) = 69.9\ \text{ksi} \quad f'_s = 60\ \text{ksi}$$

$$d''_x = 9.5\ \text{in} \quad A_s = A'_s = 3.81\ \text{in.}^2$$

$$\begin{aligned} P_{\text{bx}} &= 0.85f'_c a_x b + A'_s(f_y - 0.85f'_c) - A_s f_y \\ &= 0.85(5)(10.18)(16) + 3.81(60 - 0.85 \times 5) - 3.81(60) \\ &= 676.1\ \text{K} \end{aligned}$$

$$\begin{aligned} M_{\text{bx}} &= 0.85(5)(10.18)(16)\left(21.5 - \frac{10.18}{2} - 9.5\right) \\ &\quad + 3.81(60 - 0.85 \times 5) \times (21.5 - 2.5 - 9.5) + 3.81(60)(9.5) \\ &= 8973\ \text{K·in.} = 747.8\ \text{K·ft} \end{aligned}$$

b. About the y-axis: $d_t = 13.5$ in., $d''_y = 5.5$ in., $A_s = A'_s = 3.81$ in.^{2}

$$c_{\text{by}} = \frac{87(13.5)}{87 + 60} = 7.99\ \text{in.}$$

$$a_{\text{by}} = 0.8(7.99) = 6.39\ \text{in.} \quad f'_s = 87\left(\frac{c - d'}{c}\right) = 59.8\ \text{ksi}$$

$$\begin{aligned} P_{\text{by}} &= 0.85(5)(6.39)(24) + 3.81(59.8 - 0.85 \times 5) - 3.81(60) \\ &= 634.8\ \text{K} \end{aligned}$$

$$\begin{aligned} M_{\text{by}} &= 0.85(5)(6.39)(24)\left(13.5 - \frac{6.39}{2} 5.5\right) \\ &\quad + 3.81(59.8 - 0.85 \times 5)(13.5 - 2.5 - 5.5) + 3.81(60)(5.5) \\ &= 5557.3\ \text{K} \cdot \text{in.} = 463\ \text{K} \cdot \text{ft} \end{aligned}$$

3. Compute the nominal balanced load for biaxial bending, P_{bb}:

$$\tan\alpha = \frac{M_{ny}}{M_{nx}} = \frac{P_n \cdot e_x}{P_n \cdot e_y} = \frac{e_x}{e_y} = \frac{8}{12} \qquad \alpha = 33.7^\circ$$

$$\frac{P_{bx} - P_{by}}{90^\circ} = \frac{\Delta P_b}{90^\circ - \alpha^\circ} \quad \text{or} \quad \frac{676.1 - 634.8}{90} = \frac{\Delta P_b}{90 - 33.7}$$

$$\Delta P_b = 25.8 \text{ K}$$

$$P_{bb} = P_{by} + \Delta P_b = 634.8 + 25.8 = 660.6 \text{ K}$$

4. Compute P_n from the equation of failure surface:

$$\frac{P_n - 660.6}{2198.4 - 660.6} + \left(\frac{P_n \times 12}{8973}\right)^{1.5} + \left(\frac{P_n \times 8}{5557.3}\right)^{1.5} = 1.0$$

Multiply by 1000 and solve for P_n:

$$(0.65P_n - 429.85) + 0.0489P_n^{1.5} + 0.0546P_n^{1.5} = 1000$$

$$0.65P_n + 0.1035P_n^{1.5} = 1429.85$$

By trial, $P_n = 487$ K. Because $P_n < P_{bb}$, it is a tension failure case for biaxial bending, and thus $P_o = -2198.4$ K (to keep the first term positive).

$$1000\left(\frac{P_n - 660.9}{-2198.4 - 660.9}\right) + 0.0489P_n^{1.5} + 0.0546P_n^{1.5} = 1000$$

$$0.35P_n + 0.1035P_n^{1.5} = 769.1$$

$$P_n = 429 \text{ K} \quad \text{and} \quad P_u = 0.65P_n = 278.8 \text{ K}$$

Note: The strength capacity, ϕP_n, of the same rectangular section was calculated using the Bresler reciprocal equation (Example 11.18), Parme method (Example 11.19), and Hsu method (Example 11.20) to get $\phi P_n = 421.5$ K, 455 K, and 429 K, respectively. The Parme method gave the highest value for this example.

11.19 SI EXAMPLE

Example 11.21

Determine the balanced compressive forces P_b, e_b, and M_b for the section shown in Fig. 11.30. Use $f'_c = 30$ MPa and $f_y = 400$ MPA ($b = 350$ mm, $d = 490$ mm).

Solution

1. For a balanced condition, the strain in the concrete is 0.003 and the strain in the tension steel is $\varepsilon_y = f_y/E_s = 400/200{,}000 = 0.002$, where $E_s = 200{,}000$ MPa.

$$A_s = A'_s = 4(700) = 2800 \text{ mm}^2$$

2. Locate the neutral axis depth, c_b:

$$c_b = \left(\frac{600}{600 + f_y}\right) d_t \quad (\text{where } f_y \text{ is in MPa})$$

$$= \left(\frac{600}{600 + 420}\right)(490) = 288 \text{ mm}$$

$$a_b = 0.85c_b = 0.85 \times 288 = 245 \text{ mm}$$

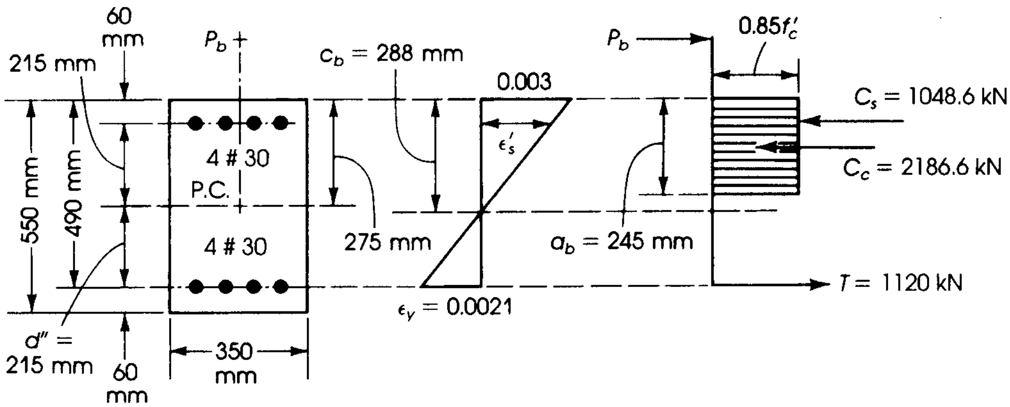


Figure 11.30 Example 11.21.

3. Check if compression steel yields. From the strain diagram,

$$\frac{\varepsilon_s'}{0.003} = \frac{c - d'}{c} = \frac{288 - 60}{288}$$
$$\varepsilon_s' = 0.00238 > \varepsilon_y$$

Therefore, compression steel yields.

4. Calculate the forces acting on the section:

$$C_c = 0.85 f_c' ab = \frac{0.85}{1000} \times 30 \times 245 \times 350 = 2186.6 \text{ kN}$$
$$T = A_s f_y = 2800 \times 0.400 \times 1120 \text{ kN}$$
$$C_s = A_s'(f_y - 0.85 f_c') = \frac{2800 \text{ mm}^2}{1000}(400 - 0.85 \times 30) = 1048.6 \text{ kN}$$

5. Calculate P_b and M_b:

$$P_b = C_c + C_s - T = 2115.2 \text{ kN}$$

From Eq. 11.10,

$$M_b = P_b e_b = C_c\left(d - \frac{a}{2} - d''\right) + C_s(d - d' - d'') + T d''$$

The plastic centroid is at the centroid of the section and $d'' = 215$ mm.

$$M_b = 2186.6\left(490 - \frac{245}{2} - 215\right) + 1048.6(490 - 60 - 215)$$
$$+ 1120 \times 215 - 799.7 \text{ kN} \cdot \text{m}$$
$$e_b = \frac{M_b}{P_b} = \frac{799.7}{2115.2} = 0.378 \text{ m} = 378 \text{ mm}$$

SUMMARY

Sections 11.1–11.3

1. The plastic centroid can be obtained by determining the location of the resultant force produced by the steel and the concrete, assuming both are stressed in compression to f_y and $0.85f'_c$, respectively.
2. On a load–moment interaction diagram the following cases of analysis are developed:
   - **a.** Axial compression, P_o
   - **b.** Maximum nominal axial load, $P_{n\ max} = 0.8\ P_o$ (for tied columns) and $P_{n\ max} = 0.85P_o$ (for spiral columns)
   - **c.** Compression failure occurs when $P_n > P_b$ or $e < e_b$
   - **d.** Balanced condition, P_b and M_b
   - **e.** Tension failure occurs when $P_n < P_b$ or $e > e_b$
   - **f.** Pure flexure

Section 11.4

1. For compression-controlled sections, $\phi = 0.65$, while for tension-controlled section, $\phi = 0.9$.
2. For the transition region,

$$\phi = 0.65 + (\varepsilon_t - 0.002)\left(\frac{250}{3}\right) \quad \text{(for tied columns)}$$

$$\phi = 0.75 + (\varepsilon_t - 0.002)(50) \quad \text{(for spiral columns)}$$

Section 11.5

For a balanced section,

$$c_b = \frac{87d_t}{87 + f_y} \quad \text{and} \quad a_b = \beta_1 c_b$$

$$\beta_1 = 0.85 \text{ for} f'_c \leq 4 \text{ ksi}$$

$$P_b = C_c + C_s - T = 0.85f'_c ab + A'_s(f_y - 0.85f'_c) - A_s f_y$$

$$M_b = P_b e_b = C_c\left(d - \frac{a}{2} - d''\right) + Td'' + C_s(d - d' - d'')$$

$$e_b = \frac{M_b}{P_b}$$

Section 11.6

The equations for the general analysis of rectangular sections under eccentric forces are summarized.

Sections 11.7–11.8

Examples for the cases when tension and compression controls are given.

Sections 11.9–11.10

Examples are given for the interaction diagram and for the case when side bars are used.

Section 11.11

This section gives the load capacity of circular columns. The cases of a balanced section when compression controls are explained by examples.

Section 11.12

This section gives examples of the analysis and design of columns using charts.

Section 11.13

This section gives examples of the design of column sections.

Sections 11.14–11.18

Biaxial bending:

1. For circular columns with uniform reinforcement,

$$M_u = \sqrt{(M_{ux})^2 + (M_{uy})^2} \qquad e = \sqrt{(e_x)^2 + (e_y)^2}$$

2. For square and rectangular sections,

$$\frac{1}{P_n} = \frac{1}{P_{nx}} + \frac{1}{P_{ny}} - \frac{1}{P_{n0}}$$

$$\frac{M_{nx}}{M_{ox}} + \frac{M_{ny}}{M_{oy}} \leq 1.0$$

3. In the Bresler load contour method,

$$\left(\frac{M_{nx}}{M_{ox}}\right)^{1.5} + \left(\frac{M_{ny}}{M_{oy}}\right)^{1.5} = 1.0$$

4. In the PCA load contour method,

$$M_{ny} + M_{nx}\left(\frac{b}{h}\right)\left(\frac{1-\beta}{\beta}\right) = M_{oy}$$

$$M_{nx} + M_{ny}\left(\frac{h}{b}\right)\left(\frac{1-\beta}{\beta}\right) = M_{ox}$$

5. Equations of failure surface method are given with applications.

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PROBLEMS

Note: For all problems, use $f_y = 60$ ksi, $d' = 2.5$ in., and $A_s = A'_s$ where applicable. Slight variation in answers are expected.

11.1 (Rectangular sections: balanced condition) For the rectangular column sections given in Table 11.3, determine the balanced compressive load, P_b, the balanced moment, M_b, and the balanced eccentricity, e_b, for each assigned problem. (Answers are given in Table 11.3.) ($\phi = 0.65$.)

11.2 (Rectangular sections: compression failure) For the rectangular column sections given in Table 11.3, determine the load capacity, P_n, for each assigned problem when the eccentricity is $e = 6$ in. (Answers are given in Table 11.3.)

11.3 (Rectangular sections: tension failure) For the rectangular column sections given in Table 11.3, determine the load capacity, P_n, for each assigned problem when the eccentricity is $e = 24$ in. (Answers are given in Table 11.3.)

11.4 (Rectangular sections with side bars) Determine the load capacity, ϕP_n, for the column section shown in Fig. 11.31 considering all side bars when the eccentricity is $e_y = 8$ in. Use $f'_c = 4$ ksi and $f_y = 60$ ksi. (Answer: 658 K.)

11.5 Repeat Problem 11.4 with Fig. 11.32. (Answer: 660 K.)

11.6 Repeat Problem 11.4 with Fig. 11.33. (Answer: 368 K.)

11.7 Repeat Problem 11.4 with Fig. 11.34. (Answer: 822 K.)

Table 11.3 Answers for Problems 11.1–11.3

					Answers to Problems			
					11.1		11.2	11.3
Number	f'_c(ksi)	b (in.)	h (in.)	$A_s = A'_s$	P_b	e_b	P_n (e = 6 in.)	P_n (e = 24 in.)
(a)	4	20	20	6 no. 10	572	17.4	1193	395
(b)	4	14	14	4 no. 8	249	10.9	407	93
(c)	4	24	24	8 no. 10	848	20.1	1860	696
(d)	4	18	26	6 no. 10	698	20.6	1528	591
(e)	4	12	18	4 no. 9	305	15.2	592	176
(f)	4	14	18	4 no. 10	354	16.2	715	221
(g)	5	16	16	5 no. 10	406	15.3	807	228
(h)	5	18	18	5 no. 9	540	12.5	930	230
(i)	5	14	20	4 no. 9	476	13.4	847	221
(j)	5	16	22	4 no. 10	606	14.8	1140	327
(k)	6	16	24	5 no. 10	746	16.8	1532	476
(l)	6	14	20	4 no. 9	534	12.8	944	226

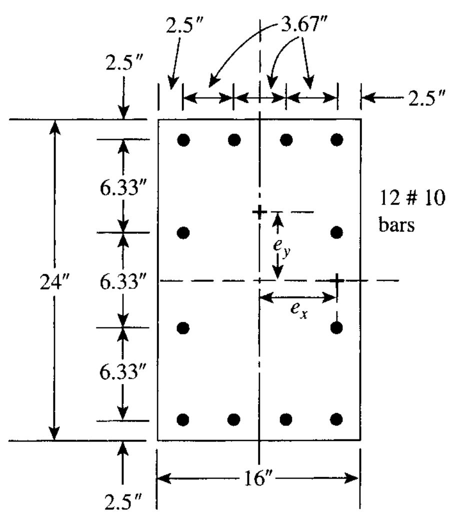


Figure 11.31 Problem 11.4.

11.8 (Design of rectangular column sections) For each assigned problem in Table 11.4, design a rectangular column section to support the factored load and moment shown. Determine A_s, A'_s, and h if not given; then choose adequate bars considering that $A_s = A'_s$. The final total steel ratio, ρ_g, should be close to the given values where applicable. Check the load capacity, ϕP_n, of the final section using statics and equilibrium equations. One solution for each problem is given in Table 11.4.

11.9 (ACI charts) Repeat Problems 11.2*b*, 11.2*d*, 11.2*f*, 11.8*a*, 11.8*c*, and 11.8*e* using the ACI charts.

11.10 (Circular columns: balanced condition) Determine the balanced load capacity, ϕP_b, the balanced moment, ϕM_b, and the balanced eccentricity, e_b, for the circular tied sections shown in Fig. 11.35. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

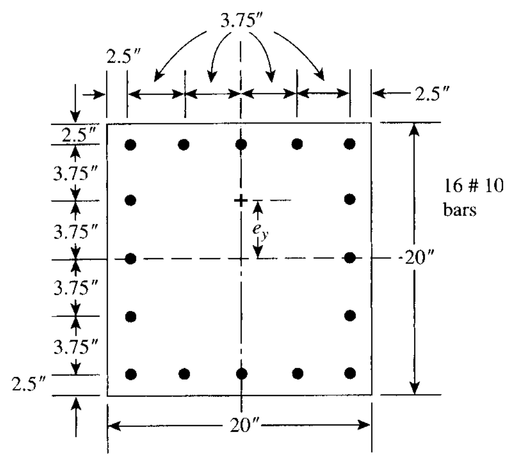


Figure 11.32 Problem 11.5.

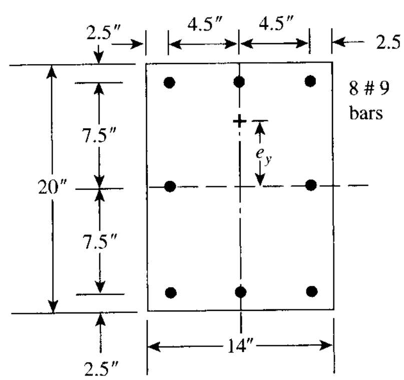


Figure 11.33 Problem 11.6.

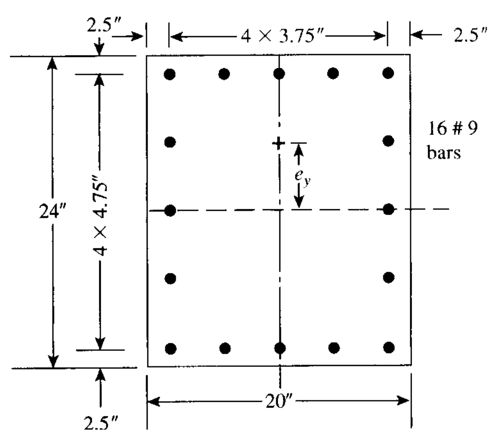


Figure 11.34 Problem 11.7.

Table 11.4 Problem 11.8

Number	f'_c(ksi)	P_u (K)	M_u (K-ft)	b (in.)	h (in.)	ρ_g %	One Solution h (in.)	$A_s = A'_s$
(a)	4	530	353	16	—	4.0	20	5 no. 10
(b)	4	410	205	14	18	—	18	5 no. 8
(c)	4	480	640	18	—	3.5	24	6 no. 10
(d)	4	440	440	20	20	—	20	6 no. 9
(e)	4	1125	375	20	24	—	24	6 no. 10
(f)	4	710	473	18	—	3.0	24	5 no. 10
(g)	5	300	300	14	—	2.0	20	3 no. 9
(h)	5	1000	665	20	26	—	26	6 no. 10
(i)	6	590	197	14	—	2.0	18	2 no. 10
(j)	6	664	332	16	20	—	20	4 no. 9

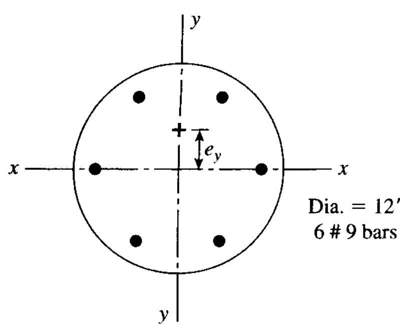


Figure 11.35 Problem 11.10.

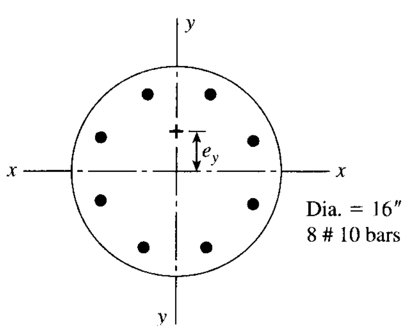


Figure 11.36 Problem 11.11.

11.11 Repeat Problem 11.10 for Fig. 11.36.

11.12 Repeat Problem 11.10 for Fig. 11.37.

11.13 Repeat Problem 11.11 for Fig. 11.38.

11.14 (Circular columns) Determine the load capacity, ϕP_n, for the circular tied column sections shown in Figs. 11.35 through 11.38 when the eccentricity is $e_y = 6$ in. Use $f'_c = 4$ ksi and $f_y = 60$ ksi.

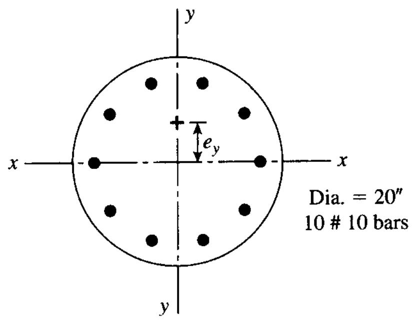


Figure 11.37 Problem 11.12.

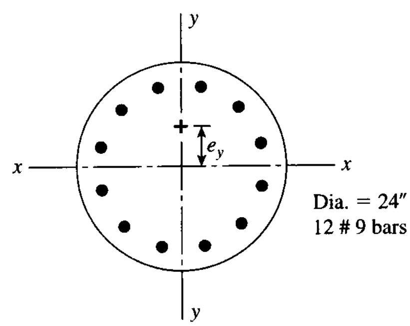


Figure 11.38 Problem 11.13.

11.15 (Biaxial bending) Determine the load capacity, P_n, for the column sections shown in Figs. 11.31 through 11.34 if $e_y = 8$ in. and $e_x = 6$ ini. using the Bresler reciprocal method. Use f_c'(4 ksi) and f_y = 60 ksi. For each problem the values of P_{nx}, P_{ny}, P_{n_0} (P_{bx}, M_{bx}), and (P_{by}, M_{by}) are as follows:

a. Figure 11.31: 952 K, 835 K, 2168 K (571 K, 792 K·ft), (536 K, 483 K·ft)

b. Figure 11.32: 930 K, 1108 K, 2505 K (577 K, 742 K·ft), (577 K, 742 K·ft)

c. Figure 11.33: 558 K, 495 K, 1408 K (408 K, 414 K·ft), (368 K, 260 K·ft)

d. Figure 11.34: 1093 K, 1145 K, 2538 K (718 K, 865 K·ft), (701 K, 699 K·ft)

11.16 Repeat Problem 11.15 using the Parme method.

11.17 Repeat Problem 11.15 using the Hsu method.

11.18 For the column sections shown in Fig. 11.31, determine

a. The uniaxial load capacities about the x- and y-axes, P_{nx} and P_{ny} using $e_y = 6$ in. and $e_x = 6$ in.

b. The uniaxial balanced load and moment capacities about the x- and y-axes, P_{bx}, P_{by}, M_{bx}, and M_{by}.

c. The axial load, P_{n_0}.

d. The biaxial load capacity P_n when $e_y = e_x = 6$ in., using the Bresler reciprocal method, the Hsu method, or both.

11.19 Repeat Problem 11.18 for Fig. 11.32.

11.20 Repeat Problem 11.18 for Fig. 11.33.

11.21 Repeat Problem 11.18 for Fig. 11.34.

CHAPTER 12

SLENDER COLUMNS

Columns in a high-rise building, Toronto, Canada.

12.1 INTRODUCTION

In the analysis and design of short columns discussed in the previous two chapters, it was assumed that buckling, elastic shortening, and secondary moment due to lateral deflection had minimal effect on the ultimate strength of the column; thus, these factors were not included in the design procedure. However, when the column is long, these factors must be considered. The extra length will cause a reduction in the column strength that varies with the column effective height, width of the section, the slenderness ratio, and the column end conditions.

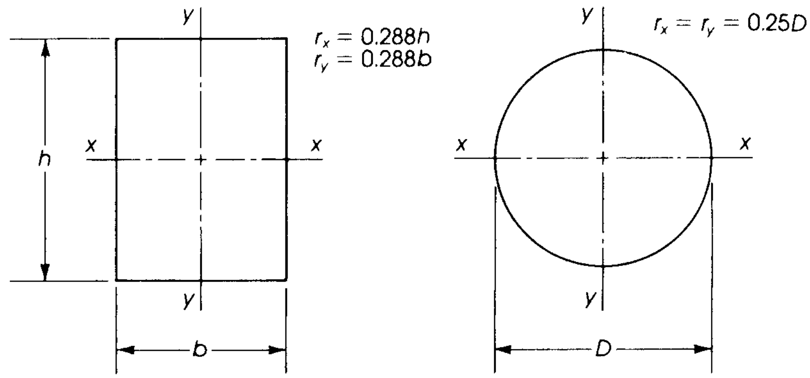


Figure 12.1 Rectangular and circular sections of columns, with radius of gyration r.

A column with a high slenderness ratio will have a considerable reduction in strength, whereas a low slenderness ratio means that the column is relatively short and the reduction in strength may not be significant. The slenderness ratio is the ratio of the column height, l, to the radius of gyration, r, where $r = I/A$, I being the moment of inertia of the section and A the sectional area.

For a rectangular section of width b and depth h (Fig. 12.1), $I_x = bh^3/12$ and $A = bh$. Therefore, $r_x = I/A = 0.288h$ (or, approximately, $r_x = 0.3h$). Similarly, $I_y = hb^3/12$ and $r_y = 0.288b$ (or, approximately, $0.3b$). For a circular column with diameter D, $I_x = I_y = \pi D^2/64$ and $A = \pi D^2/4$; therefore, $r_x = r_y = 0.25D$.

In general, columns may be considered as follows:

1. Long with a relatively high slenderness ratio, where lateral bracing or shear walls are required.
2. Long with a medium slenderness ratio that causes a reduction in the column strength. Lateral bracing may not be required, but strength reduction must be considered.
3. Short where the slenderness ratio is relatively small, causing a slight reduction in strength. This reduction may be neglected, as discussed in previous chapters.

12.2 EFFECTIVE COLUMN LENGTH (Kl_u)

The slenderness ratio l/r can be calculated accurately when the effective length of the column (Kl_u) is used. This effective length is a function of two main factors:

1. The unsupported length, l_u, represents the unsupported height of the column between two floors. It is measured as the clear distance between slabs, beams, or any structural member providing lateral support to the column. In a flat slab system with column capitals, the unsupported height of the column is measured from the top of the lower floor slab to the bottom of the column capital. If the column is supported with a deeper beam in one

direction than in the other direction, l_u should be calculated in both directions (about the x- and y-axes) of the column section. The critical (greater) value must be considered in the design.

2. The effective length factor, K, represents the ratio of the distance between points of zero moment in the column and the unsupported height of the column in one direction. For example, if the unsupported length of a column hinged at both ends, on which sidesway is prevented, is l_u, the points of zero moment will be at the top and bottom of the column—that is, at the two hinged ends. Therefore, the factor $K = l_u/l_u$ is 1.0. If a column is fixed at both ends and sidesway is prevented, the points of inflection (points of 0 moment) are at $l_u/4$ from each end. Therefore, $K = 0.5l_u/l_u = 0.5$ (Fig. 12.2). To evaluate the proper value of K, two main cases are considered.

   When structural frames are braced, the frame, which consists of beams and columns, is braced against sidesway by shear walls, rigid bracing, or lateral support from an adjoining structure. The ends of the columns will stay in position, and lateral translation of joints is prevented. The range of K in braced frames is always equal to or less than 1.0. The ACI Code, Section 10.10, recommends the use of $K = 1.0$ for braced frames.

   When the structural frames are unbraced, the frame is not supported against sidesway, and it depends on the stiffness of the beams and columns to prevent lateral deflection. Joint translations are not prevented, and the frame sways in the direction of lateral loads. The range of K for different columns and frames is given in Fig. 12.2, considering the two cases when sidesway is prevented or not prevented.

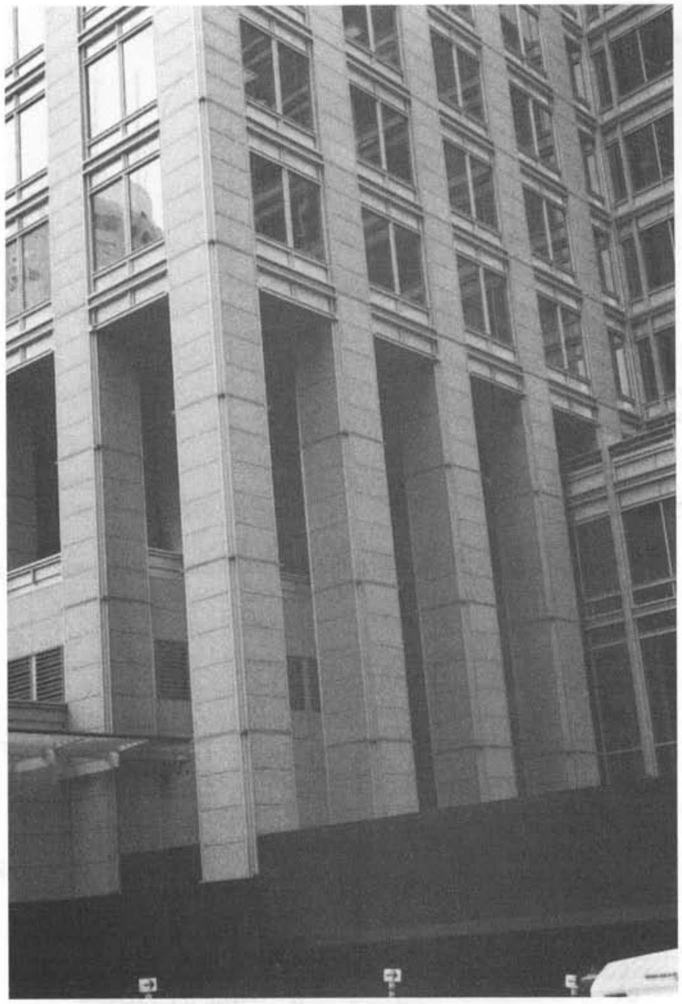
12.3 EFFECTIVE LENGTH FACTOR (*K*)

The effective length of columns can be estimated by using the alignment chart shown in Fig. 12.3 [10]. To find the effective length factor K, it is necessary first to calculate the end restraint factors Ψ_A and ψ_B at the top and bottom of the column, respectively, where

$$\psi = \frac{\Sigma EI/l_c \text{ of columns}}{\Sigma EI/l \text{ of beams}} \tag{12.1}$$

(both in the plane of bending) where l_c = length center to center of joints in a frame and l = span length, center to center of joints. The ψ factor at one end shall include all columns and beams meeting at the joint. For a hinged end, ψ is infinite and may be assumed to be 10.0. For a fixed end, ψ is zero and may be assumed to be 1.0. Those assumed values may be used because neither a perfect frictionless hinge nor perfectly fixed ends can exist in reinforced concrete frames.

The procedure for estimating K is to calculate ψ_A for the top end of the column and ψ_B for the bottom end of the column. Plot ψ_A and ψ_B on the alignment chart of Fig. 12.3 and connect the two points to intersect the middle line, which indicates the K-value. Two nomograms are shown, one for braced frames where sidesway is prevented, and the second for unbraced frames, where sidesway is not prevented. The development of the charts is based on the assumptions that (1) the structure consists of symmetrical rectangular frames, (2) the girder moment at a joint is distributed to columns according to their relative stiffnesses, and (3) all columns reach their critical loads at the same time.



Long columns in an office building.

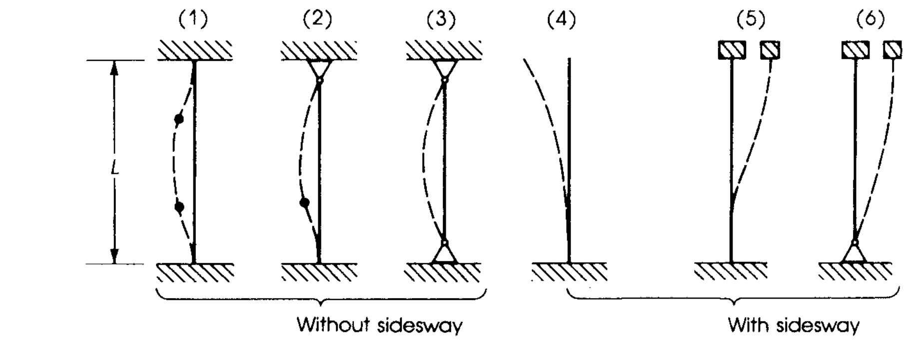
12.4 MEMBER STIFFNESS (*EI*)

The stiffness of a structural member is equal to the modulus of elasticity E times the moment of inertia I of the section. The values of E and I for reinforced concrete members can be estimated as follows:

1. The modulus of elasticity of concrete was discussed in Chapter 2; the ACI Code gives the following expression:

$$E_c = 33w^{1.5}\sqrt{f'_c} \quad \text{or} \quad E_c = 57{,}000\ \sqrt{f'_c}\ \text{(psi)}$$

for normal-weight concrete. The modulus of elasticity of steel is $E_s = 29 \times 10^6$ psi.



		(1)	(2)	(3)	(4)	(5)	(6)
Theoretical	K	0.5	0.7	1.0	2.0	1.0	2.0
Design	K	0.65	0.8	1.2	2.1	1.0	2.0

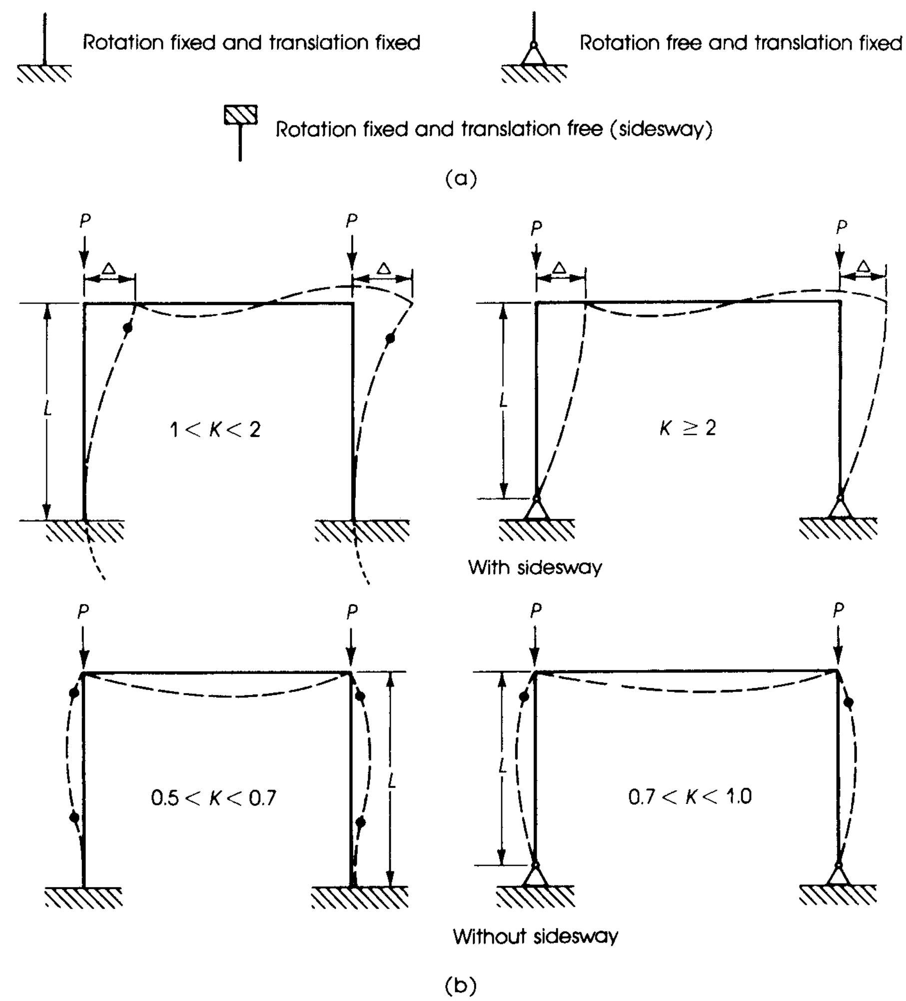


Figure 12.2 (*a*) Effective lengths of columns and length factor *K* and (*b*) effective lengths and *K* for portal columns.

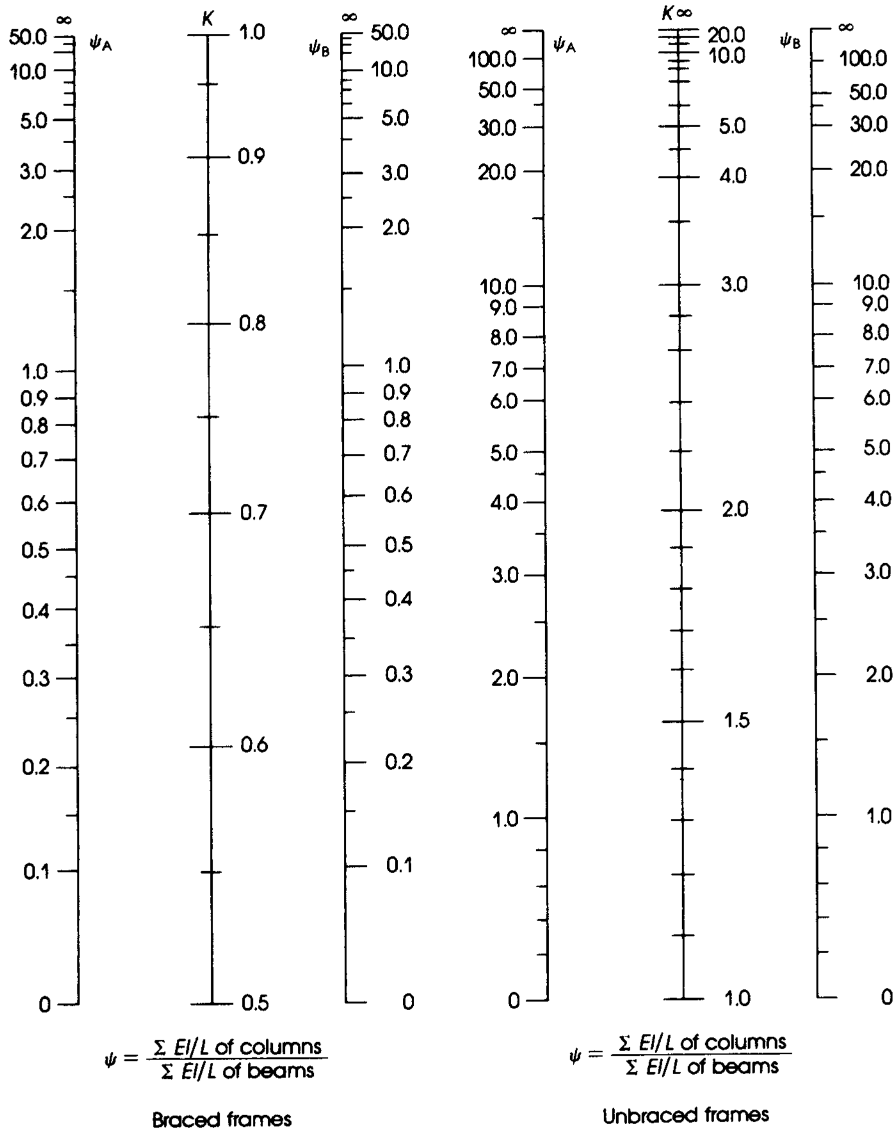


Figure 12.3 Alignment chart.

2. For reinforced concrete members, the moment of inertia I varies along the member, depending on the degree of cracking and the percentage of reinforcement in the section considered.

   To evaluate the factor ψ, EI must be calculated for beams and columns. For this purpose, I can be estimated as follows (ACI Code, Section 10.4.4.1):

   a. Compression members:

   Columns $I = 0.70 I_g$

   Walls—Uncracked $I = 0.70 I_g$

   —(Cracked) $I = 0.35 I_g$

b. Flexural members:

Beams $I = 0.35I_g$

Flat plates and flat slabs $I = 0.25I_g$

Alternatively, the moments of inertia of compression and flexural members, I shall be permitted to be computed as follows:

c. Compression members:

$$I = \left(0.80 + 25\frac{A_{st}}{A_g}\right)\left(1 - \frac{M_u}{P_u h} - 0.5\frac{P_u}{P_o}\right) I_g \leq 0.875I_g \qquad (12.2)$$

where P_u and M_u shall be determined from the particular load combination under consideration, or the combination of P_u and M_u determined in the smallest value of I, I need not be taken less than $0.35I_g$.

d. Flexural members:

$$I = (0.10 + 25\rho)\left(1.2 - 0.2\frac{b_w}{d}\right) I_g \leq 0.5I_g \qquad (12.3)$$

where I_g = the moment of inertia of the gross concrete section about the centroidal axis, neglecting reinforcement.

$$\rho = \text{ratio of } A_s/bd \text{ in cross section}$$

The moment of inertia of T-beams should be based on the effective flange width defined in Section 3.15.2. It is generally sufficiently accurate to take I_g of a T-beam as two times the I_g of the web, $2(b_w h^3/12)$.

If the factored moments and shears from an analysis based on the moment of inertia of a wall, taken equal to $0.70I_g$, indicate that the wall will crack in flexure, based on the modulus of rupture, the analysis should be repeated with $I = 0.35I_g$ in those stories where cracking is predicted using factored loads.

The values of the moments of inertia were dervied for nonprestressed members. For prestressed members, the moments of inertia may differ depending on the amount, location, and type of the reinforcement and the degree of cracking prior to ultimate. The stifness value for prestressed concrete members should include an allowance for the variabilty of the stifnesses.

For continuous flexural members, I shall be permitted to be taken as the average of values obtained from Eq. (12.3) for the critical positive and negative moment sections. I need not be taken less than $0.25I_g$.

The cross-sectional dimensions and reinforcement ratio used in the above formulas shall be within 10 percent of the dimensions and reinforcement ratio shown on the design drawings or the stiffness evaluation shall be repeated.

3. Area, $A = 1.0A_g$ (gross-sectional area).

4. The moments of inertia shall be divided by $(1 + \beta_{\text{dns}})$ when, sustained lateral loads act on the structure or for stability check, where

$$\beta_{\text{dns}} = \frac{\text{maximum factored axial sustained load}}{\text{maximum factored axial load}} = \frac{1.2D \text{ (sustained)}}{1.2D + 1.6L} \leq 1.0 \qquad (12.4)$$

12.5 LIMITATION OF THE SLENDERNESS RATIO (Kl_u/r)

12.5.1 Nonsway Frames

The ACI Code, Section 10.10.1 recommends the following limitations between short and long columns in braced (nonsway) frames:

1. The effect of slenderness may be neglected and the column may be designed as a short column when
$$\frac{Kl_u}{r} \leq 34 - \frac{12M_1}{M_2} \tag{12.5}$$
where M_1 and M_2 are the factored end moments of the column and M_2 is greater than M_1.
2. The ratio M_1/M_2 is considered positive if the member is bent in single curvature and negative for double curvature (Fig. 12.4).
3. The term $(34 - 12M_1/M_2)$ shall not be taken greater than 40.
4. If the factored column moments are zero or $e = M_u/P_u < e_{min}$, the value of M_2 should be calculated using the minimum eccentricity given by ACI Code Section 10.10.6.5:
$$e_{min} = (0.6 + 0.03h) \quad \text{(inch)} \tag{12.6}$$
$$M_2 = P_u(0.6 + 0.03h) \tag{12.7}$$
where M_2 is the minimum moment. The moment M_2 shall be considered about each axis of the column separately. The value of K may be assumed to be equal to 1.0 for a braced frame unless it is calculated on the basis of EI analysis.

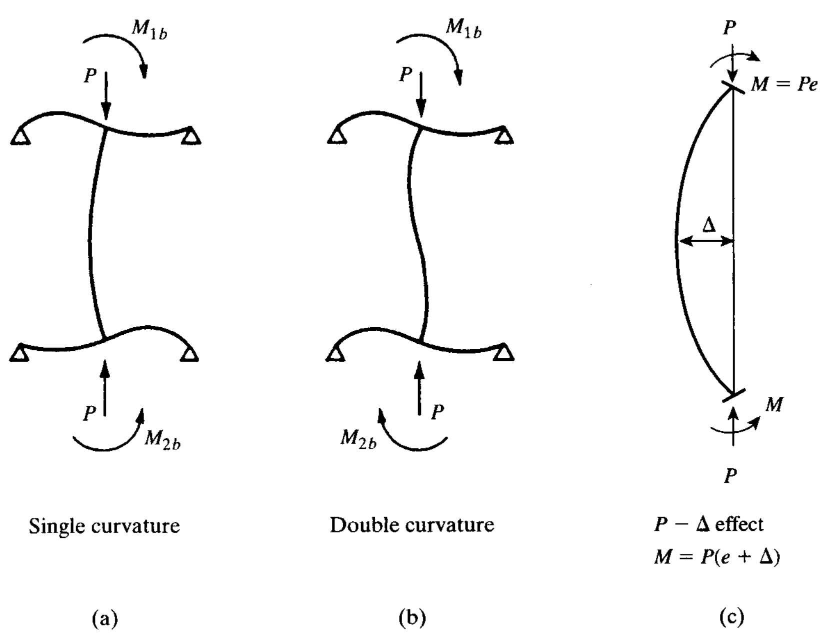


Figure 12.4 Single and double curvatures.

5. It shall be permitted to consider compression members braced against sidesway when bracing elements have a total stiffness, resisting lateral movement of that story, of at least 12 times the gross stiffness of the columns within the story.

12.5.2 Sway Frames

In compression members not braced (sway) against sidesway, the effect of the slenderness ratio may be neglected when

$$\frac{Kl_u}{r} < 22 \qquad \text{(ACI Code Section 10.10.1)} \tag{12.8}$$

12.6 MOMENT-MAGNIFIER DESIGN METHOD

12.6.1 Introduction

The first step in determining the design moments in a long column is to determine whether the frame is braced or unbraced against sidesway. If lateral bracing elements, such as shear walls and shear trusses, are provided or the columns have substantial lateral stiffness, then the lateral deflections produced are relatively small and their effect on the column strength is substantially low. It can be assumed that a story within a structure is nonsway if

$$Q = \frac{\Sigma P_u \Delta_0}{V_{us} l_c} \leq 0.05 \tag{12.9}$$

where ΣP_u and V_{us} are the story total vertical load and story shear, respectively, and Δ_0 is the first-order relative deflection between the top and bottom of the story due to V_{us}. The length l_c is that of the compression member in a frame, measured from center to center of the joints in the frame.

In general, compression members may be subjected to lateral deflections that cause secondary moments. If the secondary moment, M', is added to the applied moment on the column, M_a, the final moment is $M = M_a + M'$. An approximate method for estimating the final moment M is to multiply the applied moment M_a by a factor called the *magnifying moment factor* δ, which must be equal to or greater than 1.0, or $M_{max} = \delta M_a$ and $\delta \geq 1.0$. The moment M_a is obtained from the elastic structural analysis using factored loads, and it is the maximum moment that acts on the column at either end or within the column if transverse loadings are present.

If the P-Δ effect is taken into consideration, it becomes necessary to use a second-order analysis to account for the nonlinear relationship between the load, lateral displacement, and the moment. This is normally performed using computer programs. The ACI Code permits the use of first-order analysis of columns. The ACI Code *moment-magnifier design method* is a simplified approach for calculating the moment-magnifier factor in both braced and unbraced frames.

12.6.2 Magnified Moments in Nonsway Frames

The effect of slenderness ratio Kl_u/r in a compression member of a braced frame may be ignored if $Kl_u/r \leq 34 - 12M_1/M_2$, as given in Section 12.5.1. If Kl_u/r is greater than $(34 - 12M_1/M_2)$, then slenderness effect must be considered. The procedure for determining the magnification factor δ_{ns} in nonsway frames can be summarized as follows (ACI Code, Section 10.10.6):

1. Determine if the frame is braced against sidesway and find the unsupported length, l_u, and the effective length factor, K (K may be assumed to be 1.0).

2. Calculate the member stiffness, EI, using the reasonably approximate equation

$$EI = \frac{0.2E_c I_g + E_s I_{se}}{1 + \beta_{\text{dns}}} \tag{12.10}$$

or the more simplified approximate equation

$$EI = \frac{0.4E_c I_g}{1 + \beta_{\text{dns}}} \tag{12.11}$$

$$EI = 0.25E_c I_g \quad (\text{for } \beta_{\text{dns}} = 0.6) \tag{12.12}$$

where

$E_c = 57{,}000\sqrt{f'_c}$
$E_s = 29 \times 10^6$ psi
I_g = gross moment of inertia of the section about the axis considered, neglecting A_s
I_{se} = moment of inertia of the reinforcing steel
$\beta_{\text{dns}} = \dfrac{\text{maximum factored axial sustained load}}{\text{maximum factored axial load}} = \dfrac{1.2D\ (\text{sustained})}{1.2D + 1.6L}$

Note: The above β_{dns} is the ratio used to compute magnified moments in columns due to sustained loads.

Equations 12.11 and 12.12 are less accurate than Eq. 12.10. Moreover, Eq. 12.12 is obtained by assuming $\beta_d = 0.6$ in Eq. 12.11.

For improved accuracy EI can be approximated using suggested E and I values provided by:

$$I = \left(0.80 + 25\frac{A_{\text{st}}}{A_g}\right)\left(1 - \frac{M_u}{P_u h} - 0.5\frac{P_u}{p_o}\right) I_g \leq 0.875\, I_g$$

I need not be taken less than $0.35I_g$
where

A_{st} = Total area of longitudinal reinforcement (in.^{2})
P_o = nominal axial strenght at zero eccentricity (lb)
P_u = Factored axial force (+ve for compression) (lb)
M_u = Factored moment at section (lb.in.)
h = thikness of member (in.)

3. Determine the Euler buckling load, P_c:

$$P_c = \frac{\pi^2 EI}{(Kl_u)^2} \tag{12.13}$$

Use the values of EI, K, and l_u as calculated from steps 1 and 2.

4. Calculate the value of the factor C_m to be used in the equation of the moment-magnifier factor. For braced members without transverse loads,

$$C_m = 0.6 + \frac{0.4M_1}{M_2} \geq 0.4 \tag{12.14}$$

where M_1/M_2 is positive if the column is bent in single curvature. For members with transverse loads between supports, C_m shall be taken as 1.0.

5. Calculate the moment magnifier factor δ_{ns}:

$$\delta_{ns} = \frac{C_m}{1 - (P_u/0.75P_c)} \geq 1.0 \tag{12.15}$$

where P_u is the applied factored load and P_c and C_m are as calculated previously.

6. Design the compression member using the axial factored load, P_u, from the conventional frame analysis and a magnified moment, M_c, computed as follows:

$$M_c = \delta_{ns} M_2 \tag{12.16}$$

where M_2 is the larger factored end moment due to loads that result in no sidesway and should be $\geq P_u(0.6 + 0.03\,\text{h})$. For frames braced against sidesway, the sway factor is $\delta_s = 0$. In nonsway frames, the lateral deflection is expected to be less than or equal to $H/1500$, where H is the total height of the frame.

12.6.3 Magnified Moments in Sway Frames

The effect of slenderness may be ignored in sway (unbraced) frames when $Kl_u/r < 22$. The procedure for determining the magnification factor, δ_s, in sway (unbraced) frames may be summarized as follows (ACI Code, Section 10.10.7):

1. Determine if the frame is unbraced against sidesway and find the unsupported length l_u and K, which can be obtained from the alignment charts (Fig. 12.3).

2–4. Calculate EI, P_c, and C_m as given by Eqs. 12.2, 12.10 through 12.14. Note that β_{dns} (to calculate I) is the ratio of maximum factored sustained shear within a story to the total factored shear in that story.

5. Calculate the moment-magnifier factor, δ_s using one of the following methods:

a. Magnifier method

$$\delta_s = \frac{1}{1 - (\Sigma P_u/0.75\Sigma P_c)} \geq 1.0 \tag{12.17}$$

where $\delta_s \leq 2.5$ and ΣP_u is the summation for all the factored vertical loads in a story and ΣP_c is the summation for all sway-resisting columns in a story. Also,

$$\delta_s M_s = \frac{M_s}{1 - (\Sigma P_u/0.75\Sigma P_c)} \geq M_s \tag{12.18}$$

where M_s is the factored end moment due to loads causing appreciable sway.

b. Approximate second order analysis

$$\delta_s = \frac{1}{1 - Q} \geq 1 \quad \text{or} \quad \delta_s M_s = \frac{M_s}{1 - Q} \geq M_s \tag{12.19}$$

where

$$Q = \frac{\Sigma P_u \Delta_o}{V_{us} l_c} \tag{12.20}$$

where

P_u = Factored axial load (lb)

Δ_o = Revative lateral deflection between the top and bottom of a story due to lateral forces using a first order elastic frame analysis

V_{us} = Factored horizontal shear in a story (lb)
l_c = Length of compression member in a frame (m.)

If δ_s exceeds 1.5, δ_s shall be calculated using second order elastic analysis or the magnifier method described in a.

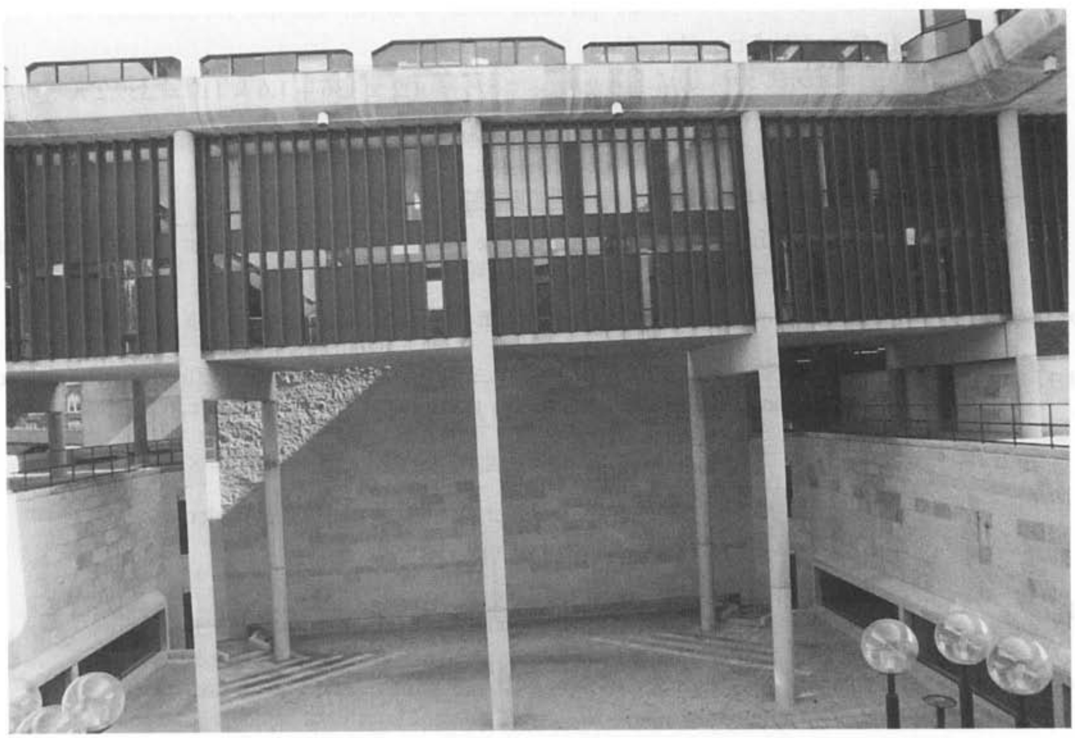
6. Calculate the magnified end moments M_1 and M_2 at the ends of an individual compression member, as follows:

$$M_1 = M_{1\,ns} + \delta_s M_{1s} \tag{12.21}$$

$$M_2 = M_{2\,ns} + \delta_s M_{2s} \tag{12.22}$$

where $M_{1\,ns}$ and $M_{2\,ns}$ are the moments obtained from the no-sway condition, whereas M_{1s} and M_{2s} are the moments obtained from the sway condition. If M_2 is greater than M_1 from structural analysis, then the design magnified moment is

$$M_c = M_{2\,ns} + \delta_s M_{2s} \tag{12.23}$$



Columns, University of Wisconsin, Madison, Wisconsin.

Example 12.1

The column section shown in Fig. 12.5 carries an axial load $P_D = 136$ K and a moment $M_D = 116$ K·ft due to dead load and an axial load $P_L = 110$ K and a moment $M_L = 93$ K·ft due to live load. The column is part of a frame that is braced against sidesway and bent in single curvature about its major axis. The unsupported length of the column is $l_c = 19$ ft, and the moments at both ends of the column are equal. Check the adequacy of the column using $f'_c = 4$ ksi and $f_y = 60$ ksi.

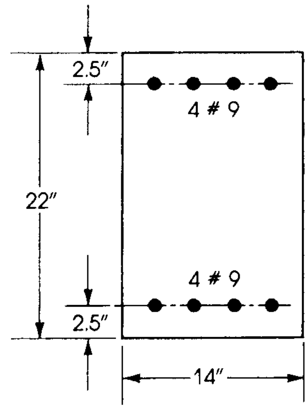


Figure 12.5 Example 12.1.

Solution

1. Calculate factored loads:

$$P_u = 1.2P_D + 1.6P_L = 1.2 \times 136 + 1.6 \times 110 = 339.2 \text{ K}$$

$$M_u = 1.2M_D + 1.6M_L = 1.2 \times 116 + 1.6 \times 93 = 288 \text{ K·ft}$$

$$e = \frac{M_u}{P_u} = \frac{288 \times 12}{339.2} = 10.2 \text{ in.}$$

2. Check if the column is long. Because the frame is braced against sidesway, assume $K = 1.0$, $r = 0.3h = 0.3 \times 22 = 6.6$ in., and $l_u = 19$ ft.

$$\frac{Kl_u}{r} = \frac{1 \times 19 \times 12}{6.6} = 34.5$$

For braced columns, if $Kl_u/r \leq 34 - 12M_1/M_2$, slenderness effect may be neglected. Given end moments $M_1 = M_2$ and M_1/M_2 positive for single curvature,

$$\text{Right-hand side} = 34 - 12\frac{M_1}{M_2} = 34 - 12 \times 1 = 22$$

Because $Kl_u/r = 34.5 > 22$, slenderness effect must be considered.

3. Calculate EI from Eq. 12.10:

a. Calculate E_c:

$$E_c = 57{,}000\sqrt{f'_c} = 57{,}000\sqrt{4000} = 3605 \text{ ksi}$$

$$E_s = 29{,}000 \text{ ksi}$$

b. The moment of inertia is

$$I_g = \frac{14(22)^3}{12} = 12{,}422 \text{ in.}^4 \quad A_s = A'_s = 4.0 \text{ in.}^2$$

$$I_{se} = 2 \times 4.0\left(\frac{22-5}{2}\right)^2 = 578 \text{ in.}^4$$

The dead-load moment ratio is

$$\beta_{dns} = \frac{1.2 \times 136}{339.2} = 0.48$$

c. The stiffness is

$$EI = \frac{0.2E_cI_g + E_sI_{se}}{1+\beta_{dns}}$$
$$= \frac{(0.2 \times 3605 \times 12{,}422) + (29{,}000 \times 578)}{1+0.48}$$
$$= 17.40 \times 10^6\ \text{K·in.}^2$$

4. Calculate P_c:

$$P_c = \frac{\pi^2 EI}{(Kl_u)^2} = \frac{\pi^2(17.40 \times 10^6)}{(12 \times 19)^2} = 3303\ \text{K}$$

5. Calculate C_m from Eq. 12.14:

$$C_m = 0.6 + \frac{0.4M_1}{M_2} \geq 0.4$$
$$= 0.6 + 0.4(1) = 1.0$$

6. Calculate the moment-magnifier factor from Eq. 12.15:

$$\delta_{ns} = \frac{C_m}{1-(P_u/0.75P_c)} = \frac{1}{1-339.2/(0.75 \times 3303)} = 1.16$$

7. Calculate the design moment and load: Assume ($\phi = 0.65$),

$$P_n = \frac{339.2}{0.65} = 522\ \text{K}$$
$$M_n = \frac{288}{0.65} = 443.1\ \text{K·ft}$$

Design $M_c = 443.1(1.16) = 514$ K·ft. Design eccentricity = 514/522 = 0.98 ft = 11.82 in., or 12 in.

8. Determine the nominal load strength of the section using $e = 12$ in. according to Example 11.4:

$$P_n = 47.6a + 226.4 - 4f_s \qquad \text{(I)}$$
$$e' = e + d - \frac{h}{2} = 12 + 19.5 - \frac{22}{2} = 20.50\ \text{in.}$$
$$P_n = \frac{1}{20.50}\left[47.6a\left(19.5 - \frac{a}{2}\right) + 226.4(19.5 - 2.5)\right]$$
$$= 45a - 1.15a^2 + 186.6 \qquad \text{(II)}$$

Solving for a from Eqs. I and II, $a = 10.6$ in. and $P_n = 535$ K. The load strength, P_n, is greater than the required load of 522 K; therefore, the section is adequate. If the section is not adequate, increase steel reinforcement.

9. Check the assumed ϕ:

$$a = 10.6\ \text{in.} \quad c = 12.47\ \text{in.} \quad d_t = 19.5\ \text{in.}$$
$$\varepsilon_t = \left(\frac{d_t - c}{c}\right) 0.003$$
$$= 0.00169 < 0.002$$
$$\phi = 0.65$$

Example 12.2

Check the adequacy of the column in Example 12.1 if the unsupported length is $l_u = 10$ ft. Determine the maximum nominal load on the column.

Solution

1. Applied loads are $P_n = 522$ K and $M_n = 443.1$ K.
2. Check if the column is long: $l_u = 10$ ft, $r = 0.3h = 0.3 \times 22 = 6.6$ in., and $K = 1.0$ (frame is braced against sidesway).

$$\frac{Kl_u}{r} = \frac{1 \times (10 \times 12)}{6.6} = 18.2$$

Check if $Kl_u/r = 34 - 12M_{1b}/M_{2b}$:

$$\text{Right-hand side} = 34 - 12 \times 1 = 22$$

$$\frac{Kl_u}{r} = 18.2 < 22$$

Therefore, the slenderness effect can be neglected.

3. Determine the nominal load capacity of the short column, as explained in Example 11.4. From Example 11.4, the nominal compressive strength is $P_n = 612.1$ K (for $e = 10$ in.), which is greater than the required load of 522 K, because the column is short with $e = 10.2$ in. (Example 12.1).

Example 12.3

Check the adequacy of the column in Example 12.1 if the frame is unbraced (sway) against sidesway, the end-restraint factors are $\psi_A = 0.8$ and $\psi_B = 2.0$, and the unsupported length is $l_u = 16$ ft.

Solution

1. Determine the value of K from the alignment chart (Fig. 12.3) for unbraced frames. Connect the values of $\psi_A = 0.8$ and $\psi_B = 2.0$, to intersect the K-line at $K = 1.4$.

$$\frac{Kl_u}{r} = \frac{1.4 \times (16 \times 12)}{6.6} = 40.7$$

2. For unbraced frames, if $Kl_u/r \leq 22$, the column can be designed as a short column. Because actual $Kl_ur = 40.7 > 22$, the slenderness effect must be considered.
3. Calculate the moment magnifier δ_{ns}, given $C_m = 1.0$, $K = 1.4$, $EI = 17.40 \times 10^6$ K·in.^{2} (from Example 12.1), and

$$P_c = \frac{\pi^2 EI}{(Kl_u)^2} = \frac{\pi^2 \times 17.40 \times 10^6}{(1.4 \times 16 \times 12)^2} = 2377 \text{ K}$$

$$\delta_{ns} = \frac{C_m}{1 - \left(\dfrac{P_u}{0.75 \times P_c}\right)} = \frac{1.0}{1 - \left(\dfrac{339.2}{0.75 \times 2377}\right)} = 1.24$$

4. From Example 12.1, the applied loads are $P_u = 339.2$ K and $M_u = 288$ K·ft, or

$$P_n = 522 \text{ K} \quad \text{and} \quad M_n = 443.1 \text{ K·ft}$$

The design moment $M_c = 1.24(443.1) = 549.4$ K·ft; hence,

$$e = \frac{\delta_{ns} M_n}{P_n} = 549.4 \times \frac{12}{522} = 12.63 \text{ in.} \qquad \text{say, 13 in.}$$

5. The requirement now is to check the adequacy of a short column for $P_n = 522\,\text{K}$, $M_c = 549.4\,\text{K·ft}$, and $e = 13\,\text{in}$. The procedure is explained in Example 11.4.
6. From Example 11.4,

$$P_n = 47.6a + 226.4 - 4f_s$$

$$e' = e + d - \frac{h}{2} = 13 + 19.5 - \frac{22}{2} = 21.5\ \text{in.}$$

$$P_n = \frac{1}{21.5}\left[47.6a\left(19.5 - \frac{a}{2}\right) + 226.4(19.5 - 2.5)\right]$$

$$= 43.16a - 1.1a^2 + 179 \quad a = 10.4\ \text{in.}$$

Thus, $c = 12.24\,\text{in.}$ and $P_n = 508\,\text{K}$. This load capacity of the column is less than the required P_n of 522 K. Therefore, the section is not adequate.

7. Increase steel reinforcement to four no. 10 bars on each side and repeat the calculations to get $P_n = 568\,\text{K}$, $\varepsilon_t < 0.002$, and $\phi = 0.65$.

Example 12.4

Design an interior square column for the first story of an eight-story office building. The clear height of the first floor is 16 ft, and the height of all other floors is 11 ft. The building layout is in 24 bays (Fig. 12.6), and the columns are not braced against sidesway. The loads acting on a first-floor interior column due to gravity and wind are as follows:

$$\text{Axial dead load} = 380\ \text{K}$$

$$\text{Axial live load} = 140\ \text{K}$$

$$\text{Axial wind load} = 0\ \text{K}$$

$$\text{Dead-load moments} = 32\ \text{K·ft (top) and } 54\ \text{K·ft (bottom)}$$

$$\text{Live-load moments} = 20\ \text{K·ft (top) and } 36\ \text{K·ft (bottom)}$$

$$\text{Wind-load moments} = 50\ \text{K·ft (top) and } 50\ \text{K·ft (bottom)}$$

$$EI/l \text{ for beams} = 360 \times 10^3\ \text{K·in.}$$

Use $f_c' = 5$ ksi, $f_y = 60$ ksi, and the ACI Code requirements. Assume an exterior column load of two-thirds the interior column load, a corner column load of one-third the interior column load, and $\beta_{\text{dns}} = 0.55$.

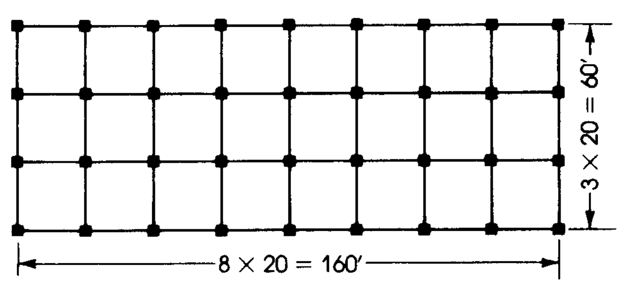


Figure 12.6 Example 12.4.

Solution

1. Calculate the factored forces using load combinations. For gravity loads,

$$P_u = 1.2D + 1.6L = 1.2(380) + 1.6(140) = 680 \text{ K}$$
$$M_u = M_{2\,\text{ns}} = 1.2M_D + 1.6M_L = 1.2(54) + 1.6(36) = 122.4 \text{ K·ft}$$

For gravity plus wind load,

$$P_u = (1.2D + 0.5L + 1.6W)$$
$$= [1.2(380) + 0.5(140 + 0)] = 526 \text{ K}$$
$$M_{\text{uns}} = M_{2\,\text{ns}} = (1.2 \times 54 + 1.6 \times 36) = 122.4 \text{ K·ft}$$
$$M_{\text{us}} = M_{2s} = (1.6M_w) = (1.6 \times 50) = 80 \text{ K·ft}$$

Other combinations are not critical:

$$P_u = 0.9D + 1.6W = 0.9(380) + 1.6(0) = 342 \text{ K}$$
$$M_2 = M_{\text{uns}} = 0.9M_D = 0.9(54) = 48.6 \text{ K·ft}$$
$$M_{2s} = 1.6M_w = 1.6(50) = 80 \text{ K·ft}$$
$$e = \frac{M_u}{P_u} = \frac{M_{2\,\text{ns}}}{P_u} = 122.4 \times \frac{12}{680} = 2.16 \text{ in.}$$
$$\min e = 0.6 + 0.03(18) = 1.14 \text{ in.} < 2.16$$

2. Select a preliminary section of column based on gravity load combination using tables or charts. Select a section 18 by 18 in. reinforced by four no. 10 bars (Fig. 12.7).

3. Check Kl_u/r:

$$I_g = \frac{(18)^4}{12} = 8748 \text{ in.}^4 \quad E_c = 4.03 \times 10^6 \text{ psi}$$

for columns, $I = 0.7\ I_g$.
For a 16-ft column,

$$\frac{EI}{l_c} = \frac{(0.7)(8748)(4.03 \times 10^6)}{16 \times 12} = 128.5 \times 10^6$$

For an 11-ft column,

$$\frac{EI}{l_c} = \frac{(0.7)(8748)(4.03 \times 10^6)}{11 \times 12} = 187 \times 10^6$$

For beams, and $EI_g/l_b = 360 \times 10^6$, $I = 0.35\ I_g$, and $EI/l_b = 0.35EI_g/l_b = 126 \times 10^6$

$$\psi(\text{top}) = \psi(\text{bottom}) = \frac{\Sigma(EI/l_c)}{\Sigma(EI/l_b)} = \frac{(128.5 + 187)}{2(126)} = 1.25$$

From the chart (Fig. 12.3), K is 1.37 for an unbraced frame and 0.8 for a braced frame.

$$\frac{Kl_u}{r} = \frac{1.37(16 \times 12)}{0.3 \times 18} = 48.7$$

which is more than 22. Therefore, the slenderness ratio must be considered.

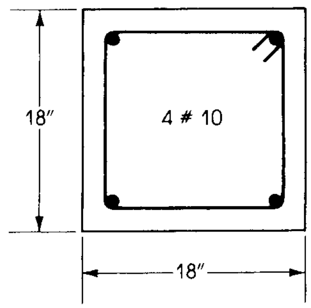


Figure 12.7 Column cross section, Example 12.4.

4. Compute P_c:

$$E_c = 4.03 \times 10^3 \text{ ksi} \quad E_s = 29 \times 10^3 \text{ ksi}$$

$$I_g = 8748 \text{ in.}^4 \quad I_{se} = 5.06\left(\frac{13}{2}\right)^2 = 214 \text{ in.}^4$$

$$\beta_{dns} = 0.55$$

$$EI = \frac{0.2E_c I_g + E_s I_{se}}{1 + \beta_{dns}}$$

$$EI = \frac{0.2(4.03 \times 10^3 \times 8748) + 29 \times 10^3(214)}{1 + 0.55} = 8.55 \times 10^6 \text{ K·in.}^2$$

For calculation of δ_s, $\beta_{dns} = 0$ and $E = 8.55 \times 10^6(1.55) = 13.25 \times 10^6$ K·in.^{2}

$$P_c = \frac{\pi^2 EI}{(Kl_u)^2} = \frac{\pi^2(8.55 \times 10^6)}{(0.8 \times 16 \times 12)^2} = 3577 \text{ K} \quad \text{(braced)}$$

$$P_c = \frac{\pi^2(13.25 \times 10^6)}{(1.37 \times 16 \times 12)^2} = 1890 \text{ K} \quad \text{(unbraced)}$$

For one floor in the building, there are 14 interior columns, 18 exterior columns, and four corner columns.

$$\Sigma P_u = 14(526) + 18\left(\frac{2}{3} \times 526\right) + 4\left(\frac{1}{3} \times 526\right) = 14{,}377 \text{ K}$$

$$\Sigma P_c = 14(1890) + 22\left(\frac{2}{3} \times 1890\right) = 54{,}180 \text{ K}$$

$$\delta_s = \frac{1.0}{1 - \left(\dfrac{14{,}377}{0.75 \times 54{,}180}\right)} = 1.54$$

which is greater than 1.0 (Eq. 12.17).

$$M_c = M_{2ns} + \delta_s M_{2s} = (122.4) + 1.54(80) = 245.6 \text{ K·ft}$$

5. Design loads are $P_u = 526$ K and $M_c = 245.6$ K·ft.

$$e = \frac{245.6(12)}{526} = 5.6 \text{ in.}$$

$$e_{\min} = 0.6 + 0.03(18) = 1.14 \text{ in.} < e$$

By analysis, for $e = 5.6$ in. and $A_s = A'_s = 2.53$ in.^{2}, ($\phi = 0.65$ in.) the load capacity of the 18 × 18-in. column is $\phi P_n = 556$ K and $\phi M_n = 259$ K-ft, so the section is adequate. (Solution steps are similar to Example 11.4. Values are $a = 10.37$ in., $c = 13$ in., $f_s = 17$ ksi, $f'_s = 60$ ksi, $\phi P_b = 385$ K, and $e_b = 8.9$ in.).

$$\varepsilon_t = 0.003\frac{(15.5 - 13)}{13} = 0.00058 < 0.002, \quad \phi = 0.65.$$

SUMMARY

Sections 12.1–12.3

1. The radius of gyration is $r = \sqrt{I/A}$, where $r = 0.3h$ for rectangular sections and $0.25D$ for circular sections.
2. The effective column length is Kl_u. For braced frames, $K = 1.0$; for unbraced frames, K varies as shown in Fig. 12.2.
3. K can be determined from the alignment chart (Fig. 12.3) or Eqs. 12.2 through 12.6.

Section 12.4

Member stiffness is EI:

$$E_c = 33w^{1.5}\sqrt{f'_c}$$

The moment of inertia, I, may be taken as $I = 0.35I_g$ for beams, $0.70I_g$ for columns, $0.70I_g$ for uncracked walls, $0.35I_g$ for cracked walls, and $0.25I_g$ for plates and flat slabs.

Alternatively, the moments of inertia of compression and flexural members, I, shall be permitted to be computed as follows:

1. Compression members:

$$I = \left(0.80 + 25\frac{A_{st}}{A_g}\right)\left(1 - \frac{M_u}{P_u h} - 0.5\frac{P_u}{P_o}\right)I_g \le 0.875I_g \qquad (12.2)$$

2. Flexural members:

$$I = (0.10 + 25\rho)\left(1.2 - 0.2\frac{b_w}{d}\right)I_g \le 0.5I_g \qquad (12.3)$$

Section 12.5

The effect of slenderness may be neglected when

$$\frac{Kl_u}{r} \le 22 \quad \text{(for unbraced frames)} \qquad (12.8)$$

$$\frac{Kl_u}{r} \le 34 - \frac{12M_1}{M_2} \quad \text{(for braced columns)} \qquad (12.5)$$

where M_1 and M_2 are the end moments and $M_2 > M_1$.

Section 12.6

1. For nonsway frames,

$$EI = \frac{0.2E_c I_g + E_s I_{se}}{1 + \beta_{dns}} \tag{12.10}$$

or the more simplified equation

$$EI = \frac{0.4E_c I_g}{1 + \beta_{dns}} \tag{12.11}$$

$$\beta_{dns} = \frac{1.2D}{1.2D + 1.6L} \tag{12.4}$$

More simply,

$$EI = 0.25E_c I_g \ (\beta_{dns} = 0.6) \tag{12.12}$$

The Euler buckling load is

$$P_c = \frac{\pi^2 EI}{(Kl_u)^2} \tag{12.13}$$

$$C_m = 0.6 + \frac{0.4M_1}{M_2} \geq 0.4 \tag{12.14}$$

The moment-magnifier factor (nonsway frames) is

$$\delta_{ns} = \frac{C_m}{1 - \frac{P_u}{0.75P_c}} \tag{12.15}$$

The design moment is

$$M_c = \delta_{ns} M_2 \tag{12.16}$$

2. For sway (unbraced) frames, the moment-magnifier factor is calculated either from
   a. Magnifier method

$$\delta_s = \frac{1.0}{1 - \frac{\Sigma P_u}{0.75 \Sigma P_c}} \geq 1.0 \tag{12.17}$$

   b. Approximate second order analysis

$$\delta_s = \frac{1}{1 - Q} \tag{12.19}$$

$$Q = \frac{\Sigma P_u \Delta_0}{V_{us} l_c} \tag{12.20}$$

the design moment is

$$M_1 = M_{1\,ns} + \delta_s \ M_{1s} \tag{12.21}$$

$$M_2 = M_{2\,ns} + \delta_s \ M_{2s} \tag{12.22}$$

If $M_2 > M_1$ then:

$$M_c = M_{2\,ns} + \delta_s \ M_{2s} \tag{12.23}$$

where $M_{2\,ns}$ is the unmagnified moment due to gravity loads (nonsway moment) and $\delta_s M_{2s}$ is the magnified moment due to sway frame loads.

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PROBLEMS

12.1 The column section in Fig. 12.8 carries an axial load $P_D = 128\,\text{K}$ and a moment $M_D = 117\,\text{K·ft}$ due to dead load and an axial load $P_L = 95\,\text{K}$ and a moment $M_1 = 100\,\text{K·ft}$ due to live load. The column is part of a frame, braced against sidesway, and bent in single curvature about its major axis. The unsupported length of the column is $l_u = 18\,\text{ft}$, and the moments at both ends are equal. Check the adequacy of the section using $f_c' = 4$ ksi and $f_y = 60\,\text{ksi}$.

12.2 Repeat Problem 12.1 if $l_u = 12\,\text{ft}$.

12.3 Repeat Problem 12.1 if the frame is unbraced against sidesway and the end-restraint factors are ψ (top) $= 0.7$ and ψ (bottom) $= 1.8$ and the unsupported height is $l_u = 14\,\text{ft}$.

12.4 The column section shown in Fig. 12.9 is part of a frame unbraced against sidesway and supports an axial load $P_D = 166\,\text{K}$ and a moment $M_D = 107\,\text{K·ft}$ due to dead load and $P_L = 115\,\text{K}$ and $M_L = 80\,\text{K·ft}$ due to live load. The column is bent in single curvature and has an unsupported length $l_u = 16\,\text{ft}$. The moment at the top of the column is $M_2 = 1.5M_1$, the moment at the bottom of the column. Check if the section is adequate using $f_c' = 5$ ksi, $f_y = 60\,\text{ksi}$, ψ (top) $= 2.0$, and ψ (bottom) $= 1.0$.

12.5 Repeat Problem 12.4 if the column length is $l_u = 14\,\text{ft}$.

12.6 Repeat Problem 12.4 if the frame is braced against sidesway and $M_1 = M_2$.

12.7 Repeat Problem 12.4 using $f_c' = 4$ ksi and $f_y = 60\,\text{ksi}$.

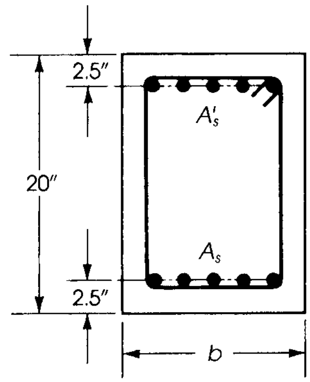


Figure 12.8 Problem 12.1 ($A_s = A'_s$) = 5 no. 9 bars and b = 14 in.

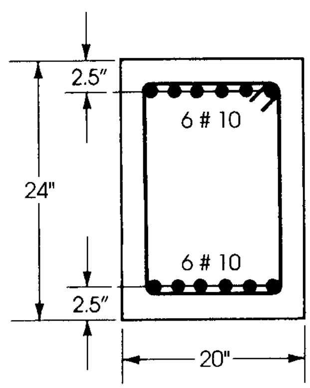


Figure 12.9 Problem 12.4.

12.8 Design a 20-ft-long rectangular tied column for an axial load $P_D = 214.5$ K and a moment $M_D = 64$ K·ft due to dead load and an axial load $P_L = 120$ K and a moment $M_L = 40$ K·ft due to live load. The column is bent in single curvature about its major axis, braced against sidesway, and the end moments are equal. The end-restraint factors are ψ (top) = 2.5 and ψ (bottom) = 1.4. Use $f'_c = 5$ ksi, $f_y = 60$ ksi, and $b = 15$ in.

12.9 Design the column in Problem 12.8 if the column length is 10 ft.

12.10 Repeat Problem 12.8 if the column is unbraced against sidesway.

CHAPTER 13

FOOTINGS

Office building under construction, New Orleans, Louisiana.

13.1 INTRODUCTION

Reinforced concrete footings are structural members used to support columns and walls and to transmit and distribute their loads to the soil. The design is based on the assumption that the footing is rigid, so that the variation of the soil pressure under the footing is linear. Uniform soil pressure is achieved when the column load coincides with the centroid of the footing. Although this assumption is acceptable for rigid footings, such an assumption becomes less accurate as the footing becomes relatively more flexible. The proper design of footings requires that

1. The load capacity of the soil is not exceeded.
2. Excessive settlement, differential settlement, or rotations are avoided.
3. Adequate safety against sliding and/or overturning is maintained.

The most common types of footings used in buildings are the single footings and wall footings (Figs. 13.1 and 13.2). When a column load is transmitted to the soil by the footing, the soil becomes compressed. The amount of settlement depends on many factors, such as the type of soil, the load intensity, the depth below ground level, and the type of footing. If different footings of the same structure have different settlements, new stresses develop in the structure. Excessive differential settlement may lead to the damage of nonstructural members in the buildings or even failure of the affected parts.

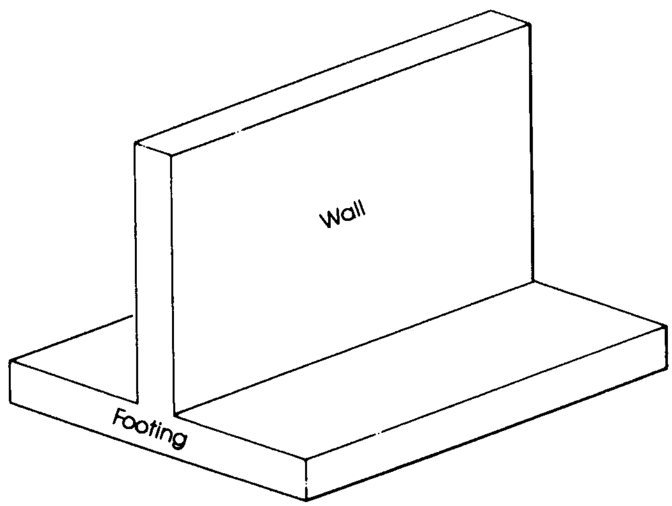


Figure 13.1 Wall footing.

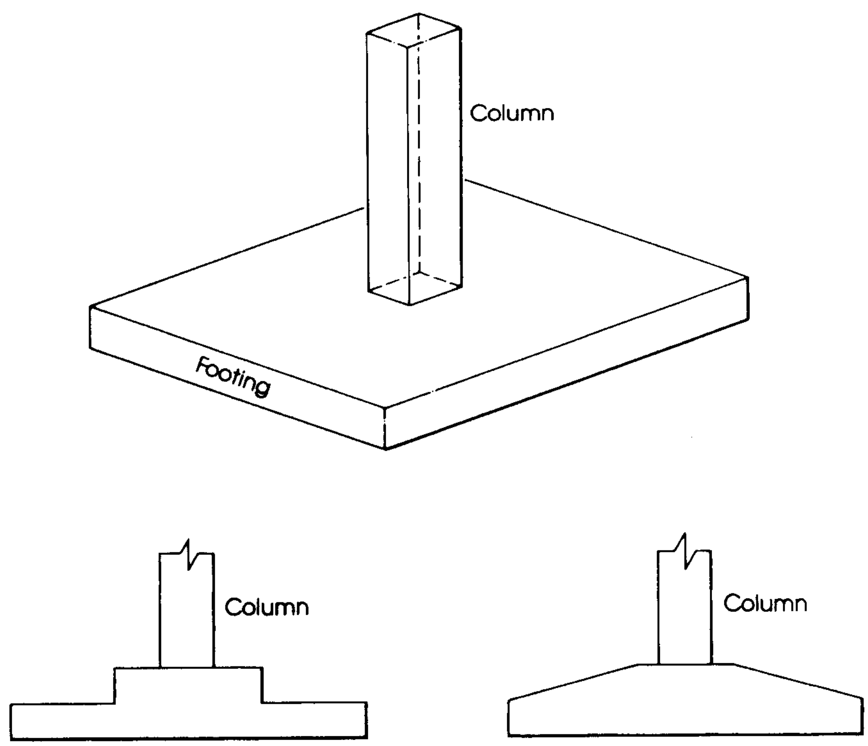


Figure 13.2 Single footing.

Vertical loads are usually applied at the centroid of the footing. If the resultant of the applied loads does not coincide with the centroid of the bearing area, a bending moment develops. In this case, the pressure on one side of the footing will be greater than the pressure on the other side.

If the bearing soil capacity is different under different footings—for example, if the footings of a building are partly on soil and partly on rock—a differential settlement will occur. It is usual in such cases to provide a joint between the two parts to separate them, allowing for independent settlement.

The depth of the footing below the ground level is an important factor in the design of footings. This depth should be determined from soil tests, which should provide reliable information on safe bearing capacity at different layers below ground level. Soil test reports specify the allowable bearing capacity to be used in the design. In cold areas where freezing occurs, frost action may cause heaving or subsidence. It is necessary to place footings below freezing depth to avoid movements.

13.2 TYPES OF FOOTINGS

Different types of footings may be used to support building columns or walls. The most common types are as follows:

1. *Wall footings* are used to support structural walls that carry loads from other floors or to support nonstructural walls. They have a limited width and a continuous length under the wall (Fig. 13.1). Wall footings may have one thickness, be stepped, or have a sloped top.
2. *Isolated*, or *single, footings* are used to support single columns (Fig. 13.2). They may be square, rectangular, or circular. Again, the footing may be of uniform thickness, stepped, or have a sloped top. This is one of the most economical types of footings, and it is used when columns are spaced at relatively long distances. The most commonly used are square or rectangular footings with uniform thickness.
3. *Combined footings* (Fig. 13.3) usually support two columns or three columns not in a row. The shape of the footing in plan may be rectangular or trapezoidal, depending on column loads. Combined footings are used when two columns are so close that single footings cannot be used or when one column is located at or near a property line.
4. *Cantilever*, or *strap, footings* (Fig. 13.4) consist of two single footings connected with a beam or a strap and support two single columns. They are used when one footing supports an eccentric column and the nearest adjacent footing lies at quite a distance from it. This type replaces a combined footing and is sometimes more economical.
5. *Continuous footings* (Fig. 13.5) support a row of three or more columns. They have limited width and continue under all columns.
6. *Raft*, or *mat, foundations* (Fig. 13.6) consist of one footing, usually placed under the entire building area, and support the columns of the building. They are used when
   - **a.** The soil-bearing capacity is low.
   - **b.** Column loads are heavy.
   - **c.** Single footings cannot be used.
   - **d.** Piles are not used.
   - **e.** Differential settlement must be reduced through the entire footing system.
7. *Pile caps* (Fig. 13.7) are thick slabs used to tie a group of piles together and to support and transmit column loads to the piles.

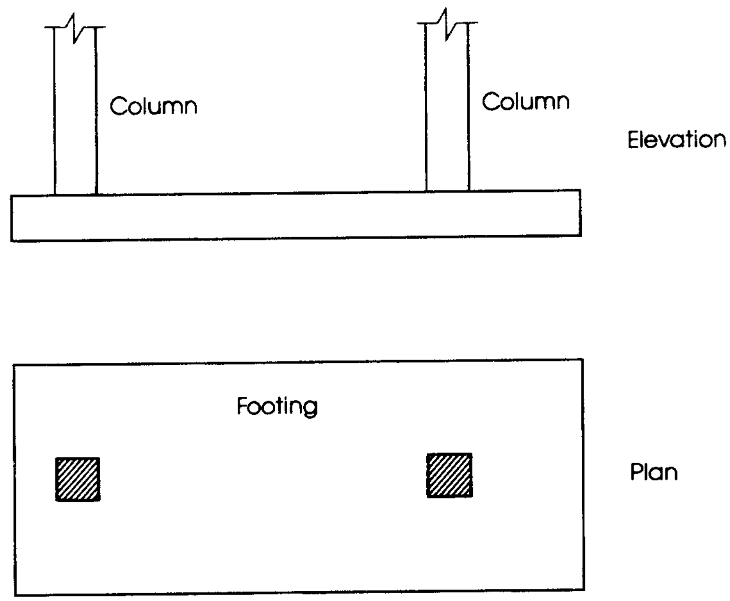


Figure 13.3 Combined footing.

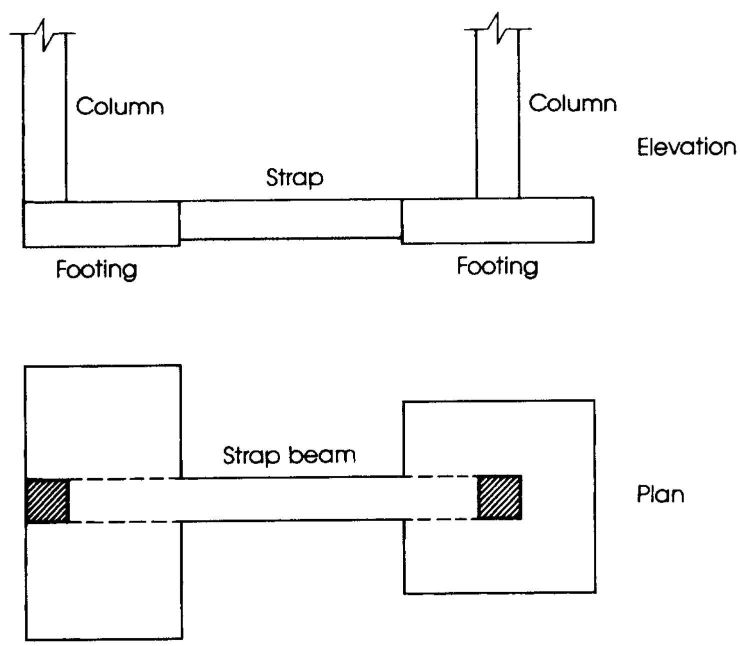


Figure 13.4 Strap footing.

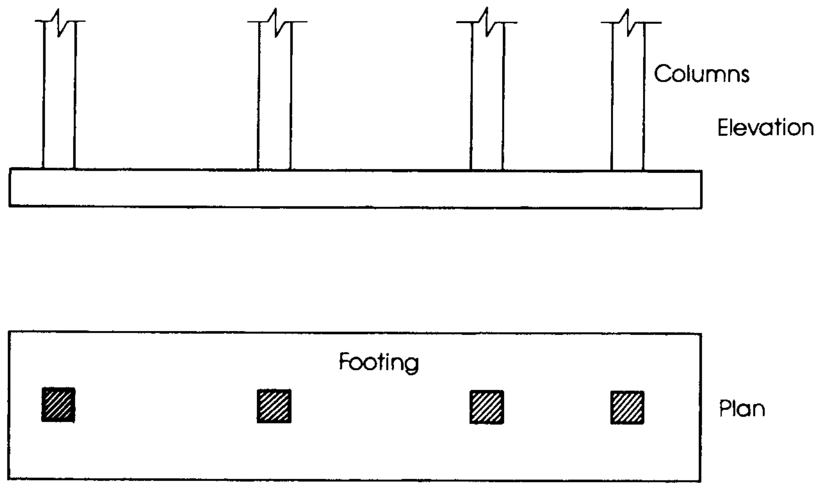


Figure 13.5 Continuous footing.

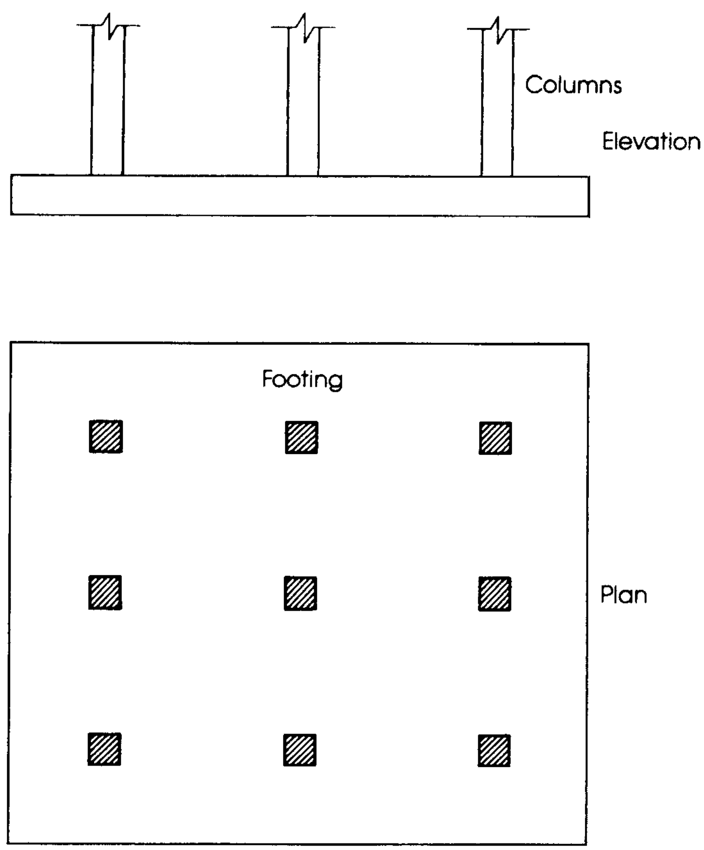


Figure 13.6 Raft, or mat, foundation.

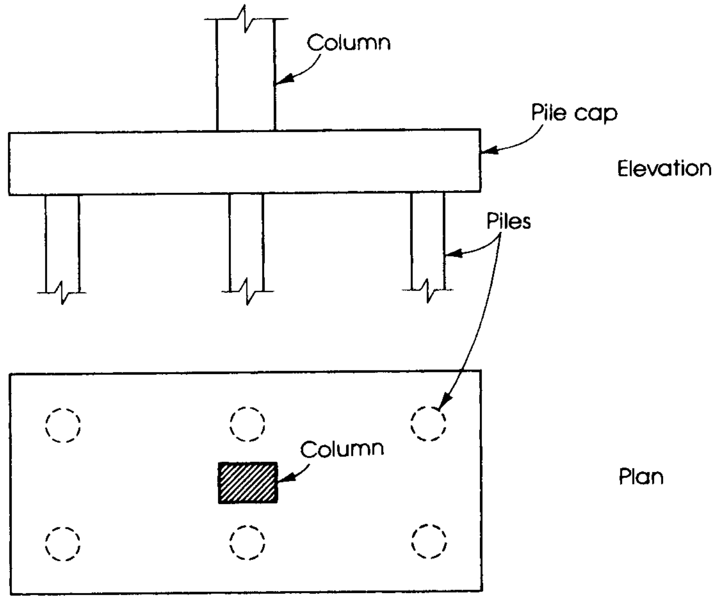


Figure 13.7 Pile cap footing.

13.3 DISTRIBUTION OF SOIL PRESSURE

Fig. 13.8 shows a footing supporting a single column. When the column load, P, is applied on the centroid of the footing, a uniform pressure is assumed to develop on the soil surface below the footing area. However, the actual distribution of soil pressure is not uniform but depends on many factors, especially the composition of the soil and the degree of flexibility of the footing.

For example, the distribution of pressure on cohesionless soil (sand) under a rigid footing is shown in Fig.13.9. The pressure is maximum under the center of the footing and decreases toward the ends of the footing. The cohesionless soil tends to move from the edges of the footing, causing a reduction in pressure, whereas the pressure increases around the center to satisfy equilibrium conditions. If the footing is resting on a cohesive soil such as clay, the pressure under the edges is greater than at the center of the footing (Fig. 13.10). The clay near the edges has a strong cohesion with the adjacent clay surrounding the footing, causing the nonuniform pressure distribution.

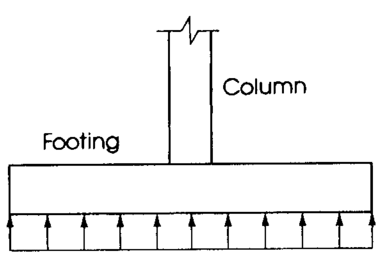


Figure 13.8 Distribution of soil pressure assuming uniform pressure.

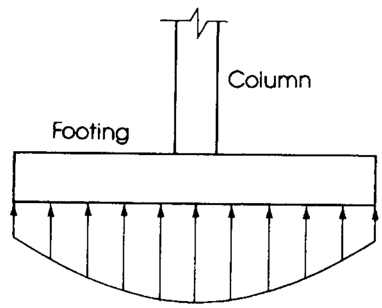


Figure 13.9 Soil pressure distribution in cohesionless soil (sand).

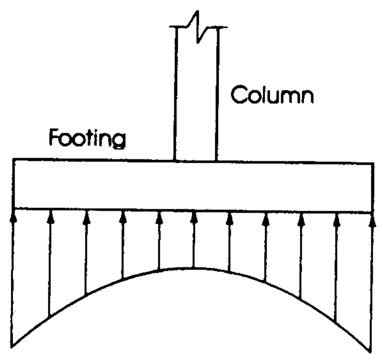


Figure 13.10 Soil pressure distribution in cohesive soil (clay).

The allowable bearing soil pressure, q_a, is usually determined from soil tests. The allowable values vary with the type of soil, from extremely high in rocky beds to low in silty soils. For example, q_a, for sedimentary rock is 30 ksf, for compacted gravel is 8 ksf, for well-graded compacted sand is 6 ksf, and for silty-gravel soils is 3 ksf.

Referring to Fig. 13.8, when the load P is applied, the part of the footing below the column tends to settle downward. The footing will tend to take a uniform curved shape, causing an upward pressure on the projected parts of the footing. Each part acts as a cantilever and must be designed for both bending moments and shearing forces. The design of footings is explained in detail later.

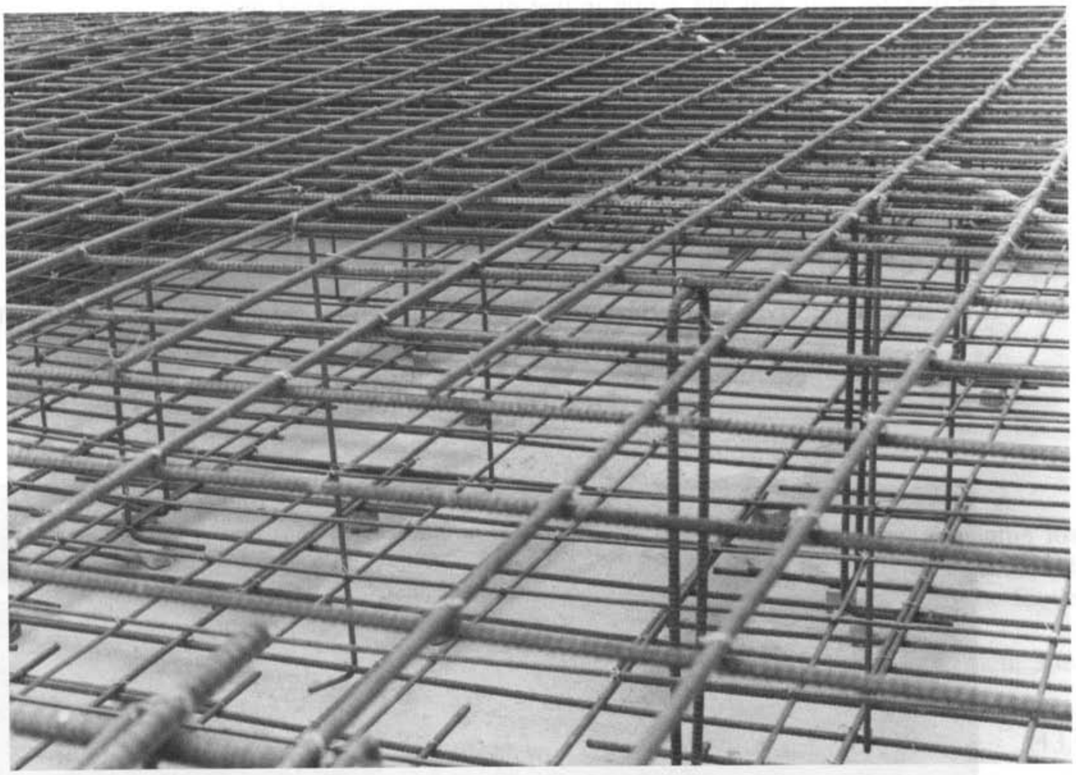
13.4 DESIGN CONSIDERATIONS

Footings must be designed to carry the column loads and transmit them to the soil safely. The design procedure must take the following strength requirements into consideration:

1. The area of the footing based on the allowable bearing soil capacity
2. On-way shear
3. Two-way shear, or punching shear
4. Bending moment and steel reinforcement required

5. Bearing capacity of columns at their base and dowel requirements
6. Development length of bars
7. Differential settlement

These strength requirements are explained in the following sections.



Reinforcing rebars placed in two layers in a raft foundation.

13.4.1 Size of Footings

The area of the footings can be determined from the actual external loads (unfactored forces and moments) such that the allowable soil pressure is not exceeded. In general, for vertical loads

$$\text{Area of footing} = \frac{\text{total service load (including self-weight)}}{\text{allowable soil pressure, } q_a} \tag{13.1}$$

or

$$\text{Area} = \frac{P(\text{total})}{q_a}$$

where the total service load is the unfactored design load. Once the area is determined, a factored soil pressure is obtained by dividing the factored load, $P_u = 1.2D + 1.6L$, by the area of the footing. This is required to design the footing by the strength design method.

$$q_u = \frac{P_u}{\text{area of footing}} \tag{13.2}$$

The allowable soil pressure, q_a, is obtained from soil test and is based on service load conditions.

13.4.2 One-Way Shear (Beam Shear) (V_{u_1})

For footings with bending action in one direction, the critical section is located at a distance d from the face of the column. The diagonal tension at section m-m in Fig. 13.11 can be checked as was done before in beams. The allowable shear in this case is equal to

$$\phi V_c = 2\phi\lambda\sqrt{f'_c}bd \quad (\phi = 0.75) \tag{13.3}$$

where b = width of section m-m. The factored shearing force at section m-m can be calculated as follows:

$$V_{u_1} = q_u b\left(\frac{L}{2} - \frac{C}{2} - d\right) \tag{13.4}$$

If no shear reinforcement is to be used, then d can be determined, assuming $V_u = \phi V_c$:

$$d = \frac{V_{u_1}}{2\phi\lambda\sqrt{f'_c}b} \tag{13.5}$$

Wall and column footings, partly covered.

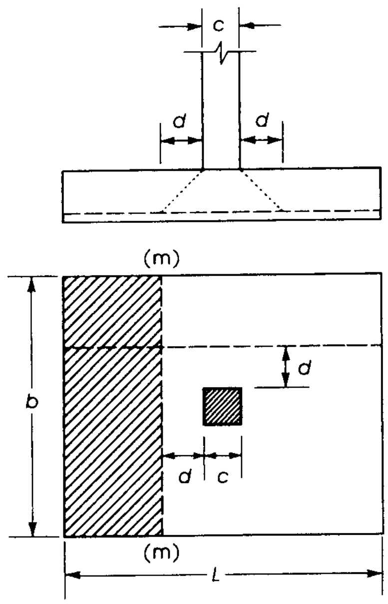


Figure 13.11 One-way shear.

13.4.3 Two-Way Shear (Punching Shear) (V_{u_2})

Two-way shear is a measure of the diagonal tension caused by the effect of the column load on the footing. Inclined cracks may occur in the footing at a distance $d/2$ from the face of the column on all sides. The footing will fail as the column tries to punch out part of the footing (Fig. 13.12).

The ACI Code, Section 11.11.2 allows a shear strength, V_c, in footings without shear reinforcement for two-way shear action, the smallest of

$$V_{c_1} = 4\lambda\sqrt{f_c'}b_0 d \tag{13.6}$$

$$V_{c_2} = \left(2 + \frac{4}{\beta}\right)\lambda\sqrt{f_c'}b_0 d \tag{13.7}$$

$$V_{c_3} = \left(\frac{\alpha_s d}{b_0} + 2\right)\lambda\sqrt{f_c'}b_0 d \tag{13.8}$$

where

β = Ratio of long side to short side of the rectangular column

b_0 = perimeter of the critical section taken at $d/2$ from the loaded area (column section) (see Fig. 13.12)

d = effective depth of footing

λ = is a modification factor for type of concrete (ACI 8.6.1)

λ = 1.0 Normal-weight concrete

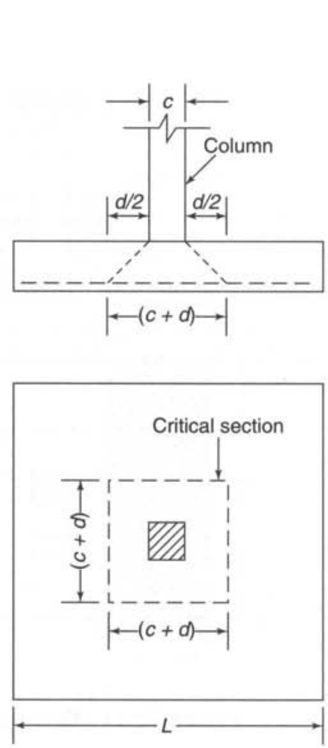
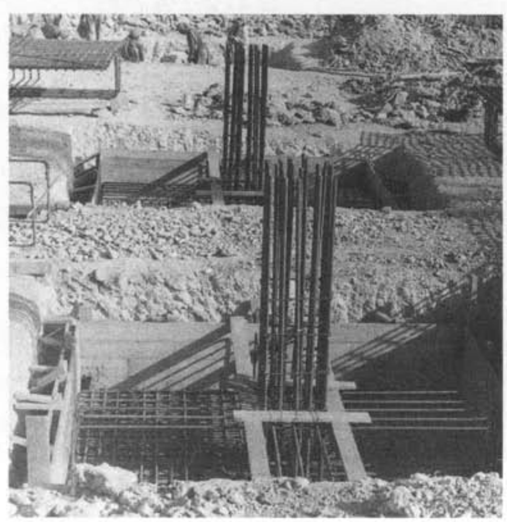


Figure 13.12 Punching shear (two-way).

$\lambda = 0.85$ sand-lightweight concrete
$\lambda = 0.75$ for all-lightweight concrete

Linear interpolation shall be permitted between 0.85 and 1.0 on the basis of volumetric fractions, for concrete containing normal-weight fine aggregate and a blend of lightweight and normal-weight coarse aggregate.



Reinforced concrete single footings.

For the values of V_{c_1} and V_{c_2} it can be observed that V_{c_1} controls (less than V_{c_2}) whenever $\beta_c \leq 2$, whereas V_{c_2} controls (less than V_{c_1}) whenever $\beta_c > 2$. This indicates that the allowable shear V_c is reduced for relatively long footings. The actual soil pressure variation along the long side increases with an increase in β. For shapes other than rectangular, β is taken to be the ratio of the longest dimension of the effective loaded area in the long direction to the largest width in the short direction (perpendicular to the long direction).

For Eq. 13.8, α_s is assumed to be 40 for interior columns, 30 for edge columns, and 20 for corner columns. The concrete shear strength, V_{c_3} represents the effect of an increase in b_0 relative to d. For a high ratio of b_0/d, V_{c_3} may control.

Based on the preceding three values of V_c, the effective depth, d, required for two-way shear is the largest obtained from the following formulas ($\phi = 0.75$):

$$d_1 = \frac{V_{u_2}}{\phi 4 \lambda \sqrt{f'_c} b_0} \quad \text{(where } \beta \leq 2) \tag{13.9}$$

or

$$d_1 = \frac{V_{u_2}}{\phi \left(2 + \frac{4}{\beta}\right) \lambda \sqrt{f'_c} b_0} \quad \text{(where } \beta > 2) \tag{13.10}$$

$$d_2 = \frac{V_{u_2}}{\phi \left(\frac{\alpha_s d}{b_0} + 2\right) \lambda \sqrt{f'_c} b_0} \tag{13.11}$$

The two-way shearing force, V_{u_2}, and the effective depth, d, required (if shear reinforcement is not provided) can be calculated as follows (refer to Fig. 13.12):

1. Assume d.
2. Determine b_0: $b_0 = 4(c + d)$ for square columns, where one side $= c$. $b_0 = 2\,(c_1 + d) + 2(c_2 + \mathrm{d})$ for rectangular columns of sides c_1 and c_2.
3. The shearing force V_{u_2} acts at a section that has a length $b_0 = 4\,(c + d)$ or $[2(c_1 + d) + 2(c_2 + d)]$ and a depth d; the section is subjected to a vertical downward load, P_u, and a vertical upward pressure, q_u (Eq. 13.2). Therefore,

$$V_{u_2} = P_u - q_u(c + d)^2 \text{ for square columns} \tag{13.12a}$$

$$V_{u_2} = P_u - q_u(c_1 + d)(c_2 + d) \text{ for rectangular columns} \tag{13.12b}$$

4. Determine the largest d (of d_1 and d_2). If d is not close to the assumed d, revise your assumption and repeat.

13.4.4 Flexural Strength and Footing Reinforcement

The critical sections for moment occur at the face of the column (section n-n, Fig. 13.13). The bending moment in each direction of the footing must be checked and the appropriate reinforcement must be provided. In square footings and square columns, the bending moments in both directions are equal. To determine the reinforcement required, the depth of the footing in each direction may be used. Because the bars in one direction rest on top of the bars in the other direction, the effective depth, d, varies with the diameter of the bars used. An average value of d may be adopted. A practical value of d may be assumed to be $(h - 4.5)$ in.

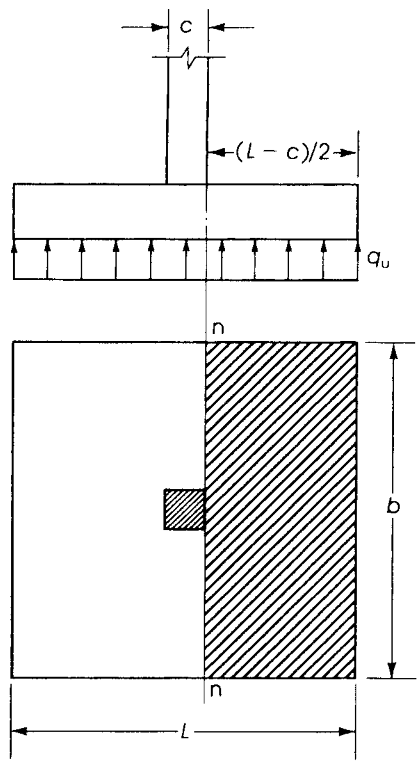


Figure 13.13 Critical section of bending moment.

The depth of the footing is often controlled by shear, which requires a depth greater than that required by the bending moment. The steel reinforcement in each direction can be calculated in the case of flexural members as follows:

$$M_u = \phi A_s f_y \left(d - \frac{A_s f_y}{1.7 f_c' b}\right) \tag{13.13}$$

Also, the steel ratio, ρ, can be determined as follows (Eq. 4.2):

$$\rho = \frac{0.85 f_c'}{f_y}\left[1 - \sqrt{1 - \frac{2R_u}{\phi(0.85 f_c')}}\right] \tag{13.14}$$

where $R_u = M_u/bd^2$. When R_u is determined, ρ can also be obtained from Eq. 13.15.

The minimum steel ratio requirement in flexural members is equal to $200/f_y$ when $f_c' <$ 4500 psi and equal to $3\sqrt{f_c'}/f_y$ when $f_c' \geq 4500$ psi. However, the ACI Code, Section 10.5, indicates that for structural slabs of uniform thickness, the minimum area and maximum spacing of steel bars in the direction of bending shall be as required for shrinkage and temperature reinforcement. This last minimum steel requirement is very small, and a higher minimum reinforcement ratio is recommended, but it should not be greater than $200/f_y$.

The reinforcement in one-way footings and two-way footings must be distributed across the entire width of the footing. In the case of two-way rectangular footings, the ACI Code, Section 15.4.4, specifies that in the long direction, a portion of the total reinforcement $\gamma_s A_s$

distributed uniformly along the width of the footing. In the short direction, a certain ratio of the total reinforcement in this direction must be placed uniformly within a bandwidth equal to the length of the short side of the footing according to

$$\gamma_s = \frac{2}{\beta + 1} \tag{13.15}$$

where

$$\beta = \frac{\text{long side of footing}}{\text{short side of footing}} \tag{13.16}$$

The bandwidth must be centered on the centerline of the column (Fig. 13.14). The remaining reinforcement in the short direction must be uniformly distributed outside the bandwidth. This remaining reinforcement percentage shall not be less than that required for shrinkage and temperature.

When structural steel columns or masonry walls are used, then the critical sections for moments in footings are taken at halfway between the middle and the edge of masonry walls and halfway between the face of the column and the edge of the steel base place (ACI Code, Section 15.4.2).

13.4.5 Bearing Capacity of Column at Base

The loads from the column act on the footing at the base of the column, on an area equal to the area of the column cross-section. Compressive forces are transferred to the footing directly by bearing on the concrete.

Forces acting on the concrete at the base of the column must not exceed the bearing strength of concrete as specified by the ACI Code, Section 10.14:

$$\text{Bearing strength } N_1 = \phi(0.85 f'_c A_1) \tag{13.17}$$

where $\phi = 0.65$ and $A_1 =$ the bearing area of the column. The value of the bearing strength given in Eq. 13.17 may be multiplied by a factor $\sqrt{A_2/A_1} \leq 2.0$ for bearing on footings when the supporting surface is wider on all sides than the loaded area. Here A_2 is the area of the part of the supporting footing that is geometrically similar to and concentric with the loaded area (Fig. 13.15). Because $A_2 > A_1$, the factor $\sqrt{A_2/A_1}$ is greater than unity, indicating that the allowable

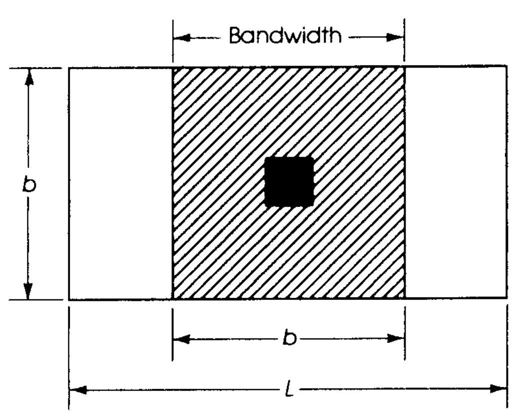


Figure 13.14 Bandwidth for reinforcement distribution.

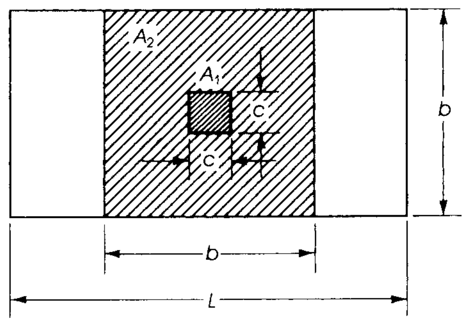


Figure 13.15 Bearing areas on footings. $A_1 = c^2$, $A_2 = b^2$.

bearing strength is increased because of the lateral support from the footing area surrounding the column base. The modified bearing strength is

$$N_2 = \phi(0.85 f_c' A_1)\sqrt{\frac{A_2}{A_1}} \leq 2\phi(0.85 f_c' A_1) \quad (13.18)$$

If the factored force, P_u, is greater than either N_1 or N_2 reinforcement must be provided to transfer the excess force. This is achieved by providing dowels or extending the column bars into the footing. The excess force is $P_{ex} = P_u - N_1$ and the area of the dowel bars is $A_{sd} = (P_{ex}/f_y) \geq 0.005\ A_1$, where A_1 is the area of the column section. At least four bars should be used at the four corners of the column. If the factored force is less than either N_1 or N_2, then minimum reinforcement must be provided. The ACI Code, Section 15.8.2, indicates that the minimum area of the dowel reinforcement is at least $0.005A_g$ (and not less than four bars), where A_g is the gross area of the column section. The minimum reinforcement requirements apply also to the case when the factored forces are greater than N_1 and N_2. The dowel bars may be placed at the four corners of the column and extended in both the column and footing. The dowel diameter shall not exceed the diameter of the longitudinal bars in the columns by more than 0.15 in. This requirement is necessary to ensure proper action between the column and footing. The development length of the dowels must be checked to determine proper transfer of the compression force into the footing.

13.4.6 Development Length of the Reinforcing Bars

The critical sections for checking the development length of the reinforcing bars are the same as those for bending moments. The development length for compression bars was given in Chapter 7:

$$l_{dc} = \frac{0.02 f_y d_b}{\lambda\sqrt{f_c'}} \quad (7.15)$$

but this value cannot be less than $0.0003\ f_y d_b \geq 8$ in. For other values, refer to Chapter 7.

13.4.7 Differential Settlement (Balanced Footing Design)

Footings usually support the following loads:

- Dead loads from the substructure and superstructure
- Live load resulting from occupancy

- Weight of materials used in backfilling
- Wind loads

Each footing in a building is designed to support the maximum load that may occur on any column due to the critical combination of loadings, using the allowable soil pressure.

The dead load, and maybe a small portion of the live load (called the *usual* live load), may act continuously on the structure. The rest of the live load may occur at intervals and on some parts of the structure only, causing different loadings on columns. Consequently, the pressure on the soil under different footings will vary according to the loads on the different columns, and differential settlement will occur under the various footings of one structure. Because partial settlement is inevitable, the problem turns out to be the amount of differential settlement that the structure can tolerate. The amount of differential settlement depends on the variation in the compressibility of the soils, the thickness of the compressible material below foundation level, and the stiffness of the combined footing and superstructure. Excessive differential settlement results in cracking of concrete and damage to claddings, partitions, ceilings, and finishes.

Differential settlement may be expressed in terms of angular distortion of the structure. Bjerrum [5] indicated that the danger limits of distortion for some conditions vary between $\frac{1}{600}$ to $\frac{1}{150}$ depending on the damage that will develop in the building.

For practical purposes it can be assumed that the soil pressure under the effect of sustained loadings is the same for all footings, thus causing equal settlements. The sustained load (or the usual load) can be assumed to be equal to the dead load plus a percentage of the live load, which occurs very frequently on the structure. Footings then are proportioned for these sustained loads to produce the same soil pressure under all footings. In no case is the allowable soil bearing capacity to be exceeded under the dead load plus the maximum live load for each footing. Example 13.4 explains the procedure for calculating the areas of footings, taking into consideration the effect of differential settlement.

13.5 PLAIN CONCRETE FOOTINGS

Plain concrete footings may be used to support masonry walls or other light loads and transfer them to the supporting soil. The ACI Code Section 22.7 allows the use of plain concrete pedestals and footings on soil, provided that the design stresses shall not exceed the following:

1. Maximum flexural stress in tension is less than or equal to $5\phi\lambda\sqrt{f'_c}$ (where $\phi = 0.60$).
2. Maximum stress in one-way shear (beam action) is less than or equal to $\frac{4}{3}\,\phi\lambda\sqrt{f'_c}$ (where $\phi = 0.60$).
3. Maximum shear stress in two-way action according to ACI Code Section 22.5.4 is

$$\left(\frac{4}{3} + \frac{8}{3\beta}\right)\phi\lambda\sqrt{f'_c} \le 2.66\,\phi\sqrt{f'_c} \quad \text{(where } \phi = 0.60\text{)} \tag{13.19}$$

   where

   β = Ratio of long side to short side of the rectangular column
   λ = modification factor described in 13.4.3.

4. Maximum compressive strength shall not exceed the concrete bearing strengths specified; f'_c of plain concrete should not be less than 2500 psi.

5. The minimum thickness of plain concrete footings shall not be less than 8 in.
6. The critical sections for bending moments are at the face of the column or wall.
7. The critical sections for one-way shear and two-way shear action are at distances d and $d/2$ from the face of the column or wall, respectively. Although plain concrete footings do not require steel reinforcement, it will be advantageous to provide shrinkage reinforcement in the two directions of the footing.
8. Stresses due to factored loads are computed assuming a linear distribution in concrete.
9. The effective depth, d, must be taken equal to the overall thickness minus 3 in.
10. For flexure and one-way shear, use a gross section bh, whereas for two-way shear, use b_0h to calculate ϕV_c.

Example 13.1

Design a reinforced concrete footing to support a 20-in.-wide concrete wall carrying a dead load of 26 K/ft, including the weight of the wall, and a live load of 20 K/ft. The bottom of the footing is 6 ft below final grade. Use normal-weight concrete with $f_c' = 4$ ksi, $f_y = 60$ ksi, and an allowable soil pressure of 5 ksf.

Solution

1. Calculate the effective soil pressure. Assume a total depth of footing of 20 in. Weight of footing is $(\frac{20}{12})(150) = 250$ psf. Weight of the soil fill on top of the footing, assuming that soil weighs 100 lb/ft^3, is $(6 - \frac{20}{12}) \times 100 = 433$ psf. Effective soil pressure at the bottom of the footing is $5000 - 250 - 433 = 4317\,\text{psf} = 4.32\,\text{ksf}$.
2. Calculate the width of the footing for a 1-ft length of the wall:

$$\text{Width of footing} = \frac{\text{total load}}{\text{effective soil pressure}}$$

$$= \frac{26 + 20}{4.32} = 10.7\text{ft}$$

Use 11 ft.

3. Net upward pressure = (factored load)/(footing width) (per 1 ft):

$$P_u = 1.2D + 1.6L = 1.2 \times 26 + 1.6 \times 20 = 63.2 \text{ K}$$

$$\text{Net pressure} = q_u = \frac{63.2}{11} = 5.745 \text{ ksf}$$

4. Check the assumed depth for shear requirements. The concrete cover in footings is 3 in., and assume no. 8 bars; then $d = 20 - 3.5 = 16.5$. The critical section for one-way shear is at a distance d from the face of the wall:

$$V_u = q_u\left(\frac{B}{2} - d - \frac{c}{2}\right) = 5.745\left(\frac{11}{2} - \frac{16.5}{12} - \frac{20}{2 \times 12}\right) = 18.91 \text{ K}$$

$$\text{Allowable one-way shear} = 2\lambda\sqrt{f_c'} = (2)(1)\sqrt{4000} = 126.5 \text{ psi}$$

$$\text{Required } d = \frac{V_u}{\phi(2\sqrt{f_c'})b} = \frac{18.91 \times 1000}{0.75(126.5)(12)} = 16.6 \text{ in.}$$

$$b = 1\text{-ft length of footing} = 12 \text{ in.}$$

Total depth is $16.6 + 3.5 = 20.1$ in., or 20 in. Actual d is $20 - 3.5 = 16.5$ in. (as assumed). Note that few trials are needed to get the assumed and calculated d quite close.

5. Calculate the bending moment and steel reinforcement. The critical section is at the face of the wall:

$$M_u = \frac{1}{2}q_u\left(\frac{B}{2} - \frac{c}{2}\right)^2 = \frac{5.745}{2}\left(\frac{11}{2} - \frac{20}{24}\right)^2 = 62.6 \text{ K·ft}$$

$$R_u = \frac{M_u}{bd^2} = \frac{62.6 \times 12{,}000}{12(16.5)^2} = 230 \text{ psi}$$

From Table A.1 in Appendix A, for $R_u = 230$ psi, $f'_c = 4$ ksi, and $f_y = 60$ ksi, the steel percentage is $\rho = 0.0045$ (or from Eq. 13.14). Minimum steel percentage for flexural members is

$$\rho_{min} = \frac{200}{f_y} = \frac{200}{60{,}000} = 0.0033$$

Percentage of shrinkage reinforcement is 0.18% (for $f_y = 60$ ksi). Therefore, use $\rho = 0.0045$ as calculated.

$$A_s = 0.0045 \times 12 \times 16.5 = 0.89 \text{ in.}^2$$

Use no. 8 bars spaced at 9 in. ($A_s = 1.05$ in.^{2}) (Table A.14).

6. Check the development length for no. 8 bars:

$$l_d = 48d_b = 48(1) = 48 \text{ in. (Refer to Chapter 7).}$$

Provided

$$l_d = \frac{B}{2} - \frac{c}{2} - 3 \text{ in.} = \frac{11(12)}{2} - \frac{20}{2} - 3 = 53 \text{ in.}$$

7. Calculate secondary reinforcement in the longitudinal direction: $A_s = 0.0018(12)(20) = 0.43$ in.^{2}/ft. Choose no. 5 bars spaced at 8 in. ($A_s = 0.46$ in.^{2}). Details are shown in Fig. 13.16.

Example 13.2

Design a square single footing to support an 18-in.-square tied interior column reinforced with eight no. 9 bars. The column carries an unfactored axial dead load of 245 K and an axial live load of 200 K. The base of the footing is 4 ft below final grade and the allowable soil pressure is 5 ksf. Use normal-weight concrete, with $f'_c = 4$ ksi and $f_y = 60$ ksi.

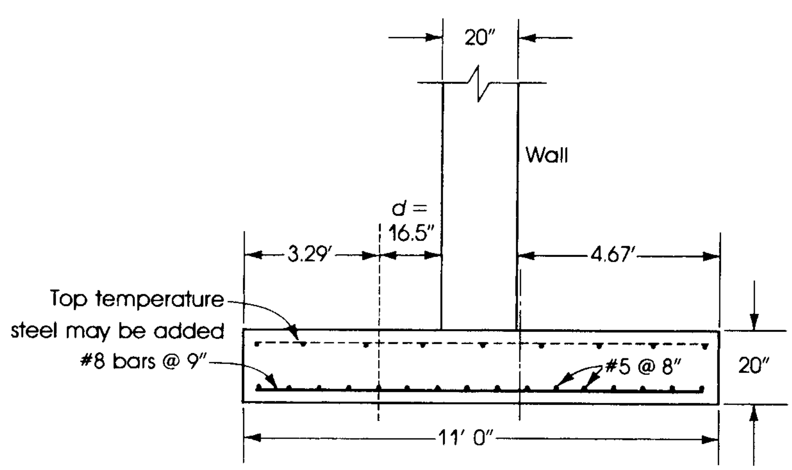


Figure 13.16 Example 13.1: Wall footing.

Solution

1. Calculate the effective soil pressure. Assume a total depth of footing of 2 ft. The weight of the footing is $2 \times 150 = 300$ psf. The weight of the soil on top of the footing (assuming the weight of soil = 100 pcf) is $2 \times 100 = 200$ psf.

$$\text{Effective soil pressure} = 5000 - 300 - 200 = 4500 \text{ psf}$$

2. Calculate the area of the footing:

$$\text{Actual loads} = D + L = 245 + 200 = 445 \text{ K}$$

$$\text{Area of footing} = \frac{445}{4.5} = 98.9 \text{ ft}^2$$

$$\text{Side of footing} = 9.94 \text{ ft.}$$

Thus, use 10 ft (Fig.13.17).

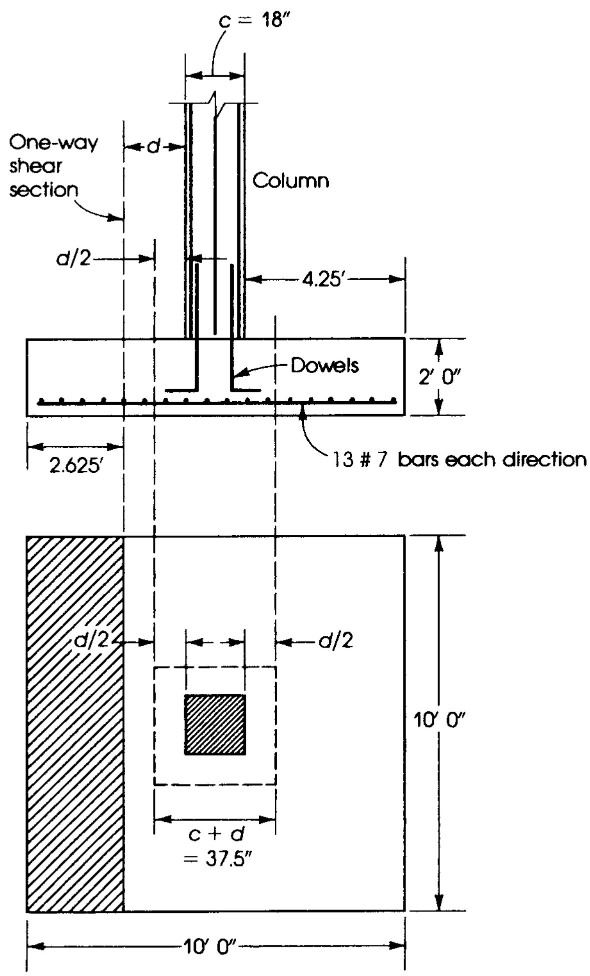


Figure 13.17 Example 13.2: Square footing.

3. Net upward pressure equals (factored load)/(area of footing).

$$P_u = 1.2D + 1.6L$$
$$= 1.2 \times 245 + 1.6 \times 200 = 614 \text{ K}$$
$$\text{Net upward pressure, } q_u = \frac{614}{10 \times 10} = 6.14 \text{ ksf}$$

4. Check depth due to two-way shear. If no shear reinforcement is used, two-way shear determines the critical footing depth required. For an assumed total depth of 24 in., calculate d to the centroid of the top layer of the steel bars to be placed in the two directions within the footing. Let the bars to be used be no. 8 bars for calculating d.

$$d = 24 - 3 \text{ (cover)} - 1.5 \text{ (bar diameters)} = 19.5 \text{ in.}$$

It is quite practical to assume $d = h - 4.5$ in.

$$b_0 = 4(c + d) = 4(18 + 19.5) = 150 \text{ in.}$$
$$c + d = 18 + 19.5 = 37.5 \text{ in.} = 3.125 \text{ ft}$$
$$V_{u_2} = P_u - q_u(c + d)^2 = 614 - 6.14(3.125)^2 = 554 \text{ K}$$
$$\text{Required } d_1 = \frac{V_{u_2}}{4\phi\lambda(\sqrt{f'_c}b_0)}$$
$$= \frac{554(1000)}{(4)(0.75)(1)\sqrt{4000}(150)} = 19.5 \text{ in.} \quad (\beta = 1; \text{Eq. } 13.9)$$
$$\text{Required } d_2 = \frac{554(1000)}{0.75\left(\frac{40 \times 19.5}{150} + 2\right)(\sqrt{4000})(150)}$$
$$= 10.8 \text{ in.} \quad \text{(not critical)}$$

($\alpha_s = 40$ for interior columns.) Thus, the assumed depth is adequate. Two or more trials may be needed to reach an acceptable d that is close to the assumed one.

5. Check depth due to one-way shear action: The critical section is at a distance d from the face of the column.

$$\text{Distance from edge of footing} = \left(\frac{L}{2} - \frac{c}{2} - d\right) = 2.625 \text{ ft}$$
$$V_{u_1} = 6.14 \times (2.625)(10) = 161.2 \text{ K}$$

The depth required for one-way shear is

$$d = \frac{V_{u_1}}{(0.75)(2)\lambda\sqrt{f'_c}b}$$
$$= \frac{161.2(1000)}{(0.75)(2)(1)(\sqrt{4000})(10 \times 12)} = 14.2 \text{ in.} < 19.5 \text{ in.}$$

6. Calculate the bending moment and steel reinforcement. The critical section is at the face of the column. The distance from edge of footing is

$$\left(\frac{L}{2} - \frac{c}{2}\right) = 5 - \frac{1.5}{2} = 4.25 \text{ ft}$$

$$M_u = \frac{1}{2} q_u \left(\frac{L}{2} - \frac{c}{2}\right)^2 b = \frac{1}{2}(6.14)(4.25)^2(10) = 554.5 \text{ K·ft}$$

$$R_u = \frac{M_u}{bd^2} = \frac{554.5(12{,}000)}{(10 \times 12)(19.5)^2} = 145.8 \text{ psi}$$

Applying Eq. 13.14, $\rho = 0.0028$.

$$A_s = \rho bd = 0.0028(10 \times 12)(19.5) = 6.55 \text{ in.}^2$$

$$\text{Minimum } A_s \text{ (shrinkage steel)} = 0.0018(10 \times 12)(24) = 5.18 \text{ in.}^2 < 6.55 \text{ in.}^2$$

$$\text{Minimum } A_s \text{ (flexure)} = 0.0033(10 \times 12)(19.5) = 7.72 \text{ in.}^2$$

Therefore, $A_s = 7.72 \text{ in.}^2$ can be adopted. Use 13 no. 7 bars ($A_s = 7.82 \text{ in.}^2$), spaced at $s = (120 - 6)/12 = 9.5$ in. in both directions.

7. Check bearing stress:

a. Bearing strength, N_1 at the base of the column ($A_1 = 18 \times 18$ in.) is

$$N_1 = \phi(0.085 f_c' A_1) = 0.65(0.85 \times 4)(18 \times 18) = 716 \text{ K}$$

b. Bearing strength, N_2, at the top of footing ($A_2 = 10 \times 10$ ft) is

$$N_2 = N_1 \sqrt{\frac{A_2}{A_1}} \leq 2N_1$$

$$A_2 = 10 \times 10 = 100 \text{ ft}^2 \quad A_1 = \frac{18 \times 18}{144} = 2.25 \text{ ft}^2$$

$$\sqrt{\frac{A_2}{A_1}} = 6.67 > 2$$

Therefore, $N_2 = 2N_1 = 1432$ K. Because $P_u = 614 \text{ K} < N_1$, bearing stress is adequate. The minimum area of dowels required is 0.005 $A_1 = 0.005\ (18 \times 18) = 1.62 \text{ in.}^2$. The minimum number of bars is four, so use four no. 8 bars placed at the four corners of the column.

c. Development length of dowels in compression:

$$l_{dc} = \frac{0.02 d_b f_y}{\lambda \sqrt{f_c'}} = \frac{0.02(1)(60{,}000)}{(1)\sqrt{4000}} = 19 \text{ in.}$$

(controls). Minimum l_{dc} is $0.0003 d_b f_y = 0.0003(1)\ (60{,}000) = 18 \text{ in.} \geq 8$ in. Therefore, use four no. 8 dowels extending 19 in. into column and footing. Note that l_d is less than d of 19.5 in., which is adequate.

8. The development length of main bars in footing for no. 7 bars is $l_d = 48 d_b = 42$ in. (refer to Chapter 7), provided $l_d = L/2 - c/2 - 3 \text{ in.} = 48$ in. Details of the footing are shown in Fig. 13.17 on page 430.

Example 13.3

Design a rectangular footing for the column of Example 13.2 if one side of the footing is limited to 8.5 ft.

Solution

1. The design procedure for rectangular footings is similar to that of square footings, taking into consideration the forces acting on the footing in each direction separately.
2. From the previous example, the area of the footing required is $98.9\,\text{ft}^2$:

$$\text{Length of footing} = \frac{98.9}{8.5} = 11.63 \quad \text{ft}$$

so use 12 ft (Fig. 13.18). Footing dimensions are 8.5×12 ft.

3. $P_u = 614$ K. Thus, net upward pressure is

$$q_u = \frac{614}{8.5 \times 12} = 6.02 \text{ ksf}$$

4. Check the depth due to one-way shear. The critical section is at a distance d from the face of the column. In the longitudinal direction,

$$V_{u_1} = \left(\frac{L}{2} - \frac{c}{2} - d\right) \times q_u b$$

$$= \left(\frac{12}{2} - \frac{1.5}{2} - \frac{19.5}{12}\right) \times 6.02 \times 8.5 = 185.5 \text{ K}$$

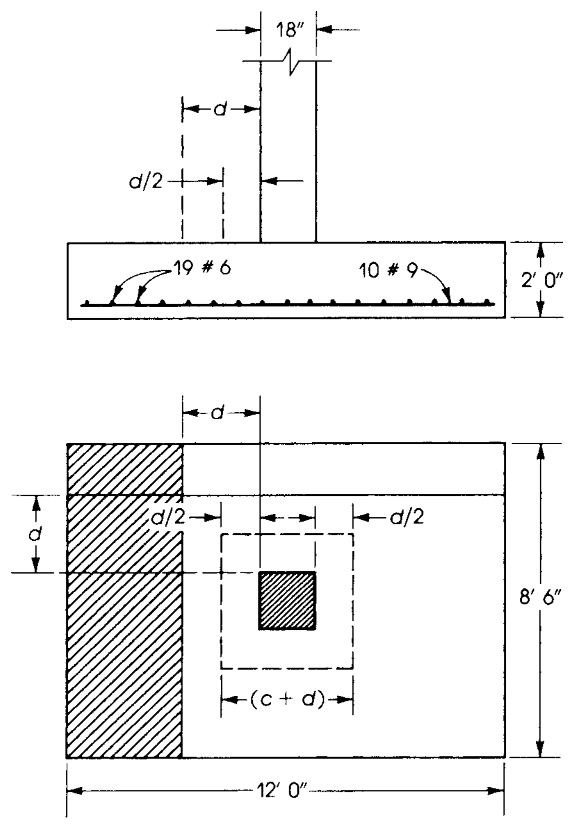


Figure 13.18 Example 13.3: Rectangular footing.

This shear controls. In the short direction, $V_u = 135.4\text{ K}$ (not critical).

$$\text{Required } d = \frac{V_{u_1}}{2\phi\lambda\sqrt{f_c'}b} = \frac{185.5 \times 1000}{(2)(0.75)(1)\sqrt{4000} \times (8.5 \times 12)} = 19.2 \text{ in.}$$

$$d \text{ provided} = 19.5 \text{ in.} > 19.2 \text{ in.}$$

5. Check the depth for two-way shear action (punching shear). The critical section is at a distance $d/2$ from the face of the column on four sides.

$$b_0 = 4(18 + 19.5) = 150 \text{ in.}$$

$$(c + d) = 18 + 19.5 = 37.5 \text{ in.} = 3.125 \text{ ft}$$

$$\beta = \frac{12}{8.5} = 1.41 < 2$$

(Use $V_c = 4\phi\lambda\sqrt{f_c'}b_0d$.)

$$V_{u_2} = P_u - q_u(c + d)^2 = 614 - 6.02(3.125)^2 = 555.2 \text{ K}$$

$$d_1 = \frac{V_{u_2}}{4\phi\lambda\sqrt{f_c'}b_0} = \frac{555.2 \times 1000}{4(0.75)(1)\sqrt{4000} \times 150} = 19.5 \text{ in.}$$

$$d_2 = 10.6 \text{ in.} \quad \text{(Does not control.)}$$

6. Design steel reinforcement in the longitudinal direction. The critical section is at the face of the support. The distance from the edge of the footing is

$$\frac{L}{2} - \frac{c}{2} = \frac{12}{2} - \frac{1.5}{2} = 5.25 \text{ ft}$$

$$M_u = \frac{1}{2}(6.02)(5.25)^2(8.5) = 705.2 \text{ K·ft}$$

$$R_u = \frac{M_u}{bd^2} = \frac{705.2(12{,}000)}{(8.5 \times 12)(19.5)^2} = 218 \text{ psi}$$

Applying Eq. 13.14, $\rho = 0.0042$:

$$A_s = 0.0042(8.5 \times 12)(19.5) = 8.35 \text{ in.}^2$$

$$\text{Min } A_s \text{ (shrinkage)} = 0.0018(8.5 \times 12)(24) = 4.4 \text{ in.}^2$$

$$\text{Min } A_s \text{ (flexure)} = 0.0033(8.5 \times 12)(19.5) = 6.56 \text{ in.}^2$$

Use $A_s = 8.35$ in. and 10 no. 9 bars ($A_s = 10\text{ in.}^2$) spaced at $S = (102 - 6)/9 = 10.7$ in.

7. Design steel reinforcement in the short direction. The distance from the face of the column to the edge of the footing is

$$\frac{8.5}{2} - \frac{1.5}{2} = 3.5 \text{ ft}$$

$$M_u = \frac{1}{2}(6.02)(3.5)^2(12) = 422.5 \text{ K·ft}$$

$$R_u = \frac{M_u}{bd^2} = \frac{442.5(12{,}000)}{(12 \times 12)(19.5)^2} = 97 \text{ psi}$$

Applying Eq. 13.4, $\rho = 0.0019$:

$$A_s = 0.0019(12 \times 12)(19.5) = 5.34 \text{ in.}^2$$

$$\text{Min } A_s \text{ (shrinkage)} = 0.0018(12 \times 12)(24) = 6.22 \text{ in.}^2$$

$$\text{Min } A_s \text{ (flexure)} = 0.0033(12 \times 12)(19.5) = 9.26 \text{ in.}^2$$

The value of A_s to be used must be greater than or equal to 6.22 in.^{2} Use 18 no. 6 bars (A_s = 7.92 in.^{2}).

$$\gamma_s = \frac{2}{\beta + 1} = \frac{2}{\left(\frac{12}{8.5}\right) + 1} = 0.83$$

The number of bars in an 8.5-ft band is 18(0.83) = 15 bars. The number of bars left on each side is (18 − 15)/2 = 2 bars. Therefore, place 15 no. 6 bars within the 8.5-ft band; then place two no. 6 bars (A_s = 0.88 in.^{2}) within (12 − 8.5)/2 = 1.625 ft on each side of the band. The total number of bars is 19 no. 6 bars (A_s = 8.36 in.^{2}). In this example, the bars may be distributed at equal spacings all over the 12-ft length; S = (144 − 6)/18 = 7.6 in. Details of reinforcement are shown in Fig.13.18.

8. Check the bearing stress at the base of the column, as explained in the previous example. Use four no. 8 dowel bars.
9. Development length of the main reinforcement: l_d = 29 in. for no. 6 bars and 54 in. for no. 9 bars.

$$\text{Provided } l_d \text{ (long direction)} = \left(\frac{l}{2} - \frac{c}{2} - 3 \text{ in.}\right) = 60 \text{ in.}$$

$$\text{Provided } l_d \text{ (short direction)} = 39 \text{ in.} > 29 \text{ in.}$$

Example 13.4

Determine the footing areas required for equal settlement (balanced footing design) if the usual live load is 20% for all footings. The footings are subjected to dead loads and live loads as indicated in the following table. The allowable net soil pressure is 6 ksf.

	Footing Number				
	1	2	3	4	5
Dead load	120 K	180 K	140 K	190 K	210 K
Live load	150 K	220 K	200 K	170 K	240 K

Solution

1. Determine the footing that has the largest ratio of live load to dead load. In this example, the footing 3 ratio of 1.43 is higher than the other ratios.
2. Calculate the usual load for all footings. The usual load is the dead load and the portion of live load that most commonly occurs on the structure. In this example,

$$\text{Usual load} = \text{D.L.} + 0.2(\text{L.L.})$$

The values of the usual loads are shown in the following table.

3. Determine the area of the footing that has the highest ratio of L.L./D.L.

$$\text{Area of footing 3} = \frac{\text{D.L.} + \text{L.L}}{\text{allowable soil pressure}} = \frac{140 + 200}{6} = 56.7 \text{ ft}^2$$

The usual soil pressure under footing 3 is

$$\frac{\text{Usual load}}{\text{Area of footing}} = \frac{180}{56.7} = 3.18 \text{ ksf}$$

4. Calculate the area required for each footing by dividing its usual load by the soil pressure of footing 3. The areas are tabulated in the following table. For footing 1, for example, the required area is $150/3.18 = 47.2\,\text{ft}^2$.
5. Calculate the maximum soil pressure under each footing:

$$q_{\max} = \frac{D+L}{\text{area}} \leq 6 \text{ ksf} \quad \text{(allowable soil pressure)}$$

Description	Footing Number				
	1	2	3	4	5
$\frac{\text{Live load}}{\text{Dead load}}$	1.25	1.22	1.43	0.90	1.14
Usual load = D.L. + 0.2 (L.L.) (kips)	150	224	180	224	258
Area required = $\frac{\text{usual load}}{3.18 \text{ ksf}} (\text{ft}^2)$	47.2	70.4	56.7	70.4	81.1
Max. soil pressure = $\frac{\text{D}+\text{L}}{\text{area}}$ (ksf)	5.72	5.68	6.00	5.11	5.55

Example 13.5

Design a plain concrete footing to support a 16-in.-thick concrete wall. The loads on the wall consist of a 16-K/ft dead load (including the self-weight of wall) and a 10-K/ft live load. The base of the footing is 4 ft below final grade. Use $f'_c = 3$ ksi and an allowable soil pressure of 5 ksf.

Solution

1. Calculate the effective soil pressure. Assume a total depth of footing of 28 in.

$$\text{Weight of footing} = \frac{28}{12} \times 145 = 338 \text{ psf}$$

The weight of the soil, assuming that soil weighs 100 pcf, is $(4 - 2.33) \times 100 = 167$ psf. Effective soil pressure is $5000 - 338 - 167 = 4495$ psf.

2. Calculate the width of the footing for a 1-ft length of the wall ($b = 1$ ft):

$$\text{Width of footing} = \frac{\text{total load}}{\text{effective soil pressure}}$$

$$= \frac{16+10}{4.495} = 5.79 \text{ ft}$$

Use 6.0 ft (Fig. 13.19).

3. $U = 1.2D + 1.6L = 1.2 \times 16 + 1.6 \times 10 = 35.2$ K/ft. The net upward pressure is $q_u = 35.2/6 = 5.87$ ksf.
4. Check bending stresses. The critical section is at the face of the wall. For a 1-ft length of wall and footing,

$$M_u = \frac{1}{2} q_u \left(\frac{L}{2} - \frac{c}{2}\right)^2 = \frac{1}{2}(5.87)\left(\frac{6}{2} - \frac{16}{2 \times 12}\right)^2 = 16 \text{ K·ft}$$

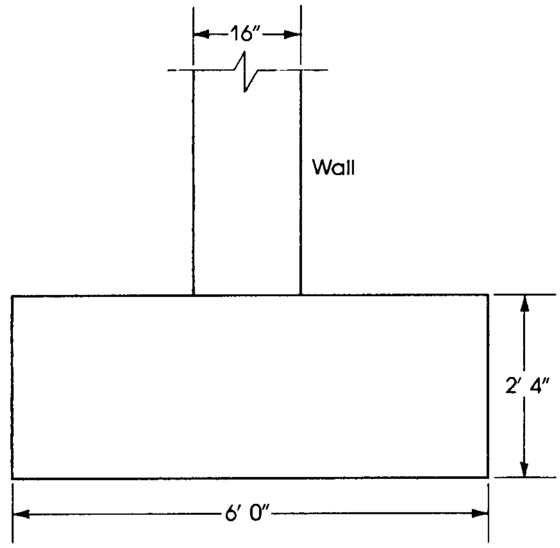


Figure 13.19 Example 13.5: Plain concrete wall footing.

Let the effective depth, d, be 28 − 3 = 25 in., assuming that the bottom 3 in. is not effective.

$$I_g = \frac{bd^3}{12} = \frac{12}{12}(25)^3 = 15{,}625 \text{ in.}^4$$

The flexural tensile stress is

$$f_t = \frac{M_u c}{I} = \frac{(16 \times 12{,}000)}{15{,}625}\left(\frac{25}{2}\right) = 153 \text{ psi}$$

The allowable flexural tensile stress is $5\phi\sqrt{f'_c} = 5 \times 0.55\sqrt{3000} = 151$ psi (close).

5. Check shear stress: The critical section is at a distance $d = 25$ in. from the face of the wall.

$$V_u = q_u\left(\frac{L}{2} - \frac{c}{2} - d\right) = 5.87\left(\frac{6}{2} - \frac{16}{2 \times 12} - \frac{25}{12}\right) = 1.47 \text{ K}$$

$$\phi V_c = \phi\left(\frac{4}{3}\right)\lambda\sqrt{f'_c}bd = \frac{(0.55)\left(\frac{4}{3}\right)(1)\sqrt{3000}(12)(25)}{1000} = 12.05 \text{ K}$$

Therefore, the section is adequate. It is advisable to use minimum reinforcement in both directions.

13.6 COMBINED FOOTINGS

When a column is located near a property line, part of the single footing might extend into the neighboring property. To avoid this situation, the column may be placed on one side or edge of the footing, causing eccentric loading. This may not be possible under certain conditions, and sometimes it is not an economical solution. A better design can be achieved by combining the footing with the nearest internal column footing, forming a combined footing. The center of gravity of the combined footing coincides with the resultant of the loads on the two columns.